Annals of the University of Craiova
Mathematics and Computer Science Series

Vol. XLIV Issue 1, June 2017

Editorial Board

Viorel Barbu, Romanian Academy, Romania
Dumitru Buşneag, University of Craiova, Romania
Philippe G. Ciarlet, French Academy of Sciences, France
Nicolae Constantinescu, University of Craiova, Romania
Jesus Ildefonso Diaz, Universidad Complutense de Madrid, Spain
Massimiliano Ferrara, Mediterranea University of Reggio Calabria, Italy
George Georgescu, University of Bucharest, Romania
Olivier Goubet, Université de Picardie Jules Verne, France
Ion Iancu, University of Craiova, Romania
Marius Iosifescu, Romanian Academy, Romania
Giovanni Molica Bisci, Mediterranea University of Reggio Calabria, Italy
Sorin Micu, University of Craiova, Romania
Gheorghe Moroşanu, Central European University Budapest, Hungary
Constantin Năstăsescu, Romanian Academy, Romania
Constantin P. Niculescu, University of Craiova, Romania
Patrizia Pucci, University of Perugia, Italy
Vicenţiu Rădulescu, University of Craiova, Romania
Dušan Repovš, University of Ljubljana, Slovenia
Sergiu Rudeanu, University of Bucharest, Romania
Mircea Sofonea, Université de Perpignan, France
Michel Willem, Université Catolique de Louvain, Belgium
Tudor Zamfirescu, Universitat Dortmund, Germany
Enrique Zuazua, Basque Center for Applied Mathematics, Spain

Managing Editor

Mihaela Sterpu, University of Craiova, Romania

Assistant Editor

Mihai Gabroveanu, University of Craiova, Romania

Information for authors. The journal is publishing all papers using electronic production methods and therefore needs to receive the electronic files of your article. These files can be submitted preferably by online submission system:

http://inf.ucv.ro/∼ami/index.php/ami/about/submissions

by e-mail at *office.annals@inf.ucv.ro* or by mail on the address:

Analele Universităţii din Craiova, Seria Matematică-Informatică
A. I. Cuza 13
Craiova, 200585, Romania
Web: http://inf.ucv.ro/∼ami/

The submitted paper should contain original work which was not previously published, is not under review at another journal or conference and does not significantly overlap with other previous papers of the authors. Each paper will be reviewed by independent reviewers. The results of the reviewing process will be transmitted by e-mail to the first author of the paper. The acceptance of the papers will be based on their scientific merit. Upon acceptance, the papers will be published both in hard copy and on the Web page of the journal, in the first available volume.

The journal is abstracted/indexed/reviewed by *Mathematical Reviews*, *Zentralblatt MATH*, *SCOPUS*. This journal is also included in many digital directories of open resources in mathematics and computer science as *Index Copernicus*, *Open J-Gate*, *AMS Digital Mathematics Registry*, *Directory of Open Access Journals*, *CENTRAL EUROPEAN UNIVERSITY - Catalogue*, etc.

Volume Editors: Vicenţiu Rădulescu, Mihaela Sterpu

Layout Editors: Mihai Gabroveanu

ISSN 1223-6934

Online ISSN 2246-9958

Printed in Romania: Editura Universitaria, Craiova, 2017.

http://www.edituraun iversitaria.ro

Table of contents

Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 1–20
ISSN: 1223-6934

Existence and uniqueness of entropy solution for some nonlinear elliptic unilateral problems in Musielak-Orlicz-Sobolev spaces

Mohammed Al-Hawmi, Abdelmoujib Benkirane, Hassane Hjiaj, and Abdelfattah Touzani

---

Abstract. In this paper, we study the existence and uniqueness of entropy solution for some quasilinear degenerate elliptic unilateral problems of the type

$$\begin{cases} -\operatorname{div} a(x,\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

in the Musielak-Orlicz-Sobolev spaces $W_0^1L_\varphi(\Omega)$, with $f \in L^1(\Omega)$ and by assuming that the conjugate function of the Musielak-Orlicz function $\varphi(x,t)$ satisfies the Δ_2-condition. An example of such equation is given by

$$\begin{cases} -\operatorname{div}\Big(|\nabla u|^{p(x)-2}\log^{\sigma}(1+|\nabla u|)\ \nabla u\Big) = f & \text{in} \ \ \Omega, \\ u \ = 0 & \text{on} \ \ \partial\Omega, \end{cases} \tag{1}$$

for $1 \leq p(x) < \infty$ and $0 < \sigma < \infty$.

2010 Mathematics Subject Classification. 35J62, 35J25.
Key words and phrases. Musielak-Orlicz-Sobolev spaces, quasilinear degenerate elliptic equations, unilateral problem, entropy solutions, truncations.

---

1. Introduction

Let Ω be a bounded open subset of $I\!\!R^N$ $(N \geq 2)$, with smooth boundary conditions.

For $2-\frac{1}{N} < p < N$, Boccardo and Gallouët have studied in [11] the elliptic problem of the type

$$\begin{cases} Au = f & \text{in} \quad \Omega, \\ \ u = \ 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $Au = -\operatorname{div} a(x,u,\nabla u)$ is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into its dual, and f is a bounded Radon measure on Ω. They have proved the existence of solutions $u \in W_0^{1,q}(\Omega)$ for all $1 < q < \bar{q} = \frac{N(p-1)}{N-1}$. Also they proved some regularity results.

Aharouch and Bennouna have treated in [1] the quasilinear elliptic of unilateral problem

$$\begin{cases} -\operatorname{div}\ (a(x,\nabla u)) = f & \text{in} \quad \Omega, \\ \ u = \ 0 \text{ on} & \partial\Omega, \end{cases} \tag{2}$$

Received March 17, 2015.

where $f \in L^1(\Omega)$. They have proved the existence and uniqueness of entropy solutions in the framework of Orlicz Sobolev spaces $W_0^1L_M(\Omega)$ without assuming the Δ_2-condition on the $N-$function M of the Orlicz spaces, (see also. [6, 7, 13]).

In [5], Bendahmane and Wittbold have proved existence and uniqueness of a renormalized solution to the nonlinear elliptic equation

$$\begin{cases} -\text{div } \left(|\nabla u|^{p(x)-2}\nabla u\right) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3}$$

where the right-hand side $f \in L^1(\Omega)$ and the exponent $p(\cdot) : \overline{\Omega} \mapsto (1, +\infty)$ is continuous, for some related results we refer to [2, 4, 12, 22].

In the recent years, Musielak-Orlicz-Sobolev spaces have attracted the attention of mainly researchers, the impulse for this manly comes from there physical applications, such in electro-rheological fluids, (see [23]). The purpose of this paper is to prove the existence and uniqueness of entropy solutions for some quasilinear unilateral elliptic problem of the form

$$\begin{cases} Au = f \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

in Musielak-Orlicz-Sobolev spaces, where $f \in L^1(\Omega)$ and $A : D(A) \subset W_0^1L_\varphi(\Omega) \mapsto W^{-1}L_\psi(\Omega)$ is the Leray-Lions operator defined as:

$$A(u) = -\text{div } a(x, \nabla u),$$

by assuming that the conjugate function of Musielak-Orlicz function $\varphi(x,t)$ satisfies Δ_2-condition, and by using corollary 1 of [9] to construct a complementary system $(W_0^1L_\varphi(\Omega), W_0^1E_\varphi(\Omega);\ W^{-1}L_\psi(\Omega),\ W^{-1}E_\psi(\Omega))$.

Note that, the second author has studded in [9] the existence of solution for the problem (4) where f is assumed to be in the dual, and only strict monotonicity is assumed, we refer also to [19] for the elliptic case with large monotonicity, and the interesting works of Gwiazda el al. [16, 17, 18] in the generalized Orlicz Sobolev spaces, also [14] where the author has proved the Poincaré inequality under the Δ_2-condition.

This paper is organized as follows. In the section 2 we recall some definitions and basic properties of Musielak-Orlicz-Sobolev. We introduce in the section 3 the assumptions on $a(x, \xi)$ under which our problem has at least one solution. The section 4 contains some useful lemmas for proving our main results. The section 5 will be devoted to show the existence and uniqueness of entropy solutions for our main problem (4).

2. Preliminaries

In this section, we introduce some definitions and known facts about Musielak-Orlicz-Sobolev spaces. The standard reference is [24].

2.1. Musielak-Orlicz function. Let Ω be an open bounded subset of $I\!R^N$ $(N \geq 2)$ with smooth boundary conditions, and let $\varphi(x,t)$ be a real-valued function defined on $\Omega \times I\!R^+$, and satisfying the following two conditions :

(a): $\varphi(x,\cdot)$ is an N-function, *i.e.* convex, nondecreasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0$ for all $t>0$, and :

$$\lim_{t\to 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t}=0 \quad , \quad \lim_{t\to\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t}=\infty,$$

(b): $\varphi(\cdot,t)$ is a measurable function.

A function $\varphi(x,t)$ which satisfies conditions (a) and (b) is called a Musielak-Orlicz function.

For every Musielak-Orlicz function $\varphi(x,t)$, we set $\varphi_x(t)=\varphi(x,t)$ and let $\varphi_x^{-1}(t)$ the reciprocal function with respect to t of $\varphi_x(t)$, i.e.

$$\varphi_x^{-1}(\varphi(x,t))=\varphi(x,\varphi_x^{-1}(t))=t.$$

For any two Musielak-Orlicz functions $\varphi(x,t)$ and $\gamma(x,t)$, we introduce the following ordering:

(c): If there exist two positive constants c and T such that for almost everywhere $x\in\Omega$:

$$\varphi(x,t)\leq\gamma(x,ct) \quad \text{for} \quad t\geq T,$$

we write $\varphi\prec\gamma$, and we say that γ dominate φ globally if $T=0$, and near infinity if $T>0$.

(d): For every positive constant c and almost everywhere $x\in\Omega$, if

$$\lim_{t\to 0}(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)})=0 \quad or \quad \lim_{t\to\infty}(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)})=0,$$

we write $\varphi\prec\prec\gamma$ at 0 or near ∞ respectively, and we say that φ increases essentially more slowly than γ at 0 or near ∞ respectively.

The Musielak-Orlicz function $\psi(x,t)$ complementary to (or conjugate of) $\varphi(x,t)$, in the sense of Young with respect to the variable t, is given by

$$\psi(x,s)=\sup_{t\geq 0}\{st-\varphi(x,t)\}, \tag{5}$$

and we have

$$st\leq\psi(x,s)+\varphi(x,t) \qquad \forall s,t\in I\!R^+. \tag{6}$$

The Musielak-Orlicz function $\varphi(x,t)$ is said to satisfy the Δ_2-condition if, there exists $k>0$ and a nonnegative function $h(\cdot)\in L^1(\Omega)$, such that

$$\varphi(x,2t)\leq k\varphi(x,t)+h(x) \qquad \text{a.e.} \quad x\in\Omega,$$

for large values of t, or for all values of t.

2.2. Musielak-Orlicz Lebesgue spaces. In this paper, the measurability of a function $u:\Omega\mapsto I\!R$ means the Lebesgue measurability.

We define the functional

$$\varrho_{\varphi,\Omega}(u)=\int_\Omega\varphi(x,|u(x)|)\,dx,$$

where $u:\Omega\mapsto I\!R$ is a measurable function. The set

$$K_\varphi(\Omega)=\{u:\Omega\longmapsto I\!R \quad \text{measurable} \ \ / \ \ \varrho_{\varphi,\Omega}(u)<+\infty\}$$

is called the Musielak-Orlicz class (or the generalized Orlicz class). The Musielak-Orlicz spaces (or the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$; equivalently

$$L_\varphi(\Omega)=\Big\{u:\Omega\longmapsto I\!R \quad \text{measurable} \quad / \quad \varrho_{\varphi,\Omega}(\frac{|u(x)|}{\lambda})<+\infty, \quad \text{for some } \lambda>0\Big\}.$$

In the space $L_\varphi(\Omega)$, we define the following two norms:

$$||u||_{\varphi,\Omega}=\inf\Big\{\lambda>0 \ \ / \ \ \int_\Omega \varphi(x,\frac{|u(x)|}{\lambda})\,dx\leq 1\Big\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm is given by:

$$|||u|||_{\varphi,\Omega}=\sup_{||v||_{\psi,\Omega}\leq 1}\int_\Omega |u(x)v(x)|\,dx,$$

where $\psi(x,t)$ is the Musielak-Orlicz function complementary (or conjugate) to $\varphi(x,t)$. These two norms are equivalent on the Musielak-Orlicz space $L_\varphi(\Omega)$.

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $(E_\varphi(\Omega))^*=L_\psi(\Omega)$.

We have $E_\varphi(\Omega)=K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega)=L_\varphi(\Omega)$ if and only if $\varphi(x,t)$ has the Δ_2-condition for large values of t, or for all values of t.

2.3. Musielak-Orlicz-Sobolev spaces. We now turn to the Musielak-Orlicz-Sobolev space $W^1L_\varphi(\Omega)$ (resp. $W^1E_\varphi(\Omega)$) is the space of all measurable functions u such that u and its distributional derivatives up to order 1 lie in $L_\varphi(\Omega)$ (resp. $E_\varphi(\Omega)$). Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ with nonnegative integers α_i, $|\alpha|=|\alpha_1|+|\alpha_2|+...+|\alpha_n|$ and $D^\alpha u$ denotes the distributional derivatives.
We define the convex modular and the norm on the Musielak-Orlicz-Sobolev spaces $W^1L_\varphi(\Omega)$ respectively by,

$$\overline{\varrho}_{\varphi,\Omega}(u)=\sum_{|\alpha|\leq 1}\varrho_{\varphi,\Omega}(D^\alpha u) \quad \textit{and} \quad ||u||_{1,\varphi,\Omega}=\inf\Big\{\lambda>0:\overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda})\leq 1\Big\},$$

for any $u\in W^1L_\varphi(\Omega)$.
The pair $\langle W^1L_\varphi(\Omega),||u||_{1,\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition

$$\text{there exists a constant} \quad c>0 \quad \text{such that} \quad \inf_{x\in\Omega}\varphi(x,1)\geq c.$$

The spaces $W^1L_\varphi(\Omega)$ and $W^1E_\varphi(\Omega)$ can be identified with subspaces of the product of $n+1$ copies of $L_\varphi(\Omega)$. Denoting this product by $\Pi L_\varphi(\Omega)$, we will use the weak topologies $\sigma(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega))$ and $\sigma(\Pi E_\psi(\Omega),\Pi L_\varphi(\Omega))$.

The space $W_0^1E_\varphi(\Omega)$ is defined as the (norm) closure of the Schwartz space D(Ω) in $W^1E_\varphi(\Omega)$, and the space $W_0^1L_\varphi(\Omega)$ as the $\sigma(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega))$ closure of D(Ω) in $W^1L_\varphi(\Omega)$, (for more details on Musielak-Orlicz-Sobolev spaces we refer to [24]).

2.4. Dual space. Let $W^{-1}L_\psi(\Omega)$ (resp. $W^{-1}E_\psi(\Omega)$) denotes the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_\psi(\Omega)$ (resp. $E_\psi(\Omega)$). It is a Banach space under the usual quotient norm.

If $\psi(x,t)$ has the Δ_2-condition, then the space D(Ω) is dense in $W_0^1L_\varphi(\Omega)$ for the topology $\sigma(\Pi L_\varphi(\Omega),\Pi L_\psi(\Omega))$ (see corollary 1 of [9]).

3. Essential assumptions

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$ with smooth boundary conditions. Let $\varphi(x,t)$ be a Musielak-Orlicz function and $\psi(x,t)$ the Musielak-Orlicz function complementary (or conjugate) to $\varphi(x,t)$. We assume here that $\psi(x,t)$ satisfying the Δ_2-condition near infinity, therefore $L_\psi(\Omega) = E_\psi(\Omega)$.
We assume that there exists an Orlicz function $M(t)$ such that $M(t) \prec \varphi(x,t)$ near infinity, i.e. there exist two constants $c > 0$ and $T \geq 0$ such that

$$M(t) \leq \varphi(x, ct) \qquad \text{a.e. in} \quad \Omega \qquad \text{for} \quad t \geq T. \tag{7}$$

Let $\Psi(\cdot)$ be a measurable function on Ω, such that

$$\Psi^+(\cdot) \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega),$$

and we consider the convex set

$$K_\Psi = \Big\{ v \in W_0^1 L_\varphi(\Omega) \ such \ \ that \, v \geq \Psi \ a.e. \ in \ \Omega \Big\}.$$

The Leray-Lions operator $A : D(A) \subset W_0^1 L_\varphi(\Omega) \longmapsto W^{-1} L_\psi(\Omega)$ given by

$$A(u) = -\text{div } a(x, \nabla u)$$

where $a : \Omega \times I\!R^N \longmapsto I\!R$ is a *Carathéodory* function (measurable with respect to x in Ω for every ξ in $I\!R^N$, and continuous with respect to ξ in $I\!R^N$ for almost every x in Ω) which satisfies the following conditions

$$|a(x,\xi)| \leq \beta\big(K(x) + k_1 \psi_x^{-1}(\varphi(x, k_2|\xi|))\big), \tag{8}$$

$$\big(a(x,\xi) - a(x,\xi^*)\big) \cdot \big(\xi - \xi^*\big) > 0 \qquad \text{for} \quad \xi \neq \xi^*, \tag{9}$$

$$a(x,\xi) \cdot \xi \geq \alpha \ \varphi(x, |\xi|), \tag{10}$$

for a.e. $x \in \Omega$ and all $\xi \in I\!R^N$, where $K(x)$ is a nonnegative function lying in $E_\psi(\Omega)$ and $\alpha, \beta > 0$ and $k_1, k_2 \geq 0$.
We consider the quasilinear unilateral elliptic problem

$$\begin{cases} -\text{div}\, a(x, \nabla u) = f & \text{in } \ \Omega, \\ u = 0 & \text{in } \ \partial\Omega, \end{cases} \tag{11}$$

with $f \in L^1(\Omega)$. We study the existence of entropy solution in the Musielak-Orlicz-Sobolev spaces.

4. Some technical lemmas

Now, we present some lemmas useful in the proof of our main results.

Lemma 4.1. *(see* [20], *Theorem 13.47) Let* $(u_n)_n$ *be a sequence in* $L^1(\Omega)$ *and* $u \in L^1(\Omega)$ *such that*

(i): $u_n \to u$ *a.e. in* Ω,
(ii): $u_n \geq 0$ *and* $u \geq 0$ *a.e. in* Ω,
(iii): $\displaystyle\int_\Omega u_n \, dx \to \int_\Omega u \, dx$,

then $u_n \to u$ *in* $L^1(\Omega)$.

Lemma 4.2. *Assuming that* (8)$-$(10) *hold, and let* $(u_n)_n$ *be a sequence in* $W_0^1 L_\varphi(\Omega)$ *such that*

(i): *$u_n \rightharpoonup u$ weakly in $W_0^1L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega))$,*
(ii): *$(a(x, \nabla u_n))_n$ is bounded in $(L_\psi(\Omega))^N = (E_\psi(\Omega))^N$,*
(iii): *Let $\Omega_s = \big\{x \in \Omega, \quad |\nabla u| \le s\big\}$ and χ_s his characteristic function, with*

$$\int_\Omega (a(x, \nabla u_n) - a(x, \nabla u\chi_s)) \cdot (\nabla u_n - \nabla u\chi_s)\, dx \longrightarrow 0 \quad as \quad n, s \to \infty, \tag{12}$$

then $\varphi(x, |\nabla u_n|) \longrightarrow \varphi(x, |\nabla u|)$ in $L^1(\Omega)$ for a subsequence.

Proof. Taking $s \ge r > 0$, we have :

$$\begin{aligned} 0 &\le \int_{\Omega_r} (a(x, \nabla u_n) - a(x, \nabla u)) \cdot (\nabla u_n - \nabla u)\, dx \\ &\le \int_{\Omega_s} (a(x, \nabla u_n) - a(x, \nabla u)) \cdot (\nabla u_n - \nabla u)\, dx \\ &= \int_{\Omega_s} (a(x, \nabla u_n) - a(x, \nabla u\chi_s)) \cdot (\nabla u_n - \nabla u\chi_s)\, dx \\ &\le \int_{\Omega} (a(x, \nabla u_n) - a(x, \nabla u\chi_s)) \cdot (\nabla u_n - \nabla u\chi_s)\, dx. \end{aligned} \tag{13}$$

thanks to (12), we obtain

$$\lim_{n\to\infty} \int_{\Omega_r} (a(x, \nabla u_n) - a(x, \nabla u)) \cdot (\nabla u_n - \nabla u)\, dx = 0. \tag{14}$$

Using the same argument as in [15], we claim that,

$$\nabla u_n \longrightarrow \nabla u \qquad \text{a.e. in} \quad \Omega. \tag{15}$$

On the other hand, we have

$$\begin{aligned} \int_\Omega a(x, \nabla u_n) \cdot \nabla u_n\, dx &= \int_\Omega (a(x, \nabla u_n) - a(x, \nabla u\chi_s)) \cdot (\nabla u_n - \nabla u\chi_s)\, dx \\ &\quad + \int_\Omega a(x, \nabla u\chi_s) \cdot (\nabla u_n - \nabla u\chi_s)\, dx + \int_\Omega a(x, \nabla u_n) \cdot \nabla u\chi_s\, dx. \end{aligned} \tag{16}$$

For the second term on the right-hand side of (16), having in mind that $\psi(x, s)$ verify Δ_2-condition, then $L_\psi(\Omega) = E_\psi(\Omega)$, and thanks to (8) we have $a(x, \nabla u\chi_s) \in (E_\psi(\Omega))^N$. Moreover, we have $\nabla u_n \rightharpoonup \nabla u$ weakly in $(L_\varphi(\Omega))^N$ for $\sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega))$, then

$$\begin{aligned} \lim_{s,n\to\infty} \int_\Omega a(x, \nabla u\chi_s) \cdot (\nabla u_n - \nabla u\chi_s)\, dx &= \lim_{s\to\infty} \int_\Omega a(x, \nabla u\chi_s) \cdot (\nabla u - \nabla u\chi_s)\, dx \\ &= \lim_{s\to\infty} \int_{\Omega/\Omega_s} a(x, 0) \cdot \nabla u\, dx = 0. \end{aligned} \tag{17}$$

Concerning the last term on the right-hand side of (16), since $(a(x, \nabla u_n))_n$ is bounded in $(E_\psi(\Omega))^N$ and using (15), we obtain

$$a(x, \nabla u_n) \rightharpoonup a(x, \nabla u) \quad \text{weakly in} \quad (E_\psi(\Omega))^N \quad \text{for} \quad \sigma(\Pi E_\psi(\Omega), \Pi L_\varphi(\Omega)),$$

which implies that

$$\begin{aligned} \lim_{s,n\to\infty} \int_\Omega a(x, \nabla u_n) \cdot \nabla u\chi_s\, dx &= \lim_{s\to\infty} \int_\Omega a(x, \nabla u) \cdot \nabla u\chi_s\, dx \\ &= \int_\Omega a(x, \nabla u) \cdot \nabla u\, dx. \end{aligned} \tag{18}$$

By combining (12) and (16) – (18), we conclude that

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla u_n\,dx \longrightarrow \int_\Omega a(x,\nabla u)\cdot\nabla u\,dx \quad \text{as} \quad n\to\infty. \tag{19}$$

On the other hand, we have $\varphi(x,|\nabla u_n|)\geq 0$ and $\varphi(x,|\nabla u_n|)\to\varphi(x,|\nabla u|)$ a.e. in Ω, by using the Fatou's Lemma we obtain

$$\int_\Omega \varphi(x,|\nabla u|)\,dx \leq \liminf_{n\to\infty}\int_\Omega \varphi(x,|\nabla u_n|)\,dx. \tag{20}$$

Moreover, since $a(x,\nabla u_n)\cdot\nabla u_n-\alpha\varphi(x,|\nabla u_n|)\geq 0$ and

$$a(x,\nabla u_n)\cdot\nabla u_n-\alpha\varphi(x,|\nabla u_n|)\longrightarrow a(x,\nabla u)\cdot\nabla u-\alpha\varphi(x,|\nabla u|) \qquad \text{a.e. in} \quad \Omega,$$

Thanks to Fatou's Lemma, we get

$$\int_\Omega a(x,\nabla u)\cdot\nabla u-\alpha\varphi(x,|\nabla u|)\,dx \leq \liminf_{n\to\infty}\int_\Omega a(x,\nabla u_n)\cdot\nabla u_n-\alpha\varphi(x,|\nabla u_n|)\,dx,$$

using (19), we obtain

$$\int_\Omega \varphi(x,|\nabla u|)\,dx \geq \limsup_{n\to\infty}\int_\Omega \varphi(x,|\nabla u_n|)\,dx. \tag{21}$$

By combining (20) and (21), we deduce

$$\int_\Omega \varphi(x,|\nabla u_n|)\,dx \longrightarrow \int_\Omega \varphi(x,|\nabla u|)\,dx \qquad \text{as} \quad n\to\infty. \tag{22}$$

In view of Lemma 4.1, we conclude that

$$\varphi(x,|\nabla u_n|)\longrightarrow \varphi(x,|\nabla u|) \qquad \text{in} \quad L^1(\Omega), \tag{23}$$

which finishes our proof.

5. Main results

Let $k>0$, we define the truncation function $T_k(\cdot):I\!R\longmapsto I\!R$ by

$$T_k(s)=\begin{cases} s & \text{if} \quad |s|\leq k,\\ k\dfrac{s}{|s|} & \text{if} \quad |s|>k.\end{cases}$$

Definition 5.1. A measurable function u is called an entropy solution of the quasi-linear unilateral elliptic problem (11) if

$$\begin{cases} T_k(u)\in K_\Psi \qquad \text{for any} \quad k>\|\Psi^+\|_\infty,\\[2ex] \displaystyle\int_\Omega a(x,\nabla u)\cdot\nabla T_k(u-v)\,dx \leq \int_\Omega fT_k(u-v)\,dx \qquad \forall v\in K_\Psi\cap L^\infty(\Omega).\end{cases} \tag{24}$$

Theorem 5.1. *Assuming that* (7) – (10) *hold, and* $f\in L^1(\Omega)$, *Then, the problem* (11) *has a unique entropy solution.*

5.1. Existence of entropy solution.

Step 1 : Approximate problems. Let $(f_n)_{n\in I\!N} \in W^{-1}E_\psi(\Omega)\cap L^\infty(\Omega)$ be a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \le |f|$ (for example $f_n = T_n(f)$). We consider the approximate problem

$$(P_n)\begin{cases} u_n \in K_\Psi, \\ \displaystyle\int_\Omega a(x,\nabla u_n)\cdot\nabla(u_n - v)\,dx \le \int_\Omega f_n(u_n - v)\,dx \quad \text{for any} \quad v \in K_\Psi \cap L^\infty(\Omega). \end{cases} \tag{25}$$

Let $X = K_\Psi$, we define the operator $A : X \longmapsto X^*$ by

$$\langle Au, v\rangle = \int_\Omega a(x,\nabla u)\cdot\nabla v \;dx \qquad \forall v \in K_\Psi.$$

Using (6), we have for any $u, v \in K_\Psi$,

$$\begin{aligned} \Big|\int_\Omega a(x,\nabla u)\cdot\nabla v\,dx\Big| &\le \int_\Omega \beta\big(K(x) + k_1\psi_x^{-1}(\varphi(x,k_2|\nabla u|))\big)|\nabla v|\,dx \\ &\le \beta\int_\Omega \psi(x,K(x))\,dx + \beta k_1\int_\Omega \varphi(x,k_2|\nabla u|)\,dx + \beta(1+k_1)\int_\Omega \varphi(x,|\nabla v|)\,dx. \end{aligned} \tag{26}$$

Lemma 5.2. *The operator A acted from $W_0^1L_\varphi(\Omega)$ in to $W^{-1}L_\psi(\Omega) = W^{-1}E_\psi(\Omega)$ is bounded and pseudo-monotone. Moreover, A is coercive in the following sense : there exists $v_0 \in K_\Psi$ such that*

$$\frac{\langle Av, v - v_0\rangle}{||v||_{1,\varphi,\Omega}} \longrightarrow \infty \qquad \text{as} \quad ||v||_{1,\varphi,\Omega} \to \infty \qquad \text{for} \quad v \in K_\Psi.$$

Proof of Lemma 5.2. In view of (26), the operator A is bounded. For the coercivity, let $\varepsilon > 0$, we have for $v_0 \in K_\Psi$ and any $v \in W_0^1L_\varphi(\Omega)$

$$\begin{aligned} \big|\langle Av, v_0\rangle\big| &\le \int_\Omega \big|a(x,\nabla v)\big|\,\big|\nabla v_0\big|\,dx \le \beta\int_\Omega (K(x) + k_1\psi_x^{-1}(\varphi(x,k_2|\nabla v|)))|\nabla v_0|\,dx \\ &\le \beta\int_\Omega K(x)|\nabla v_0|\,dx + \beta k_1\varepsilon\int_\Omega \psi_x^{-1}(\varphi(x,k_2|\nabla v|))\frac{1}{\varepsilon}|\nabla v_0|\,dx \\ &\le \beta\int_\Omega \psi(x,K(x))dx + \beta\int_\Omega \varphi(x,|\nabla v_0|)dx + \beta k_1\varepsilon\int_\Omega \varphi(x,k_2|\nabla v|)\,dx \\ &\quad + \beta k_1\varepsilon\int_\Omega \varphi(x,\frac{1}{\varepsilon}|\nabla v_0|)\,dx \\ &\le c_\varepsilon\int_\Omega \varphi(x,|\nabla v|)dx + \beta(k_1\varepsilon + 1)\int_\Omega \varphi(x,(\frac{1}{\varepsilon}+1)|\nabla v_0|)\,dx + C_1, \end{aligned}$$

with c_ε is a constant depending on ε. By taking ε small enough such that $c_\varepsilon \le \frac{\alpha}{2}$, we obtain

$$\langle Av, v_0\rangle \le \frac{\alpha}{2}\int_\Omega \varphi(x,|\nabla v|)dx + \beta(k_1\varepsilon + 1)\int_\Omega \varphi(x,(\frac{1}{\varepsilon}+1)|\nabla v_0|)\,dx + C_1.$$

On the other hand, in view of (10), we have

$$\langle Av, v\rangle = \int_\Omega a(x,\nabla v)\cdot\nabla v\,dx \ge \alpha\int_\Omega \varphi(x,|\nabla v|)\,dx.$$

Therefore

$$\frac{\langle Av, v - v_0\rangle}{||v||_{1,\varphi,\Omega}} = \frac{\langle Av, v\rangle - \langle Av, v_0\rangle}{||v||_{1,\varphi,\Omega}}$$

$$\geq \frac{\alpha\displaystyle\int_\Omega \varphi(x, |\nabla v|)\, dx - \frac{\alpha}{2}\int_\Omega \varphi(x, |\nabla v|)dx - \beta(k_1\varepsilon + 1)\int_\Omega \varphi(x, (\frac{1}{\varepsilon} + 1)|\nabla v_0|)\, dx + C_1}{\|v\|_{1,\varphi,\Omega}}$$

$$= \frac{\displaystyle\frac{\alpha}{2}\int_\Omega \varphi(x, |\nabla v|)\, dx - \beta(k_1\varepsilon + 1)\int_\Omega \varphi(x, (\frac{1}{\varepsilon} + 1)|\nabla v_0|)\, dx + C_1}{\|v\|_{1,\varphi,\Omega}} \longrightarrow \infty$$

as $\|v\|_{1,\varphi,\Omega}$ goes to infinity.

It remains to show that A is pseudo-monotone. Let $(u_k)_k$ be a sequence in $W_0^1L_\varphi(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \text{ in } W_0^1L_\varphi(\Omega) & \text{for } \sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega)), \\ Au_k \rightharpoonup \chi \text{ in } W^{-1}E_\psi(\Omega) & \text{for } \sigma(\Pi E_\psi(\Omega), \Pi L_\varphi(\Omega)), \\ \limsup\limits_{k\to\infty}\langle Au_k, u_k\rangle \leq \langle \chi, u\rangle. \end{cases} \tag{27}$$

We will prove that

$$\chi = Au \text{ and } \langle Au_k, u_k\rangle \to \langle \chi, u\rangle \text{ as } k \to \infty.$$

Firstly, since $W_0^1L_\varphi(\Omega) \hookrightarrow\hookrightarrow E_\varphi(\Omega)$, then $u_k \to u$ in $E_\varphi(\Omega)$ for a subsequence still denoted $(u_k)_k$.

As $(u_k)_k$ is a bounded sequence in $W_0^1L_\varphi(\Omega)$ and thanks to the growth condition (8), it follows that $(a(x, \nabla u_k))_k$ is bounded in $(E_\psi(\Omega))^N$. Therefore, there exists a function $\xi \in (E_\psi(\Omega))^N$ such that

$$a(x, \nabla u_k) \rightharpoonup \xi \quad \text{in} \quad (E_\psi(\Omega))^N \quad \text{for} \quad \sigma(\Pi E_\psi(\Omega), \Pi L_\varphi(\Omega)) \quad \text{as} \quad k \to \infty. \tag{28}$$

It is clear that, for all $v \in W_0^1L_\varphi(\Omega)$, we have

$$\langle \chi, v\rangle = \lim_{k\to\infty}\langle Au_k, v\rangle = \lim_{k\to\infty}\int_\Omega a(x, \nabla u_k)\cdot\nabla v\, dx = \int_\Omega \xi\cdot\nabla v\, dx. \tag{29}$$

By using (27) and (29), we obtain

$$\limsup_{k\to\infty}\langle Au_k, u_k\rangle = \limsup_{k\to\infty}\int_\Omega a(x, \nabla u_k)\cdot\nabla u_k\, dx \leq \int_\Omega \xi\cdot\nabla u\, dx. \tag{30}$$

On the other hand, thanks to (9), we have

$$\int_\Omega \Big(a(x, \nabla u_k) - a(x, \nabla u)\Big)\cdot(\nabla u_k - \nabla u)\, dx \geq 0, \tag{31}$$

then

$$\int_\Omega a(x, \nabla u_k)\cdot\nabla u_k\, dx \geq \int_\Omega a(x, \nabla u_k)\cdot\nabla u\, dx + \int_\Omega a(x, \nabla u)\cdot(\nabla u_k - \nabla u)\, dx.$$

In view of (28), we have

$$\liminf_{k\to\infty}\int_\Omega a(x, \nabla u_k)\cdot\nabla u_k\, dx \geq \int_\Omega \xi\cdot\nabla u\, dx$$

and (30) yields

$$\lim_{k\to\infty}\int_\Omega a(x,\nabla u_k)\cdot\nabla u_k\,dx=\int_\Omega \xi\cdot\nabla u\,dx. \tag{32}$$

Combining (29) and (32), we find:

$$\langle Au_k,u_k\rangle\to\langle\chi,u\rangle \qquad \text{as}\quad k\to\infty. \tag{33}$$

In view of (32), we have

$$\lim_{k\to\infty}\int_\Omega \Big(a(x,\nabla u_k)-a(x,\nabla u)\Big)\cdot(\nabla u_k-\nabla u)\,dx\to\ 0$$

which implies, thanks to Lemma 4.2, that

$$u_k\to u\quad\text{in}\quad W_0^1L_\varphi(\Omega)\qquad\text{and}\qquad \nabla u_k\to\nabla u\quad\text{a.e. in}\quad\Omega,$$

then

$$a(x,\nabla u_k)\rightharpoonup a(x,\nabla u)\quad\text{in}\quad (E_\psi(\Omega))^N,$$

we deduce that $\chi=Au$, which completes the proof the Lemma 5.2. □

In view of Lemma 5.2, there exists at least one weak solution $u_n\in W_0^1L_\varphi(\Omega)$ of the problem (25), (cf. [10], Lemma 6).

Step 2 : A priori estimates. Taking $v=u_n-\eta T_k(u_n-\Psi^+)\in W_0^1L_\varphi(\Omega),$ for η small enough we have $v\geq\Psi,$ thus v is an admissible test function in (25), and we obtain

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla T_k(u_n-\Psi^+)\,dx\leq\int_\Omega f_nT_k(u_n-\Psi^+)\,dx,$$

Since $\nabla T_k(u_n-\Psi^+)$ is identically zero on the set $\{|u_n-\Psi^+|>k\}$, we can write

$$\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla(u_n-\Psi^+)\ dx\leq\int_\Omega f_nT_k(u_n-\Psi^+)\,dx\leq C_2k,$$

with $C_2=\|f\|_1,$ it follows that

$$\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx\leq C_2k+\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla\Psi^+\,dx.$$

Let $0<\lambda<\dfrac{\alpha}{\alpha+1}$, it's clear that

$$\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx\leq C_2k+\lambda\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\frac{\nabla\Psi^+}{\lambda}\,dx. \tag{34}$$

Thanks to (9), we have

$$\int_{\{|u_n-\Psi^+|\leq k\}}\Big(a(x,\nabla u_n)-a(x,\frac{\nabla\Psi^+}{\lambda})\Big)\cdot(\nabla u_n-\frac{\nabla\Psi^+}{\lambda})\,dx\geq 0,$$

then

$$\begin{aligned}\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\frac{\nabla\Psi^+}{\lambda}\,dx\ &\leq\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx\\ &\quad-\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\frac{\nabla\Psi^+}{\lambda})\cdot(\nabla u_n-\frac{\nabla\Psi^+}{\lambda})\,dx.\end{aligned}$$

Which yields thanks to (34), that

$$\begin{aligned}\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx &\leq C_2 k+\lambda\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx\\ &\quad -\lambda\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\frac{\nabla\Psi^+}{\lambda})\cdot(\nabla u_n-\frac{\nabla\Psi^+}{\lambda})\,dx.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}(1-\lambda)\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx &\leq C_2 k+\lambda\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\frac{\nabla\Psi^+}{\lambda})\cdot\frac{\nabla\Psi^+}{\lambda}\,dx\\ &\quad -\lambda\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\frac{\nabla\Psi^+}{\lambda})\cdot\nabla u_n\,dx,\end{aligned}\tag{35}$$

In view of (6), we have

$$\begin{aligned}\Big|\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\frac{\nabla\Psi^+}{\lambda})\cdot\nabla u_n\,dx\Big| &\leq \int_{\{|u_n-\Psi^+|\leq k\}} \psi(x,|a(x,\frac{\nabla\Psi^+}{\lambda})|)\,dx\\ &\quad +\int_{\{|u_n-\Psi^+|\leq k\}} \varphi(x,|\nabla u_n|)\,dx.\end{aligned}$$

Having in mind (10) and (35), we obtain

$$\begin{aligned}&\Big(\alpha(1-\lambda)-\lambda\Big)\int_{\{|u_n-\Psi^+|\leq k\}} \varphi(x,|\nabla u_n|)\,dx\\ &\leq (1-\lambda)\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx-\lambda\int_{\{|u_n-\Psi^+|\leq k\}} \varphi(x,|\nabla u_n|)\,dx\\ &\leq C_2 k+\lambda\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\frac{\nabla\Psi^+}{\lambda})\cdot\frac{\nabla\Psi^+}{\lambda}\,dx+\lambda\int_{\{|u_n-\Psi^+|\leq k\}} \psi(x,|a(x,\frac{\nabla\Psi^+}{\lambda})|)\,dx,\end{aligned}$$

then,

$$\int_{\{|u_n-\Psi^+|\leq k\}} \varphi(x,|\nabla u_n|)\,dx\leq C_3 k \qquad \text{for} \quad k\geq 1. \tag{36}$$

On the other hand, since $\{|u_n|\leq k\}\subset\{|u_n-\Psi^+|\leq k+||\Psi^+||_\infty\}$, then

$$\begin{aligned}\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx &= \int_{\{|u_n|\leq k\}} \varphi(x,|\nabla u_n|)\,dx\\ &\leq \int_{\{|u_n-\Psi^+|\leq k+||\Psi^+||_\infty\}} \varphi(x,|\nabla u_n|)\,dx\\ &\leq C_3(k+||\Psi^+||_\infty),\end{aligned}$$

which implies that

$$\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx\leq C_4 k \qquad \text{for} \quad k\geq\max(1,\|\Psi^+\|_\infty), \tag{37}$$

with C_4 is a constant that does not depend on n and k.
Thus $(T_k(u_n))_n$ is bounded in $W_0^1L_\varphi(\Omega)$ uniformly in n, then there exists a subsequence still denoted $(T_k(u_n))_{n\in I\!N}$ and $v_k\in W_0^1L_\varphi(\Omega)$ such that

$$\begin{cases} T_k(u_n)\rightharpoonup v_k & \text{weakly in } W_0^1L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega)),\\ T_k(u_n)\to v_k & \text{strongly in } E_\varphi(\Omega) \text{ and a.e in } \Omega.\end{cases}\tag{38}$$

Step 3 : Convergence in measure of u_n. In view of (7), we have

$$M(t) \leq \varphi(x, ct) \quad \text{a.e. in} \quad \Omega \qquad \text{with} \qquad \lim_{t\to 0} \frac{M(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{M(t)}{t} = \infty.$$

In view of ([15], Lemma 5.7), there exists two positive constants C_5 and C_6, and a function $q(\cdot) \in L^1(\Omega)$ such that

$$C_5\int_\Omega M(|T_k(u_n)|)dx+\int_\Omega q(x)dx \leq \int_\Omega M(C_6|\nabla T_k(u_n)|)+q(x)dx \leq \int_\Omega \varphi(x, |\nabla T_k(u_n)|)dx.$$

So, in virtue of (37), we obtain

$$\int_\Omega M(|T_k(u_n)|)\, dx \leq kC_7 \qquad \text{for} \quad k \geq \max(1, \|\Psi^+\|_\infty). \tag{39}$$

Then, we deduce that,

$$\begin{aligned} M(k)\ \text{meas}(\{|u_n| > k\}) &= \int_{\{|u_n|>k\}} M(|T_k(u_n)|)\, dx \\ &\leq \int_\Omega M(|T_k(u_n)|)\, dx \leq kC_7, \end{aligned}$$

hence,

$$\text{meas}(\{|u_n| > k\}) = \frac{kC_7}{M(k)} \longrightarrow 0 \quad \text{as} \quad k \to +\infty. \tag{40}$$

For all $\delta > 0$, we have

$$\begin{aligned} \text{meas}\{|u_n - u_m| > \delta\} &\leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ &\quad +\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}. \end{aligned}$$

Let $\varepsilon > 0$, using (40) we may choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{41}$$

Moreover, in view of (38) we have $T_k(u_n) \to v_k$ strongly in $E_\varphi(\Omega)$, then, we can assume that $(T_k(u_n))_{n\in I\!N}$ is a Cauchy sequence in measure. Thus, for all $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \qquad \text{for all } m, n \geq n_0(k, \delta, \varepsilon). \tag{42}$$

By combining (41) – (42), we conclude that

$$\forall \delta, \varepsilon > 0 \quad \text{there exists} \quad n_0 = n_0(\delta, \varepsilon) \quad \text{such that} \quad \text{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon$$

for any $n, m \geq n_0(\delta, \varepsilon)$. It follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u. Consequently, we have

$$\left\{ \begin{array}{lll} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in} \quad W_0^1L_\varphi(\Omega) & \text{for} \quad \sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega)), \\ T_k(u_n) \to T_k(u) & \text{strongly in} \quad E_\varphi(\Omega). & \end{array} \right. \tag{43}$$

Step 4 : Strong convergence of truncations. In the sequel, we denote by $\varepsilon_i(n)$, $i = 1, 2, \dots$ various real-valued functions of real variables that converges to 0 as n tends to infinity.
Let $h > k > 0$, we define

$$M := 4k + h, \qquad z_n := u_n - T_h(u_n) + T_k(u_n) - T_k(u) \qquad \text{and} \qquad \omega_n := T_{2k}(z_n).$$

Taking $v = u_n - \eta\omega_n$, we have $v \geq \Psi$ for η small enough, thus v is an admissible test function in (25), and since $\nabla\omega_n = 0$ on $\{|u_n| \geq M\}$, we obtain

$$\int_{\{|u_n|\leq M\}} a(x, \nabla T_M(u_n)) \cdot \nabla\omega_n \, dx \leq \int_\Omega f_n\omega_n \, dx.$$

We have $\omega_n = T_k(u_n) - T_k(u)$ on $\{|u_n| \leq k\}$, we conclude that

$$\begin{aligned} &\int_{\{|u_n|\leq k\}} a(x, \nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \\ &\qquad + \int_{\{k<|u_n|\leq M\}} a(x, \nabla T_M(u_n)) \cdot \nabla\omega_n \, dx \leq \int_\Omega f_n\omega_n \, dx. \end{aligned} \tag{44}$$

Concerning the second term on the left-hand side of (44), we have

$$\begin{aligned} &\int_{\{k<|u_n|\leq M\}} a(x, \nabla T_M(u_n)) \cdot \nabla\omega_n \, dx \\ &\quad = \int_{\{k<|u_n|\leq M\}\cap\{|z_n|\leq 2k\}} a(x, \nabla T_M(u_n)) \cdot \nabla(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \, dx \\ &\quad \geq - \int_{\{k<|u_n|\leq M\}} |a(x, \nabla T_M(u_n))| \, |\nabla T_k(u)| \, dx, \end{aligned}$$

We have $\nabla T_k(u) \in (L_\varphi(\Omega))^N$, and since $(|a(x, \nabla T_M(u_n))|)_n$ is bounded in $L_\psi(\Omega) = E_\psi(\Omega)$, there exists $\zeta \in E_\psi(\Omega)$ such that $|a(x, \nabla T_M(u_n))| \rightharpoonup \zeta$ weakly in $E_\psi(\Omega)$ for $\sigma(E_\psi(\Omega), L_\varphi(\Omega))$. Therefore,

$$\int_{\{k<|u_n|\leq M\}} |a(x, \nabla T_M(u_n))| \, |\nabla T_k(u)| \, dx \longrightarrow \int_{\{k<|u|\leq M\}} \zeta \, |\nabla T_k(u)| \, dx = 0. \tag{45}$$

It follows that

$$\int_{\{k<|u_n|\leq M\}} a(x, \nabla T_M(u_n)) \cdot \nabla\omega_n \, dx \geq \varepsilon_1(n). \tag{46}$$

Then, since $f_n \to f$ in $L^1(\Omega)$ and $\omega_n \rightharpoonup T_{2k}(u - T_h(u))$ weak$-*$ in $L^\infty(\Omega)$, and using (44), we deduce that

$$\int_{\{|u_n|\leq k\}} a(x, \nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \leq \int_\Omega f T_{2k}(u - T_h(u)) \, dx + \varepsilon_2(n). \tag{47}$$

We define $\Omega_s = \big\{x \in \Omega : |\nabla T_k(u(x))| \leq s\big\}$ and denote by χ_s the characteristic function of Ω_s. For the term on the left-hand side of (47), we have

$$\begin{aligned}
&\int_{\{|u_n|\leq k\}} a(x,\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,dx\\
&=\int_\Omega a(x,\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx\\
&\quad+\int_\Omega a(x,\nabla T_k(u_n))\cdot(\nabla T_k(u)\chi_s-\nabla T_k(u))\,dx+\int_{\{|u_n|>k\}} a(x,\nabla T_k(u_n))\cdot\nabla T_k(u)\,dx\\
&=\int_\Omega \big(a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u)\chi_s)\big)\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx\\
&\quad+\int_\Omega a(x,\nabla T_k(u)\chi_s)\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx\\
&\quad-\int_{\Omega\setminus\Omega_s} a(x,\nabla T_k(u_n))\cdot\nabla T_k(u)\,dx+\int_{\{|u_n|>k\}} a(x,\nabla T_k(u_n))\cdot\nabla T_k(u)\,dx.
\end{aligned}\tag{48}$$

For the second term on the right-hand side of (48), we have $a(x,\nabla T_k(u)\chi_s) \in (E_\psi(\Omega))^N$, and since $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_\varphi(\Omega))^N$ for $\sigma(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega))$, then

$$\begin{aligned}
\lim_{n\to\infty}&\int_\Omega a(x,\nabla T_k(u)\chi_s)\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx\\
&=\int_\Omega a(x,\nabla T_k(u)\chi_s)\cdot(\nabla T_k(u)-\nabla T_k(u)\chi_s)\,dx\\
&=\int_{\Omega\setminus\Omega_s} a(x,0)\cdot\nabla T_k(u)\,dx.
\end{aligned}\tag{49}$$

Concerning the third term on the right-hand side of (48), since $(a(x,\nabla T_k(u_n))_n$ is bounded in $(E_\psi(\Omega))^N$, there exists $\xi \in (E_\psi(\Omega))^N$ such that $a(x,\nabla T_k(u_n)) \rightharpoonup \xi$ weakly in $(E_\psi(\Omega))^N$ for $\sigma(\Pi E_\psi(\Omega),\Pi L_\varphi(\Omega))$, it follows that

$$\lim_{n\to\infty}\int_{\Omega\setminus\Omega_s} a(x,\nabla T_k(u_n))\cdot\nabla T_k(u)\,dx=\int_{\Omega\setminus\Omega_s}\xi\cdot\nabla T_k(u)\,dx.\tag{50}$$

For the last term on the right-hand side of (48), we obtain

$$\lim_{n\to\infty}\int_{\{|u_n|>k\}} a(x,\nabla T_k(u_n))\cdot\nabla T_k(u)\,dx=\int_{\{|u|>k\}}\xi\cdot\nabla T_k(u)\,dx=0.\tag{51}$$

By combining (48) – (51), we deduce that

$$\begin{aligned}
&\int_{\{|u_n|\leq k\}} a(x,\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,dx\\
&=\int_\Omega \big(a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u)\chi_s)\big)\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx\\
&\qquad+\int_{\Omega\setminus\Omega_s}\big(a(x,0)-\xi\big)\cdot\nabla T_k(u)\,dx+\varepsilon_3(n)
\end{aligned}\tag{52}$$

and since $\big(a(x,0)-\eta\big)\cdot\nabla T_k(u)\in L^1(\Omega)$, then

$$\int_{\Omega\setminus\Omega_s}\big(a(x,0)-\xi\big)\cdot\nabla T_k(u)\,dx\longrightarrow 0\qquad\text{as}\quad s\to\infty.$$

Therefore, using (47) we conclude that

$$\begin{aligned}\int_\Omega \big(a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)\big)\cdot(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx \\ \leq \int_\Omega fT_{2k}(u-T_h(u))\,dx + \varepsilon_4(n,s).\end{aligned} \tag{53}$$

We have

$$\int_\Omega fT_{2k}(u-T_h(u))\,dx \longrightarrow 0 \quad \text{as} \quad h\to\infty.$$

It follows that

$$\lim_{n,s\to\infty}\int_\Omega \big(a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)\big)\cdot\big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big)\,dx = 0. \tag{54}$$

In view of Lemma 4.2, we deduce that

$$\nabla u_n \longrightarrow \nabla u \qquad \text{a.e. in} \quad \Omega, \tag{55}$$

and

$$\varphi(x,|\nabla T_k(u_n)|) \longrightarrow \varphi(x,|\nabla T_k(u)|) \qquad \text{in} \quad L^1(\Omega). \tag{56}$$

Step 5 : Passage to the limit. Let $v\in K_\Psi\cap L^\infty(\Omega)$ and $\eta>0$, we have $u_n-\eta T_k(u_n-v)\ \in K_\Psi$ is an admissible test function in (25) for η small enough, and we obtain

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla T_k(u_n-v)\,dx \leq \int_\Omega f_nT_k(u_n-v)\,dx. \tag{57}$$

Choosing $M=k+\|v\|_\infty$, then $\{|u_n-v|\leq k\}\subseteq\{|u_n|\leq M\}$. Firstly, we can write the term on the left-hand side of the above relation as

$$\begin{aligned}\int_\Omega a(x,\nabla u_n)\cdot\nabla T_k(u_n-v)\,dx &= \int_\Omega a(x,\nabla T_M(u_n))\cdot(\nabla T_M(u_n)-\nabla v)\chi_{\{|u_n-v|\leq k\}}\,dx\\ &= \int_\Omega (a(x,\nabla T_M(u_n))-a(x,\nabla v))\cdot(\nabla T_M(u_n)-\nabla v)\chi_{\{|u_n-v|\leq k\}}\,dx\\ &\quad+\int_\Omega a(x,\nabla v)\cdot(\nabla T_M(u_n)-\nabla v)\chi_{\{|u_n-v|\leq k\}}\,dx.\end{aligned} \tag{58}$$

We have

$$\begin{aligned}&(a(x,\nabla T_M(u_n))-a(x,\nabla v))\cdot(\nabla T_M(u_n)-\nabla v)\chi_{\{|u_n-v|\leq k\}}\\ &\longrightarrow (a(x,\nabla T_M(u))-a(x,\nabla v))\cdot(\nabla T_M(u)-\nabla v)\chi_{\{|u-v|\leq k\}} \quad \text{a.e. in} \quad \Omega.\end{aligned} \tag{59}$$

According to (9) and Fatou's lemma, we obtain

$$\begin{aligned}\liminf_{n\to\infty}&\int_\Omega a(x,\nabla u_n)\cdot\nabla T_k(u_n-v)\,dx\\ &\geq \int_\Omega (a(x,\nabla T_M(u))-a(x,\nabla v))\cdot(\nabla T_M(u)-\nabla v)\chi_{\{|u-v|\leq k\}}\,dx\\ &\quad+\lim_{n\to\infty}\int_\Omega a(x,\nabla v)\cdot(\nabla T_M(u_n)-\nabla v)\chi_{\{|u_n-v|\leq k\}}\,dx.\end{aligned} \tag{60}$$

For the second term on the right-hand side of (60), we have $a(x,\nabla v)\ \in\ (E_\psi(\Omega))^N$ and $\nabla T_M(u_n)\rightharpoonup\nabla T_M(u)$ weakly in $(L_\varphi(\Omega))^N$ for $\sigma(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega))$, then

$$\begin{aligned}\lim_{n\to\infty}&\int_\Omega a(x,\nabla v)\cdot(\nabla T_M(u_n)-\nabla v)\chi_{\{|u_n-v|\leq k\}}\,dx\\ &=\int_\Omega a(x,\nabla v)\cdot(\nabla T_M(u)-\nabla v)\chi_{\{|u-v|\leq k\}}\,dx.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\liminf_{n\to\infty}\int_\Omega a(x,\nabla u_n)\cdot\nabla T_k(u_n-v)\,dx &\geq \int_\Omega a(x,\nabla T_M(u))\cdot(\nabla T_M(u)-\nabla v)\chi_{\{|u-v|\leq k\}}\,dx\\ &=\int_\Omega a(x,\nabla u)\cdot\nabla T_k(u-v)\,dx.\end{aligned} \tag{61}$$

On the other hand, being $T_k(u_n-v)\rightharpoonup T_k(u-v)$ weak-$\star$ in $L^\infty(\Omega)$ we deduce that

$$\int_\Omega f_n T_k(u_n-v)\,dx \longrightarrow \int_\Omega f\ T_k(u-v)\,dx. \tag{62}$$

By combining (61) and (62), we conclude the existence of entropy solution for our problem.

5.2. Uniqueness of entropy solution.

Let u_1, u_2 be two entropy solutions of the problems (24), we shall prove that $u_1=u_2$.

By using the test function $v=T_h(u_2)\in K_\Psi\cap L^\infty(\Omega)$ in (24) for the equation with solution u_1, we have

$$\int_\Omega a(x,\nabla u_1)\cdot\nabla T_k(u_1-T_h(u_2))\,dx\leq\int_\Omega fT_k(u_1-T_h(u_2))\,dx.$$

Similarly, by using $v=T_h(u_1)\in K_\Psi\cap L^\infty(\Omega)$ as a test function for the equation (24) with solution u_2, we obtain

$$\int_\Omega a(x,\nabla u_2)\cdot\nabla T_k(u_2-T_h(u_1))\,dx\leq\int_\Omega fT_k(u_2-T_h(u_1))\,dx.$$

By adding these two inequalities, we get

$$\begin{aligned}&\int_\Omega a(x,\nabla u_1)\cdot\nabla T_k(u_1-T_h(u_2))\,dx+\int_\Omega a(x,\nabla u_2)\cdot\nabla T_k(u_2-T_h(u_1))\,dx\\ &\leq\int_\Omega f[T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))]\,dx.\end{aligned} \tag{63}$$

We decompose the first integral of the left-hand side of (63) as

$$\begin{aligned}\int_\Omega a(x,\nabla u_1)\cdot\nabla T_k(u_1-T_h(u_2))\,dx &= \int_{\{|u_1-T_h(u_2)|\leq k\}} a(x,\nabla u_1)\cdot\nabla(u_1-T_h(u_2))\,dx\\ &= \int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}} a(x,\nabla u_1)\cdot(\nabla u_1-\nabla u_2)\,dx\\ &\quad+\int_{\{|u_1-T_h(u_2)|\leq k\}\cap\{|u_2|>h\}} a(x,\nabla u_1)\cdot\nabla u_1\,dx,\\ &\geq \int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|\leq h\}} a(x,\nabla u_1)\cdot(\nabla u_1-\nabla u_2)\,dx\\ &\quad+\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\}} a(x,\nabla u_1)\cdot(\nabla u_1-\nabla u_2)\,dx.\end{aligned} \tag{64}$$

Similarly, we have

$$\begin{aligned}\int_\Omega a(x,\nabla u_2)\cdot\nabla T_k(u_2-T_h(u_1))\,dx \geq &\int_{\{|u_2-u_1|\leq k\}\cap\{|u_1|\leq h\}\cap\{|u_2|\leq h\}} a(x,\nabla u_2)\cdot(\nabla u_2-\nabla u_1)\,dx\\ &+\int_{\{|u_2-u_1|\leq k\}\cap\{|u_1|\leq h\}\cap\{|u_2|>h\}} a(x,\nabla u_2)\cdot(\nabla u_2-\nabla u_1)\,dx.\end{aligned} \tag{65}$$

By combining (64) – (65), we obtain

$$\begin{aligned}&\int_\Omega a(x,\nabla u_1)\cdot\nabla T_k(u_1-T_h(u_2))\,dx+\int_\Omega a(x,\nabla u_2)\cdot\nabla T_k(u_2-T_h(u_1))\,dx\\ &\quad\geq\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|\leq h\}}\big(a(x,\nabla u_1)-a(x,\nabla u_2)\big)\cdot(\nabla u_1-\nabla u_2)\,dx\\ &\qquad+\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\}} a(x,\nabla u_1)\cdot(\nabla u_1-\nabla u_2)\,dx\\ &\qquad+\int_{\{|u_2-u_1|\leq k\}\cap\{|u_1|\leq h\}\cap\{|u_2|>h\}} a(x,\nabla u_2)\cdot(\nabla u_2-\nabla u_1)\,dx.\end{aligned}$$

In view of (63), we conclude that

$$\begin{aligned}&\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|\leq h\}}\big(a(x,\nabla u_1)-a(x,\nabla u_2)\big)\cdot(\nabla u_1-\nabla u_2)\,dx\\ &\quad\leq\int_\Omega f[T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))]\,dx\\ &\qquad-\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\}} a(x,\nabla u_1)\cdot(\nabla u_1-\nabla u_2)\,dx\\ &\qquad-\int_{\{|u_2-u_1|\leq k\}\cap\{|u_1|\leq h\}\cap\{|u_2|>h\}} a(x,\nabla u_2)\cdot(\nabla u_2-\nabla u_1)\,dx.\end{aligned} \tag{66}$$

For the first term on the right-hand side of (66), we have

$$\begin{aligned}&\Big|\int_\Omega f[T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))]\,dx\Big|\\ &\quad\leq\int_{\{|u_1|\leq h,|u_2|\leq h\}}|f|\;|T_k(u_1-u_2)+T_k(u_2-u_1)|\,dx.\\ &\qquad+\int_{\{|u_1|>h\}}|f|\;|T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))|\,dx\\ &\qquad+\int_{\{|u_2|>h\}}|f|\;|T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))|\,dx\\ &\quad\leq 2k\int_{\{|u_1|>h\}}|f|\,dx+2k\int_{\{|u_2|>h\}}|f|\,dx.\end{aligned}$$

since $f\in L^1(\Omega)$ and $\mathrm{meas}\{|u_i|\geq h\}\to 0$ when $h\to\infty$ for $i=1,2,$ it follows that

$$\int_\Omega f[T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))]\,dx\longrightarrow 0\qquad\text{as}\quad h\to\infty. \tag{67}$$

Concerning the third term on the right-hand side of (66). By taking $T_h(u_1)$ as a test function in (24) for the equation with solution u_1, we obtain

$$\int_\Omega a(x,\nabla u_1)\cdot\nabla T_k(u_1-T_h(u_1))\,dx\leq\int_\Omega fT_k(u_1-T_h(u_1))\,dx,$$

in view of (10), we obtain

$$\begin{aligned} \alpha \int_{\{h<|u_1|\leq h+k\}} \varphi(x,|\nabla u_1|)\,dx &\leq \int_{\{h<|u_1|\leq h+k\}} a(x,\nabla u_1)\cdot\nabla u_1\,dx \\ &\leq k\int_{\{|u_1|\geq h\}} |f|\,dx \to 0 \quad \text{as} \quad h\to\infty. \end{aligned} \tag{68}$$

Also, we prove can that

$$\alpha \int_{\{h<|u_2|\leq h+k\}} \varphi(x,|\nabla u_2|)\,dx \to 0 \quad \text{as} \quad h\to\infty. \tag{69}$$

On the other hand, we have

$\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\} \subseteq \{h<|u_1|\leq h+k\}\cap\{h-k<|u_2|\leq h\}$,

In view of Young's inequality, we obtain

$$\begin{aligned} &\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\}} a(x,\nabla u_1)\cdot(\nabla u_1-\nabla u_2)\,dx \\ &\leq \beta\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\}} \big(K(x)+k_1\psi_x^{-1}(\varphi(x,k_2|\nabla u_1|))\big)(|\nabla u_1|+|\nabla u_2|\,dx \\ &\leq 2\beta\int_{\{|u_1|>h\}} \psi(x,K(x))\,dx + 2\beta k_1\int_{\{h<|u_1|\leq h+k\}} \varphi(x,k_2|\nabla u_1|)\,dx \\ &\quad +\beta(k_1+1)\int_{\{h<|u_1|\leq h+k\}} \varphi(x,|\nabla u_1|)\,dx \\ &\quad +\beta(k_1+1)\int_{\{h-k<|u_2|\leq h\}} \varphi(x,|\nabla u_2|)\,dx \longrightarrow 0 \quad \text{as} \quad h\to\infty, \end{aligned} \tag{70}$$

Similarly, we can prove that

$$\int_{\{|u_2-u_1|\leq k\}\cap\{|u_1|\leq h\}\cap\{|u_2|>h\}} a(x,\nabla u_2)\cdot(\nabla u_2-\nabla u_1)\,dx \longrightarrow 0 \quad \text{as} \quad h\to\infty, \tag{71}$$

By combining (66), (67) and (70) – (71), we conclude that

$$\begin{aligned} &\int_{\{|u_1-u_2|\leq k\}} \big(a(x,\nabla u_1)-a(x,\nabla u_2)\big)\cdot(\nabla u_1-\nabla u_2)\,dx \\ &= \lim_{h\to\infty}\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|\leq h\}} \big(a(x,\nabla u_1)-a(x,\nabla u_2)\big)\cdot(\nabla u_1-\nabla u_2)\,dx = 0, \end{aligned} \tag{72}$$

Since (72) is true for all $k>0$ and thanks to (9), we conclude that $\nabla(u_1-u_2)=0$ a.e.in Ω, and since $u_1=u_2=0$ on $\partial\Omega$, thus $u_1=u_2$ a.e. in Ω, which conclude the proof of uniqueness of entropy solutions.

Example 5.1. Taking $\varphi(x,t)=|t|^{p(x)}\log^{\sigma}(1+|t|)$ for $1\leq p(x)<\infty$ and $0<\sigma<\infty$. Let $f\in L^1(\Omega)$ and the obstacle $\Psi=0$. We consider the following Carathéodory function

$$a(x,\nabla u)=|\nabla u|^{p(x)-2}\log^{\sigma}(1+|\nabla u|)\ \nabla u.$$

It is clear that $a(x,\nabla u)$ verifies (8) – (10). In view of the Theorem 5.1, the problem

$$\begin{cases} -\mathrm{div}\Big(|\nabla u|^{p(x)-2}\log^{\sigma}(1+|\nabla u|)\ \nabla u\Big)=f & \text{in } \Omega, \\ u\ =0 & \text{on } \partial\Omega, \end{cases} \tag{73}$$

has one entropy solution, i.e.

$$u \geq 0 \quad \text{a.e. in } \ \Omega \qquad \text{and} \qquad T_k(u) \in W_0^1 L_\varphi(\Omega),$$

and

$$\int_\Omega |\nabla u|^{p(x)-2} \log^\sigma (1 + |\nabla u|) \nabla u \cdot \nabla T_k(u_n - \nu)\, dx \leq \int_\Omega f T_k(u_n - \nu)\, dx, \tag{74}$$

for any $\nu \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ with $v \geq 0$ a.e. in Ω.

References

[1] L. Aharouch, J. Bennouna, Existence and uniqueness of solutions of unilateral problems in Orlicz spaces, *Nonlinear Analysis* **72** (2010), 3553–3565.

[2] E. Azroul, M. B. Benboubker, M. Rhoudaf, On some p(x)-quasilinear problem with right-hand side measure, *Math. Comput. Simulation* **102** (2014), 117–130.

[3] E. Azroul, A. Benkirane, M. Rhoudaf, On some strongly nonlinear elliptic problems in L1-data with a nonlinearity having a constant sign in Orlicz spaces via penalization methods, *Aust. J. Math. Anal. Appl.* **7** (2010), no. 1, Art. 5, 1–25.

[4] E. Azroul, H. Hjiaj, A. Touzani, Existence and Regularity of Entropy solutions For Strongly Nonlinear $p(x)$-elliptic equations, *Electronic J. Diff. Equ.* **68** (2013), 1–27.

[5] M. Bendahmane, P. Wittbold, Renormalized solutions for nonlinear elliptic equations with variable exponents and L1 data, *Nonlinear Analysis* **70** (2009), no. 2, 567–583.

[6] A. Benkirane, J. Bennouna, Existence of entropy solutions for some nonlinear problems in Orlicz spaces, *Abstr. Appl. Anal.* **7** (2002), 85–102.

[7] A. Benkirane, A. Elmahi, An existence theorem for a strongly nonlinear elliptic problem in Orlicz spaces, *Nonlinear Analysis* **36** (1999), 11–24.

[8] A. Benkirane, A. Elmahi, Almost everywhere convergence of gradients of solutions to elliptic equations in Orlicz spaces and application, *Nonlinear Analysis T.M.A.* **28** (1997), No.11, 1769–1784.

[9] A. Benkirane, M. Ould Mohamedhen Val, An existence result for nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces, *Bull. Belg. Math. Soc. Simon Stevin* **20** (2013), no. 1, 57–75.

[10] A. Benkirane, M. Ould Mohamedhen Val, Variational inequalities in Musielak-Orlicz-Sobolev spaces, *Bull. Belg. Math. Soc. Simon Stevin* **21** (2014), no. 5 , 787 – 811 .

[11] L. Boccardo, T. Gallouët, Nonlinear elliptic equations with right-hand side measures, *Comm. Partial Differential Equations* **17** (1992), no. 3–4, 641–655.

[12] R. Di Nardo, F. Feo, O. Guibé, Uniqueness result for nonlinear anisotropic elliptic equations, *Adv. Differential Equations* **18** (2013), no. 5-6, 433-458.

[13] A. Elmahi, D. Meskine, Existence of solutions for elliptic equations having natural growth terms in Orlicz spaces, *Abstr. Appl. Anal.* **12** (2004), 1031–1045.

[14] X. Fan, An imbedding theorem for Musielak-Sobolev spaces, *Nonlinear Analysis* **75** (2012), no. 4, 1959-1971.

[15] J.P. Gossez, Nonlinear elliptic boundary value prolems for equations with rapidly (or slowly) increasing coefficients, *Trans. Am. Math. Soc.* **190** (1974), 163–205.

[16] P. Gwiazda, A. Swierczewska-Gwiazda, A. Wroblewska, Monotonicity methods in generalized Orlicz spaces for a class of non- Newtonian fluids, *Math. Methods Appl. Sci.* **33** (2010), no. 2, 125-137.

[17] P. Gwiazda, A. Swierczewska-Gwiazda, A. Wroblewska, Generalized Stokes system in Orlicz spaces. *Discrete Contin. Dyn. Syst.* **32** (2012), no. 6, 2125-2146.

[18] P. Gwiazda, P. Minakowski, A. Wroblewska-Kaminska, Elliptic problems in generalized Orlicz-Musielak spaces, *Cent. Eur. J. Math.* **10** (2012), no. 6, 2019-2032.

[19] P. Gwiazda, P. Wittbold, A. Wroblewska, A. Zimmermann, Renormalized solutions of nonlinear elliptic problems in generalized Orlicz spaces, *J. Diff. Equ.* **253** (2012), no. 2, 635-666

[20] E. Hewitt, K. Stromberg, *Real and abstract analysis*, Springer-Verlag, Berlin Heidelberg New York, 1965.

[21] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod et Gauthiers-Villars, Paris, 1969.

[22] M. Sanchón, J.M. Urbano, Entropy solutions for the p(x)-Laplace equation, *Trans. Amer. Math. Soc.* **361** (2009), 6387–6405.

[23] M. Růžička, *Electrorheological fluids: modeling and mathematical theory,* Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2000.

[24] J. Musielak, *Orlicz spaces and Modular spaces*, Lecture Notes in Math. 1034, Springer-Verlag, Berlin Heidelberg 1983.

(Mohammed Al-Hawmi) LAMA Laboratory, Department of Mathematics, University of Sidi Mohamed Ben Abdellah, B. P. 1796 Atlas Fez, Morocco
E-mail address: `m.alhomi2011@gmail.com`

(Abdelmoujib Benkirane) LAMA Laboratory, Department of Mathematics, University of Sidi Mohamed Ben Abdellah, B. P. 1796 Atlas Fez, Morocco
E-mail address: `abd.benkirane@gmail.com`

(Hassane Hjiaj) Department of Mathematics, Faculty of Sciences Tetouan, University Abdelmalek Essaadi, B. P. 2121, Tetouan, Morocco
E-mail address: `hjiajhassane@yahoo.fr`

(Abdelfattah Touzani) LAMA Laboratory, Department of Mathematics, University of Sidi Mohamed Ben Abdellah, B. P. 1796 Atlas Fez, Morocco
E-mail address: `atouzani07@gmail.com`

Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 21–29
ISSN: 1223-6934

Existence and multiplicity of solutions for $p(x)-$ Kirchhoff-type problem

Zehra Yücedag

Abstract. In the present paper, by using the Mountain Pass theorem and the Fountain theorem, we obtain the existence and multiplicity of solutions to a class of $p(x)$-Kirchhoff-type problem under Dirichlet boundary condition.

2010 Mathematics Subject Classification. 35B38,35D05,35J60,35J70.
Key words and phrases. $p(x)$-Kirchhoff-type equation; Mountain-Pass theorem; Nonlocal problem; Fountain theorem.

---

1. Introduction

In this paper, we are concerned with the following problem

$$\begin{cases} -M\left(A\left(x,\nabla u\right)\right)\ \operatorname{div}\left(a\left(x,\nabla u\right)\right)=f\left(x,u\right) & \text{in } \Omega\ , \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a smooth bounded domain, $p\in C\left(\overline{\Omega}\right)$ for any $x\in\overline{\Omega}$ and $\operatorname{div}(a\left(x,\nabla u\right))$ is a $p\left(x\right)-$Laplace type operator. Moreover $M:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, satisfying some certain conditions.

The nonlinear problems involving the $p\left(x\right)$-Laplace type operator are extremely attractive because they can be used to model dynamical phenomena which arise from the study of electrorheological fluids or elastic mechanics. Problems with variable exponent growth conditions also appear in the modelling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium. The detailed application backgrounds of the $p(x)$-Laplace type operators can be found in [3, 5, 7, 10, 13, 20, 18] and references therein.

Problem (1.1) is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff [15]. To be more precise, Kirchhoff established a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

where ρ, P_0, h, E, L are constants, which extends the classical D'Alambert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. For some interesting results we refer to [4, 6, 9, 11, 14]. Moreover,

---

Received March 19, 2015. Accepted December, 2016.

nonlocal boundary value problems like (1.2) can be used for modelling several physical and biological systems where u describes a process which depend on the average of itself, such as the population density [1, 2, 8].

In the present paper, we deal a more general Kirchhoff function M, and as a consequence the operator $div\left(a\left(x,\nabla u\right)\right)$ appears in problem (1.1), a more general operator than $p\left(x\right)$ −Laplace operator $\Delta_{p(x)}u := div(\left|\nabla u\right|^{p(x)-2}\nabla u)$ where $p\left(x\right)>1$. This caused some difficulties in calculations and required more general conditions. Moreover, thanks to the Mountain-Pass theorem and Fountain theorem, we show the existence and multiplicity of nontrivial weak solutions in the present paper. To our best knowledge, the present papers results are not covered in the literature.

This paper is organized as follows. In Section 2, we present some necessary preliminary results. In Section 3, using the variational method, we give the existence results of problem (1.1).

2. Preliminaries

We recall some basic properties of variable exponent Lebesgue-Sobolev spaces $L^{p(x)}\left(\Omega\right)$, $W^{1,p(x)}\left(\Omega\right)$ (for details, see e.g., [12, 16, 17])

Set,

$$C_+\left(\overline{\Omega}\right)=\left\{p;\ p\in C\left(\overline{\Omega}\right),\ \min p\left(x\right)>1,\forall x\in\overline{\Omega}\right\}.$$

For any $p\left(x\right)\in C_+\left(\overline{\Omega}\right)$, denote $p^- := \min_{x\in\overline{\Omega}} p\left(x\right)$, $p^+ := \max_{x\in\overline{\Omega}} p\left(x\right)<\infty$, and define the variable exponent Lebesgue space by

$$L^{p(x)}\left(\Omega\right)=\left\{u\left|u:\Omega\to\mathbb{R}\text{ is measurable, }\int_\Omega\left|u\left(x\right)\right|^{p(x)}dx<\infty\right.\right\}.$$

We define a norm, the so-called Luxemburg norm, on this space by the formula

$$|u|_{p(x)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}dx\leq 1\right\},$$

and $\left(L^{p(x)}\left(\Omega\right),\ |.|_{p(x)}\right)$ becomes a Banach spaces.

Proposition 2.1 [12, 16] *The conjugate space of* $L^{p(x)}\left(\Omega\right)$ *is* $L^{p'(x)}\left(\Omega\right)$, *where* $\frac{1}{p'(x)}+\frac{1}{p(x)}=1$. *For any* $u\in L^{p(x)}\left(\Omega\right)$ *and* $v\in L^{p'(x)}\left(\Omega\right)$, *we have*

$$\left|\int_\Omega uvdx\right|\leq\left(\frac{1}{p^-}+\frac{1}{\left(p^-\right)'}\right)|u|_{p(x)}\,|v|_{p'(x)}\,.$$

Proposition 2.2 [12, 16]*Denote* $\rho\left(u\right)=\int_\Omega\left|u\left(x\right)\right|^{p(x)}dx, \forall u,u_n\in L^{p(x)}(\Omega)$, *then*

(i) $|u|_{p(x)}>1 \implies |u|_{p(x)}^{p^-}\leq\rho\left(u\right)\leq|u|_{p(x)}^{p^+}$;

(ii) $|u|_{p(x)}<1 \implies |u|_{p(x)}^{p^+}\leq\rho\left(u\right)\leq|u|_{p(x)}^{p^-}$;

(iii) $\lim\limits_{n\to\infty}|u_n|_{p(x),\Omega}=0\Leftrightarrow\lim\limits_{n\to\infty}\rho(u_n)=0$;

(iv) $\lim\limits_{n\to\infty}|u_n|_{p(x),\Omega}\to\infty\Leftrightarrow\lim\limits_{n\to\infty}\rho(u_n)\to\infty$.

Proposition 2.3 [12, 16] *If* $u, u_n \in L^{p(x)}(\Omega)$, *then the following statements are equivalent:*

$$(i) \lim_{n\to\infty} |u_n - u|_{p(x)} = 0; \quad (ii) \lim_{n\to\infty} \rho(u_n - u) = 0;$$

$$(iii)\ u_n \to u \text{ measure in } \Omega \text{ and } \lim_{n\to\infty} \rho(u_n) = \rho(u).$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \mid \ |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

with the norm $\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \forall u \in W^{1,p(x)}(\Omega)$. The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. We can define an equivalent norm $\|u\| = |\nabla u|_{p(x)}$, since Poincaré inequality holds [13], i.e. there exists a positive constant $C > 0$ such that

$$|u|_{p(x)} \leq C |\nabla u|_{p(x)}, \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Proposition 2.4 [12, 16](i) *If* $1 < p^- \leq p^+ < \infty$, *then the spaces* $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces,*

(ii) *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$, *for all* $x \in \overline{\Omega}$, *then the embedding* $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *is compact and continuous, where*

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } N \leq p(x). \end{cases}$$

3. The main results

Let X denote the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$.

We say that $u \in X$ is a weak solution of (1.1) if

$$M\left(\int_\Omega A(x, \nabla u)\right) \int_\Omega a(x, \nabla u) \nabla \varphi dx = \int_\Omega f(x, u) \varphi dx,$$

for all $\varphi \in X$.

Define the energy functional $I : X \to \mathbb{R}$ associated with (1.1) by

$$I(u) = \widehat{M}\left(\int_\Omega A(x, \nabla u) dx\right) - \int_\Omega F(x, u) dx := \widehat{M}(\Lambda(u)) - J(u),$$

where $\Lambda(u) = \int_\Omega A(x, \nabla u) dx$ and $J(u) = \int_\Omega F(x, u) dx$. Moreover, $\widehat{M}(t) = \int_0^t M(s) ds$ and $F(x, u) = \int_0^u f(x, t) dt$.

It is well known that standart arguments imply that $J \in C^1(X, \mathbb{R})$ and the derivate of J is

$$\langle J'(u), \upsilon \rangle = \int_\Omega f(x, u) \upsilon dx, \text{ for all } u, \upsilon \in X.$$

In this article, we assume that $a(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is the continuous derivative with respect to ξ of the mapping $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x, \xi)$, i.e. $a(x, \xi) = \nabla_\xi A(x, \xi)$. Suppose that the following hypotheses:

(**A1**) For all $x \in \Omega$ and $\xi \in \mathbb{R}^N$, $|a(x, \xi)| \leq c_0(h_0(x) + |\xi|^{p(x)-1})$, where $h_0(x) \in L^{p'(x)}(\Omega)$ is a nonnegative measurable function.

(**A2**) A is $p(x)$-uniformly convex: There exists a constant $k>0$ such that $A(x,\frac{\xi+\psi}{2})\leq\frac{1}{2}A(x,\xi)+\frac{1}{2}A(x,\psi)-k\left|\xi-\psi\right|^{p(x)}$, for all $x\in\Omega$ and $\xi,\psi\in\mathbb{R}^N$.

(**A3**) For all $x\in\Omega$ and $\xi\in\mathbb{R}^N$, $\left|\xi\right|^{p(x)}\leq a(x,\xi)\cdot\xi\leq p\left(x\right)A(x,\xi)$.

(**A4**) $A\left(x,0\right)=0$, for all $x\in\Omega$.

(**A5**) $A(x,-\xi)=A(x,\xi)$, for all $x\in\Omega$ and $\xi\in\mathbb{R}^N$.

Lemma 3.1. [17]

(*i*) *A verifies the growth condition;* $\left|A\left(x,\xi\right)\right|\leq c_0(h_0\left(x\right)\left|\xi\right|+\left|\xi\right|^{p(x)})$, *for all* $x\in\Omega$ *and* $\xi\in\mathbb{R}^N$;

(*ii*) *A is* $p(x)-$*homogeneous;* $A\left(x,z\xi\right)\leq A\left(x,\xi\right)z^{p(x)}$, *for all* $z\geq1$, $\xi\in\mathbb{R}^N$ *and* $x\in\Omega$.

Lemma 3.2. (*i*) *The functional* Λ *is well-defined on* X;

(*ii*) *The functional* Λ *is of class* $C^1\left(X,\mathbb{R}\right)$ *and*

$$\left\langle\Lambda'\left(u\right),v\right\rangle=\int_\Omega a\left(x,\nabla u\right)\cdot\nabla vdx,\ \textit{for all } u,v\in X;$$

(*iii*) *The functional* Λ *is weakly lower semi-continuos on* X;

(*iv*) *For all* $u,v\in X$

$$\Lambda(\frac{u+v}{2})\leq\frac{1}{2}\Lambda\left(u\right)+\frac{1}{2}\Lambda\left(v\right)-k\left\|u-v\right\|^{p^-}\ ;$$

(*v*) *For all* $u,v\in X$

$$\Lambda\left(u\right)-\Lambda\left(v\right)\geq\left\langle\Lambda'\left(v\right),u-v\right\rangle;$$

(*vi*) *I is weakly lower semi-continuos on* X;

(*vii*)*I is well-defined on* X *and of class* $C^1(X,\mathbb{R})$, *and its derivative given by*

$$\left\langle I'\left(u\right),v\right\rangle=M\left(\int_\Omega A\left(x,\nabla u\right)dx\right)\int_\Omega a\left(x,\nabla u\right)\nabla vdx-\int_\Omega f\left(x,u\right)vdx;$$

for all $u,v\in X$.

Since the proof of Lemma 3.2 is very similar to the proof of Lemma 2.2 and Lemma 2.7 given in [17], we omit it.

Theorem 3.3. *Assume that* (**A3**) *and the following conditions hold:*

(M_1) $M:\mathbb{R}^+\to\mathbb{R}^+$ *is a continuos function and satisfies the condition*

$$m_0s^{\alpha-1}\leq M(s)\leq m_1s^{\alpha-1},$$

for all $s>0$ *and* m_0,m_1 *real numbers such that* $0<m_0\leq m_1$ *and* $\alpha\geq1$.

($\mathbf{f}_1$) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ *is a Carathéodory condition and satisfies the growth condition*

$$|f(x,t)|\leq c_0\left(1+\left|t\right|^{\delta(x)-1}\right),\ \forall\left(x,t\right)\in\Omega\times\mathbb{R},$$

where c_0 *is positive constant and* $\delta(x)\in C_+\left(\overline{\Omega}\right)$ *such that* $\delta^+<\alpha p^-<p^*\left(x\right)$ *for all* $x\in\Omega$.

Then problem (1.1) *has a weak solution.*

Proof. Let $\|u\| > 1$. By $(M_1), (\mathbf{f}_1), (\mathbf{A3})$ and Proposition 2.2 (i), we get

$$\begin{aligned} I(u) &\geq \frac{m_0}{\alpha}\left(\int_\Omega A(x,\nabla u)\,dx\right)^\alpha - c_0\int_\Omega |u|^{\delta(x)}\,dx - c_0\int_\Omega |u|\,dx \\ &\geq \frac{m_0}{\alpha\,(p^+)^\alpha}\|u\|^{\alpha p^-} - c_1\|u\|^{\delta^+} - c_2\|u\| \to +\infty, \text{ as } \|u\| \to +\infty, \end{aligned}$$

Thus, I is coercive. Since I is weakly lower semi-continuous, I has a minimum point u in X, and u is a weak solution of problem (1.1). The proof is completed. □

Theorem 3.4. *Assume that* (M_1) *and the following conditions hold:*

$(\mathbf{f}_2)$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory condition and satisfies the growth condition;*

$$|f(x,t)| \leq c\left(1 + |t|^{\eta(x)-1}\right), \forall\,(x,t) \in \Omega \times \mathbb{R},$$

$(\mathbf{f}_3)$ $f(x,t) = o\left(|t|^{\alpha p^+ - 1}\right)$, $t \to 0$, *for* $x \in \Omega$ *uniformly,*

where c *is positive constant and* $\eta(x) \in C_+\left(\overline{\Omega}\right)$ *such that* $\alpha p^+ < \eta^- \leq \eta^+ < p^*(x)$ *for all* $x \in \Omega$,

$(AR) : \exists t_* > 0$, $\theta > \frac{m_1}{m_0}\alpha p^+$ *such that*

$$0 < \theta F(x,t) \leq f(x,t)t, \;\; |t| \geq t_*, \; a.e. \; x \in \Omega.$$

Then problem (1.1) *has a nontrivial weak solution.*

Definition 3.1. We say that I satisfies Palais-Smale condition in X ((PS) condition for short) if if any sequence $\{u_n\}$ in X such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

Lemma 3.5. *Suppose* $(M_1), (\mathbf{f}_1), (\mathbf{A3})$ *and* (AR) *hold. Then, I satisfies* (PS) *condition.*

Proof. Let assume that there exists a sequence $\{u_n\} \subset X$ such that

$$|I(u_n)| \leq c \;\text{ and } I'(u_n) \to 0 \;\text{ as }\; n \to \infty. \tag{3.1}$$

Then, by $(M_1), (\mathbf{A3})$ and (AR), we have

$$\begin{aligned} c + \|u_n\| &\geq I(u_n) - \frac{1}{\theta}\langle I'(u_n), u_n\rangle \\ &\geq \frac{m_0}{\alpha}\left(\int_\Omega A(x,\nabla u_n)\,dx\right)^\alpha - \frac{m_1 p^+}{\theta}\left(\int_\Omega A(x,\nabla u_n)\,dx\right)^{\alpha-1}\int_\Omega A(x,\nabla u_n)\,dx \\ &\geq \left(\frac{m_0}{\alpha} - \frac{m_1 p^+}{\theta}\right)\left(\int_\Omega A(x,\nabla u_n)\,dx\right)^\alpha \end{aligned}$$

By $(\mathbf{A3})$ and Proposition 2.2 (ii), we can write

$$c + \|u_n\| \geq \left(\frac{m_0}{\alpha} - \frac{m_1 p^+}{\theta}\right)\|u_n\|^{\alpha p^-}.$$

Since $\alpha p^- > 1$, $\{u_n\}$ is bounded in X. Therefore, there exists $u \in X$, up to a subsequence, such that $u_n \rightharpoonup u$ in X.

Moreover, since we have the compact embedding $X \hookrightarrow L^{\eta(x)}(\Omega)$, we get

$$u_n \to u \text{ in } L^{\eta(x)}(\Omega) \;\text{ and }\; u_n \to u \text{ a.e in } \Omega \tag{3.2}$$

By (3.1), we have

$$\begin{aligned}\langle I'(u_n), u_n - u\rangle &= M\left(\int_\Omega A\left(x, \nabla u_n\right) dx\right) \int_\Omega a\left(x, \nabla u_n\right)\left(\nabla u_n - \nabla u\right) dx \\ &\quad - \int_\Omega f\left(x, u_n\right)\left(u_n - u\right) dx \to 0.\end{aligned}$$

By using $(\mathbf{f}_1)$ and Proposition 2.1, it follows

$$\left|\int_\Omega f\left(x, u_n\right)\left(u_n - u\right) dx\right| \leq c_3 \left|\left|u_n\right|^{\eta(x)-1}\right|_{\eta'(x)} \left|u_n - u\right|_{\eta(x)}.$$

If we consider the relations given in (3.2), we get $\int_\Omega f\left(x, u_n\right)\left(u_n - u\right) dx \to 0$. Then, we have

$$M\left(\int_\Omega A\left(x, \nabla u_n\right) dx\right) \int_\Omega a\left(x, \nabla u_n\right)\left(\nabla u_n - \nabla u\right) dx \to 0.$$

From (M_1), it follows

$$\int_\Omega a\left(x, \nabla u_n\right)\left(\nabla u_n - \nabla u\right) dx \to 0.$$

that is, $\lim\limits_{n\to\infty} \left\langle \Lambda'\left(u_n\right), u_n - u\right\rangle = 0$. By using Lemma 3.2 (v), we get

$$0 = \lim_{n\to\infty} \left\langle \Lambda'\left(u_n\right), u - u_n\right\rangle \leq \lim_{n\to\infty}\left(\Lambda\left(u\right) - \Lambda\left(u_n\right)\right) = \Lambda\left(u\right) - \lim_{n\to\infty} \Lambda\left(u_n\right)$$

or $\lim\limits_{n\to\infty} \Lambda\left(u_n\right) \leq \Lambda\left(u\right)$. This fact and from Lemma 3.2 (iii) imply $\lim\limits_{n\to\infty} \Lambda\left(u_n\right) = \Lambda\left(u\right)$.

Now, we assume by contradiction that $\{u_n\}$ does not converge strongly to u in X.Then, there exists $\varepsilon > 0$ and a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ such that $\|u_{n_m} - u\| \geq \varepsilon$. On the other hand, by from Lemma 3.2 (iv), we have

$$\frac{1}{2}\Lambda\left(u\right) + \frac{1}{2}\Lambda\left(u_{n_m}\right) - \Lambda(\frac{u_{n_m} + u}{2}) \geq k\left\|u_{n_m} - u\right\|^{p^-} \geq k\varepsilon^{p^-}.$$

Letting $m \to \infty$ in the above inequality, we obtain

$$\limsup_{n\to\infty} \Lambda(\frac{u_{n_m} + u}{2}) \leq \Lambda\left(u\right) - k\varepsilon^{p^-}.$$

Moreover, we have $\{\frac{u_{n_m} + u}{2}\}$ converges weakly to u in X. Using Lemma 3.2 (iii), we obtain

$$\Lambda\left(u\right) \leq \liminf_{n\to\infty} \Lambda(\frac{u_{n_m} + u}{2}),$$

which is a contradiction. Therefore, it follows that $\{u_n\}$ converges strongly to u in X. The proof of Lemma 3.5 is complete. □

Lemma 3.6. *Suppose* $(M_1), (\mathbf{f}_1), (\mathbf{f}_3), (\mathbf{A3})$*and* (AR) *hold. Then the following statements hold:*

(*i*) *There exist two positive real numbers* γ *and* a *such that* $I\left(u\right) \geq a > 0$, $u \in X$ *with* $\|u\| = \gamma$.

(*ii*) *There exists* $u \in X$ *such that* $\|u\| > \gamma$, $I\left(u\right) < 0$.

Proof. (i) Let $\|u\| < 1$. Then by (M_1), we have

$$I\left(u\right) \geq \frac{m_0}{\alpha\left(p^+\right)^\alpha}\left\|u\right\|^{\alpha p^+} - \int_\Omega F(x, u) dx.$$

Using the continuous embeddings $X \hookrightarrow L^{\alpha p^+}(\Omega)$ and $X \hookrightarrow L^{\eta(x)}(\Omega)$, there exist positive constants c_4 and c_5 such that

$$|u|_{\eta(x)} \leq c_4 \|u\| \quad \text{and} \quad |u|_{\alpha p^+} \leq c_5 \|u\| \ , \ \forall \ u \in X.$$

Let $\varepsilon > 0$ be small enough such that $\varepsilon c_4^{\alpha p^+} \leq \frac{m_0}{2\alpha (p^+)^\alpha}$. By $(\mathbf{f}_1)$ and $(\mathbf{f}_3)$, we get $F(x,t) \leq \varepsilon |t|^{\alpha p^+} + c_\varepsilon |t|^{\eta(x)}, \forall (x,t) \in \Omega \times \mathbb{R}$. Therefore, Proposition 2.2 (ii), we have

$$\begin{aligned} I(u) &\geq \frac{m_0}{\alpha (p^+)^\alpha} \|u\|^{\alpha p^+} - \varepsilon \int_\Omega |u|^{\alpha p^+} dx - c_\varepsilon \int_\Omega |u|^{\eta(x)} dx \\ &\geq \frac{m_0}{\alpha (p^+)^\alpha} \|u\|^{\alpha p^+} - \varepsilon c_4^{\alpha p^+} \|u\|^{\alpha p^+} - c_\varepsilon c_5^{\eta^-} \|u\|^{\eta^-} \\ &\geq \frac{m_0}{2\alpha (p^+)^\alpha} \|u\|^{\alpha p^+} - c_\varepsilon c_5^{\eta^-} \|u\|^{\eta^-} . \end{aligned}$$

Since $\|u\| < 1$ and $\alpha p^+ < \eta^-$, there exist two positive real numbers γ and a such that $I(u) \geq a > 0$, $u \in X$ with $\|u\| = \gamma \in (0,1)$.

(ii) From (AR), one easily deduces

$$F(x,t) \geq c_6 |t|^\theta, \quad |t| \geq t_*, \ a.e. \ x \in \Omega.$$

In the other hand, when $t > t_* > 1$, from (M_1) we can easily obtain

$$\widehat{M}(t) \leq \frac{m_1}{\alpha} t^\alpha \leq \frac{m_1}{\alpha} t^{\frac{m_1}{m_0}\alpha}.$$

Thus, for any fixed $\omega \in X \backslash \{0\}$, $t > 1$ and from Lemma 3.1 (ii), we have

$$\begin{aligned} I(t\omega) &= \widehat{M}\left(\int_\Omega A(x, \nabla t\omega)\, dx\right) - \int_\Omega F(x,t\omega) dx \\ &\leq \frac{m_1}{\alpha} \left(\int_\Omega A(x, \nabla t\omega)\, dx\right)^{\frac{m_1}{m_0}\alpha} - \int_\Omega F(x,t\omega) dx \\ &\leq \frac{m_1}{\alpha (p^-)^{\frac{m_1}{m_0}\alpha p^+}} t^{\frac{m_1}{m_0}\alpha p^+} \int_\Omega A(x, \nabla\omega)\, dx - c_6 t^\theta \int_\Omega |\omega|^\theta dx. \end{aligned}$$

From (AR) ,it can be obtained that $\theta > \frac{m_1}{m_0}\alpha p^+$. Hence, $I(t\omega) \to -\infty$ as $t \to +\infty$. ☐

Proof of Theorem 3.3. From Lemma 3.5, Lemma 3.6, Lemma 3.2 (vii), $(\mathbf{A4})$ and the fact that $I(0) = 0$, I satisfies the Mountain Pass Theorem [19]. Therefore, I has at least one nontrivial critical point, i.e., problem (1.1) has a nontrivial weak solution. The proof of Theorem 3.3 is complete. ☐

Theorem 3.7. *Assume that* $(M_1), (\mathbf{f}_1), (AR)$ *and the following*

$(\mathbf{f}_4): f(x,-t) = -f(x,t)$, *for* $(x,t) \in \Omega \times \mathbb{R}$,

then I *has a sequence of critical points* $\{u_n\}$*such that* $I(u_n) \to +\infty$ *and* (1.1) *has infinite many pairs of solutions.*

In order to prove Theorem 3.7, we need Lemma 3.8.

Since X be a reflexive and separable Banach space, then there are $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{span\{e_j \mid j = 1,2,...\}}, \qquad X^* = \overline{span\left\{e_j^* \mid j = 1,2,...\right\}},$$

and

$$\langle e_i^*, e_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

For convenience, we write $X_j = span\left\{e_j\right\}$, $Y_k = \oplus_{j=1}^{k} X_j$, $Z_k = \oplus_{j=k}^{\infty} X_j$.

Lemma 3.8. *If* $\eta(x) \in C_+\left(\overline{\Omega}\right)$, $\eta(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *denote*

$$\beta_k = \sup\left\{ |u|_{\eta(x)} : \|u\| = 1, u \in Z_k \right\}.$$

Then $\lim_{k\to\infty} \beta_k = 0$.

Since the proof of Lemma 3.8 is similar to that of Lemma 4.9 in [13], we omit it.

Proof of Theorem 3.7. According to (M_1), $(\mathbf{f}_4)$ and (AR), I satisfies (PS) condition and from $(\mathbf{A5})$ it is an even functional. We only need to prove that if k is large enough, then there exist $\rho_k > \gamma_k > 0$ such that

$(\mathbf{A6})$ $b_k := \inf\left\{I(u) \mid u \in Z_k, \|u\| = \gamma_k\right\} \to \infty$ as $k \to \infty$;

$(\mathbf{A7})$ $a_k := \max\left\{I(u) \mid u \in Y_k, \|u\| = \rho_k\right\} \leq 0$.

Thus, the conclusion of Theorem 3.7 can be obtained by Fountain Theorem [19].

$(\mathbf{A6})$ For any $u \in Z_k$, $\|u\| = \gamma_k = \left(c_8 \eta^+ \beta_k^{\eta^+} m_0^{-1}\right)^{\frac{1}{\alpha p^- - \eta^+}}$, we have

$$\begin{aligned}
I(u) &\geq \frac{m_0}{\alpha}\left(\int_\Omega A(x,\nabla u)\,dx\right)^{\alpha} - c_0 \int_\Omega |u|^{\eta(x)}\,dx - c_0 \int_\Omega |u|\,dx \\
&\geq \frac{m_0}{\alpha (p^+)^{\alpha}} \|u\|^{\alpha p^-} - c_0 |u|_{\eta(x)}^{\eta(\zeta)} - c_0 \|u\|, \text{ where } \zeta \in \Omega \\
&\geq \begin{cases} \frac{m_0}{\alpha (p^+)^{\alpha}} \|u\|^{\alpha p^-} - c_0 - c_0 \|u\|, & \text{if } |u|_{\eta(x)} \leq 1 \\ \frac{m_0}{\alpha (p^+)^{\alpha}} \|u\|^{\alpha p^-} - c_0 \beta_k^{\eta^+} \|u\|^{\eta^+} - c_0 \|u\|, & \text{if } |u|_{\eta(x)} > 1 \end{cases} \\
&\geq \frac{m_0}{\alpha (p^+)^{\alpha}} \|u\|^{\alpha p^-} - c_0 \beta_k^{\eta^+} \|u\|^{\eta^+} - c_0 \|u\| - c_7 \\
&= \frac{m_0}{\alpha (p^+)^{\alpha}} \left(c_8 \eta^+ \beta_k^{\eta^+} m_0^{-1}\right)^{\frac{\alpha p^-}{\alpha p^- - \eta^+}} - c_0 \beta_k^{\eta^+} \left(c_8 \eta^+ \beta_k^{\eta^+} m_0^{-1}\right)^{\frac{\eta^+}{\alpha p^- - \eta^+}} - c_0 \|u\| - c_7 \\
&\geq \frac{m_0}{(p^+)^{\alpha}} \left(\frac{1}{\alpha} - \frac{1}{\eta^+}\right) \left(c_8 \eta^+ \beta_k^{\eta^+} m_0^{-1}\right)^{\frac{\alpha p^-}{\alpha p^- - \eta^+}} - c_0 \left(c_8 \eta^+ \beta_k^{\eta^+} m_0^{-1}\right)^{\frac{1}{\alpha p^- - \eta^+}} - c_7
\end{aligned}$$

Because $\beta_k \to 0$ and $\alpha < \alpha p^- < \eta^+$, we have $I(u) \to \infty$ as $k \to \infty$

$(\mathbf{A7})$ From (AR), we get $F(x,t) \geq c_9 |t|^{\theta} - c_{10}$. Therefore, for any $w \in Y_k$ with $\|w\| = 1$ and $1 < t = \rho_k$, we have

$$\begin{aligned}
I(t\omega) &\leq \frac{m_1}{\alpha}\left(\int_\Omega A(x,\nabla t\omega)\,dx\right)^{\frac{m_1}{m_0}\alpha} - c_9 t^{\theta} \int_\Omega |\omega|^{\theta}\,dx - c_{10} \\
&\leq \frac{m_1}{\alpha (p^-)^{\frac{m_1}{m_0}\alpha p^+}} t^{\frac{m_1}{m_0}\alpha p^+} \int_\Omega A(x,\nabla \omega)\,dx - c_9 t^{\theta} \int_\Omega |\omega|^{\theta}\,dx - c_{10}.
\end{aligned}$$

By $\theta > \frac{m_1}{m_0}\alpha p^+$ and $\dim Y_k = k$, it is easy see that $I(t\omega) \to -\infty$ as $t \to +\infty$ for $u \in Y_k$. □

References

[1] C. O. Alves, F. J. S. A. Corrêa, T. F. Ma, Positive Solutions for a Quasilinear Elliptic Equation of Kirchhoff Type, *Computers and Mathematics with Appl.* **49** (2005), 85–93.

[2] D. Andrade, T.F. Ma, An operator equation suggested by a class of stationary problems, *Comm. Appl. Nonlinear Anal.* **4** (1997), 65–71.

[3] S. N. Antontsev, S. I. Shmarev, A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions, *Nonlinear Anal.* **60** (2005), 515–545.

[4] A. Arosio, S. Pannizi, On the well-posedness of the Kirchhoff string, *Trans. Amer. Math. Soc.* **348** (1996), 305–330.

[5] M. Avci, Existence and multiplicity of solutions for Dirichlet problems involving the $p(x)$-Laplace operator, *Electron. J. Diff. Equ.* **14** (2013), 1–9.

[6] M. M. Cavalcante, V. N. Cavalcante, J. A. Soriano, Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation, *Adv. Diff. Equ.* **6** (2001), 701–730.

[7] C.Y. Chen, Y.c. Kuo, T.f Wu, The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions, *J. Differential Equations* **250** (2011), 1876–1908.

[8] M. Chipot, J.F. Rodrigues, On a class of nonlocal nonlinear elliptic problems, *RAIRO Modelisation Math. Anal.* **26** (1992), 447–467.

[9] F. J. S. A. Corrêa, G.M. Figueiredo, On a p-Kirchhoff equation via Krasnoselskii's genus, *Appl. Math. Letters* **22** (2009), 819–822.

[10] G. Dai, R. Hao, Existence of solutions for a $p(x)$-Kirchhoff-type equation, *J. Math. Anal. Appl.* **359** (2009), 275–284.

[11] P. D'Ancona, S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, *Invent. Math.* **1**(08) (1992), 247–262.

[12] X. L. Fan, J. S. Shen, D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **262** (2001), 749–760.

[13] X. L. Fan, Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problems, *Nonlinear Anal.* **52** (2003), 1843–1852.

[14] X. He, W. Zou, Infinitely many positive solutions for Kirchhoff-type problems, *Nonlinear Anal.* **70** (2009), 1407–1414.

[15] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

[16] O. Kovăčik, J. Răkosnik, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.* **41** (1991), no. 116, 592–618.

[17] R. A. Mashiyev, B. Cekic, M. Avci, Z. Yücedag, Existence and multiplicity of weak solutions for nonuniformly elliptic equations with nonstandard growth condition, *Complex Variables and Elliptic Equations* **57** (2012), no. 5, 579–595.

[18] M. Růžička, *Electrorheological fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics 1748, Springer-Verlag, Berlin, 2000.

[19] M. Willem, *Minimax Theorems*, Birkhauser, Basel, 1996.

[20] Z. Yücedag, Solutions of nonlinear problems involving $p(x)$-Laplacian operator, *Adv. Nonlinear Anal.* **4** (2015), no. 4, 285–293.

(Zehra Yücedag) Department of Mathematics, Dicle University, Turkey
E-mail address: `zyucedag@dicle.edu.tr`

Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 30–63
ISSN: 1223-6934

$p\,(.)$-parabolic capacity and decomposition of measures

Stanislas OUARO and Urbain TRAORE

Abstract. In this paper, we develop a concept of $p\,(.)$ −parabolic capacity in order to give a result of decomposition of measures(in space and time) which does not charge the sets of null capacity.

2010 Mathematics Subject Classification. Primary 35J60; Secondary 35D05.
Key words and phrases. Parabolic capacity, decomposition of measure, variable exponent, quasicontinuous function.

1. Introduction and main result

The concept of capacity play an important role in the study of solutions of partial differential equations; it permits to see that the functions in the Sobolev spaces are defined better than almost everywhere. In the elliptic case, the notion of capacity is related to the Sobolev spaces (see [4]). More precisely, let $\Omega \subset \mathbb{R}^N$, be open bounded, for $E \subset \Omega$, the Sobolev $p\,(.)$-capacity of E is defined by

$$C_{p(.)}\,(E) := \inf_{u \in S_{p(.)}(E)} \int_\Omega \left(|u|^{p(x)} + |\nabla u|^{p(x)} \right) dx, \tag{1}$$

where

$$S_{p(.)}\,(E) := \left\{ u \in W^{1,p(.)}\,(\Omega) : \ u \geq 1 \text{ in an open set containing } E \text{ and } u \geq 0 \right\}. \tag{2}$$

In the case where $S_{p(.)}\,(E) = \emptyset$, we set $C_{p(.)}\,(E) = \infty$. One of the properties of the elliptic capacity is the following: for every $u \in W^{1,p(.)}\,(\Omega)$, there exists a $p\,(.)$-quasicontinuous function $v \in W^{1,p(.)}\,(\Omega)$ such that $u = v$ almost everywhere in Ω i.e $u = v$ a.e. Ω and for every $\varepsilon > 0$, there exists an open set $U_\varepsilon \subset \Omega$ with $C_{p(.)}\,(U_\varepsilon) < \varepsilon$ such that v restricted to $\Omega \setminus U_\varepsilon$ is continuous.
The theory of capacity is an essential tool in the study of the existence and uniqueness of the solution of some elliptic and parabolic problems with measures data. Let's recall that in the context of constant exponent, the authors in [3] proved that every diffuse measure μ i.e. a measure which does not charge the sets of null $p-$capacity belongs to $L^1\,(\Omega) + W^{-1,p'}\,(\Omega)$ with p' the conjuguate of p, that permit them to prove the existence and uniqueness of entropy solution for the following problem.

$$\begin{cases} A\,(u) = \mu & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

Received April 03, 2015. Accepted December, 2016.

where A is a Leray-Lions type operator.
In the context of variable exponent, a similar approach is used in [12] for the elliptic problem

$$\begin{cases} \nabla.a\,(x,\nabla u)+\beta\,(u)\ni\mu & \text{in } \ \Omega \\ u=0 & \text{on } \ \partial\Omega, \end{cases} \tag{4}$$

where μ is a diffuse measure. In [12], the authors used the ideas of [3] to prove that for every diffuse measure μ, there exists $f\in L^1\,(\Omega)$ and $g\in W^{-1,p'(.)}\,(\Omega)$ such that $\mu=f+g$, that permits them to prove the existence and uniqueness of entropy solution of (4).
The notion of parabolic capacity have been introduced firstly in the quadratic case $p\equiv 2$. The thermal capacity related to the heat equation, and its generalizations have been studied, for example, by Lanconelli [9] and Watson [18]. In the papers [1, 6, 7], the concept of parabolic capacities for constant exponent are defined in terms of function spaces. Droniou, Porretta and Prignet in [6], introduced and studied the notion of parabolic capacity associated with the initial boundary valued problem

$$\begin{cases} u_t+A\,(u)=\mu & \text{in } \ Q=(0,T)\times\Omega \\ u=u_0 & \text{on } \ \{0\}\times\Omega \\ u=0 & \text{on } \ (0,T)\times\partial\Omega. \end{cases} \tag{5}$$

They worked with the space

$$W=\left\{u\in L^p\left(0,T;W^{1,p}\,(\Omega)\cap L^2\,(\Omega)\right);u_t\in L^{p'}\left(0,T;\left(W_0^{1,p}\,(\Omega)\cap L^2\,(\Omega)\right)'\right)\right\},$$

to get a representation theorem for measures that are zero on subsets of Q of null capacity, more precisely they proved the following result (see [6]).

Theorem 1.1. *Let μ be a bounded measure on Q which does not charge the sets of null capacity. Then there exists $g_1\in L^{p'}\left(0,T;W^{-1,p'}\,(\Omega)\right)$, $g_2\in L^p\left(0,T;W^{1,p}\,(\Omega)\cap L^2\,(\Omega)\right)$ and $h\in L^1\,(\Omega)$ such that*

$$\int_Q\varphi d\mu=\int_0^T\langle g_1,\varphi\rangle\,dt-\int_0^T\langle g_2,\varphi_t\rangle\,dt+\int_Q h\varphi dxdt, \tag{6}$$

for all $\varphi\in C_c^\infty\,([0,T]\times\Omega)$, where $\langle .,.\rangle$ denote the duality between $\left(W^{1,p}\,(\Omega)\cap L^2\,(\Omega)\right)'$ and $W^{1,p}\,(\Omega)\cap L^2\,(\Omega)$.

In this paper, we extend the theory developed in [6] in the case of variable exponents (see [14, 15] for the theory of PDEs with variable exponents). The paper is organized as follows: in Section 2, we recall some basic notations and properties of Lebesgue and Sobolev spaces with variable exponents. In Section 3, we introduce and study the notion of $p\,(.)$ −parabolic capacity. In the last section, we show the connection between measures defined on the σ-algebra of borelians of Q and the notion of $p\,(.)-$ parabolic capacity and, we prove a theorem of decomposition of measures.

2. Preliminary

In this paper, we assume that

$$\begin{cases} p\,(.):\overline{\Omega} & \rightarrow & \mathbb{R} \quad \text{is a continuous function such that} \\ 1<p_- & \leq & p_+<+\infty, \end{cases} \tag{7}$$

where $p_- := \inf_{x\in\Omega} p(x)$ and $p_+ := \sup_{x\in\Omega} p(x)$.
We denote the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ (see [4]) as the set of all measurable function $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)}\, dx$$

is finite.
If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$\|u\|_{p(.)} := \inf\left\{\lambda > 0 : \rho_{p(.)}(u/\lambda) \leq 1\right\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm.
The space $\left(L^{p(\cdot)}(\Omega), \|.\|_{p(.)}\right)$ is a separable Banach space. Moreover, if $1 < p_- \leq p_+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$, for $x \in \Omega$.
Finally, we have the Hölder type inequality

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p_-} + \frac{1}{(p')_-}\right) \|u\|_{p(.)} \|v\|_{p'(.)}, \tag{8}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.
Let

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

which is Banach space equiped with the following norm

$$\|u\|_{1,p(.)} := \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

The space $\left(W^{1,p(\cdot)}(\Omega), \|.\|_{1,p(.)}\right)$ is a separable and reflexive Banach space.
An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result.

Proposition 2.1. *(see* [8, 21]*) If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p_+ < \infty$, the following properties hold true.*
(i) $\|u\|_{p(.)} > 1 \Rightarrow \|u\|_{p(.)}^{p_-} < \rho_{p(.)}(u) < \|u\|_{p(.)}^{p_+}$;
(ii) $\|u\|_{p(.)} < 1 \Rightarrow \|u\|_{p(.)}^{p_+} < \rho_{p(.)}(u) < \|u\|_{p(.)}^{p_-}$;
(iii) $\|u\|_{p(.)} < 1$ $(\textit{respectively} = 1; > 1) \Leftrightarrow \rho_{p(.)}(u) < 1$ $(\textit{respectively} = 1; > 1)$;
(iv) $\|u_n\|_{p(.)} \to 0$ $(\textit{respectively} \to +\infty) \Leftrightarrow \rho_{p(.)}(u_n) < 1$ $(\textit{respectively} \to +\infty)$;
(v) $\rho_{p(.)}\left(u/\|u\|_{p(.)}\right) = 1$.

For a measurable function $u : \Omega \to \mathbb{R}$, we introduce the following notation.

$$\rho_{1,p(.)}(u) = \int_\Omega |u|^{p(x)}\, dx + \int_\Omega |\nabla u|^{p(x)}\, dx.$$

Proposition 2.2. *(see* [17, 19]*) If $u \in W^{1,p(\cdot)}(\Omega)$, the following properties hold true.*
(i) $\|u\|_{1,p(.)} > 1 \Rightarrow \|u\|_{1,p(.)}^{p_-} < \rho_{1,p(.)}(u) < \|u\|_{1,p(.)}^{p_+}$;
(ii) $\|u\|_{1,p(.)} < 1 \Rightarrow \|u\|_{1,p(.)}^{p_+} < \rho_{p(.)}(u) < \|u\|_{1,p(.)}^{p_-}$;
(iii) $\|u\|_{1,p(.)} < 1$ $(\textit{respectively} = 1; > 1) \Leftrightarrow \rho_{1,p(.)}(u) < 1$ $(\textit{respectively} = 1; > 1)$.

Following [2], we extend a variable exponent $p : \overline{\Omega} \to [1, +\infty)$ to $\overline{Q} = [0,T] \times \overline{\Omega}$ by setting $p\,(t,x) = p\,(x)$ for all $(t,x) \in \overline{Q}$.
We may also consider the generalized Lebesgue space

$$L^{p(.)}\,(Q) = \left\{ u : Q \to \mathbb{R} \,\text{measurable such that} \int\!\!\int_{Q} |u\,(t,x)|^{p(x)}\, d\,(t,x) < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^{p(.)}(Q)} := \inf \left\{ \lambda > 0,\ \int\!\!\int_{Q} \left| \frac{u\,(t,x)}{\lambda} \right|^{p(x)} d\,(t,x) < 1 \right\},$$

which share the same properties as $L^{p(.)}\,(\Omega)$.

3. Parabolic capacity and measures

3.1. Capacity. In this part, we introduce our notion of capacity, following the approach developed in [6].

Definition 3.1. Let us define $V = W_0^{1,p(.)}\,(\Omega) \cap L^2\,(\Omega)$, endowed with its natural norm $\|.\|_{W_0^{1,p(.)}(\Omega)} + \|.\|_{L^2(\Omega)}$ and the space

$$W_{p(.)}\,(0,T) = \left\{ u \in L^{p_-}\,(0,T;V)\,; \nabla u \in (L^{p(.)}\,(Q))^N,\ u_t \in L^{(p_-)'}\,(0,T;V') \right\}$$

endowed with its natural norm $\|u\|_{W_{p(.)}(0,T)} = \|u\|_{L^{p_-}(0,T;V)} + \|\nabla u\|_{(L^{p(.)}(Q))^N} + \|u_t\|_{L^{(p_-)'}(0,T;V')}$.

Since $W_0^{1,p(.)}\,(\Omega)$ and $L^2\,(\Omega)$ are separables and reflexives Banach spaces, it follows that V is a separable and reflexive Banach space. Consequently, the following result can be proved similarly to that in [5]; thus, we omit its proof.

Theorem 3.1. *The space $W_{p(.)}\,(0,T)$ is a separable and reflexive Banach space.*

We also have the following result.

Proposition 3.2. *i) $W_{p(.)}\,(0,T)$ is continuously embedded in $C\left(0,T;L^2\,(\Omega)\right)$.*
ii) For all $\theta \in C^\infty\left(\mathbb{R} \times \mathbb{R}^N\right)$ and $u \in W_{p(.)}\,(0,T)$, $\theta u \in W_{p(.)}\,(0,T)$ and there exists $C\,(\theta)$ not depending on u such that $\|\theta u\|_{W_{p(.)}(0,T)} \leq C\,(\theta)\,\|u\|_{W_{p(.)}(0,T)}$.

Proof. i) Since $V \hookrightarrow L^2\,(\Omega) \hookrightarrow V'$, thanks to [5], $W_{p(.)}\,(0,T)$ is continuously embedded in $C\left(0,T;L^2\,(\Omega)\right)$ i.e. there exists $C > 0$ such that, for all $u \in W_{p(.)}\,(0,T)$,

$$\|u\|_{C(0,T;L^2(\Omega))} \leq C\,\|u\|_{W_{p(.)}(0,T)}.$$

ii) The fact that θ is a smooth function implies that $\theta u \in L^{p_-}\,(0,T;V)$ and there exists $C\,(\theta) > 0$ such that $\|\theta u\|_{L^{p_-}(0,T;V)} \leq C\,(\theta)\,\|u\|_{L^{p_-}(0,T;V)}$. We know that $\nabla\,(\theta u) = u\nabla\theta + \theta\nabla u$. Since θ is a smooth function, there exists $C\,(\theta) > 0$ such that $\|\theta\nabla u\|_{\left(L^{p(.)}(Q)\right)^N} \leq C\,(\theta)\,\|\nabla u\|_{\left(L^{p(.)}(Q)\right)^N}$; moreover, using Poincaré type inequality, one shows that $\|u\nabla\theta\|_{\left(L^{p(.)}(Q)\right)^N} \leq C\,(\theta)\,\|\nabla u\|_{\left(L^{p(.)}(Q)\right)^N}$. Therefore, we can write $\|\nabla\,(u\theta)\|_{\left(L^{p(.)}(Q)\right)^N} \leq C\,(\theta)\,\|\nabla u\|_{\left(L^{p(.)}(Q)\right)^N}$. We have, in the sense of distributions, $(\theta u)_t = u\theta_t + \theta u_t$. The second term belongs to $L^{(p_-)'}\,(0,T;V')$ and

$\|\theta u_t\|_{L^{(p_-)'}(0,T;V')} \leq C(\theta)\|u_t\|_{L^{(p_-)'}(0,T;V')}$. Since $W_{p(.)}(0,T) \hookrightarrow C\left(0,T;L^2(\Omega)\right) \hookrightarrow L^{(p_-)'}\left(0,T;L^2(\Omega)\right)$, then $u\theta_t \in L^{(p_-)'}\left(0,T;L^2(\Omega)\right)$ and $\|u\theta_t\|_{L^{(p_-)'}(0,T;L^2(\Omega))} \leq C(\theta)\|u\|_{W_{p(.)}(0,T)}$. We know that $L^2(\Omega) \hookrightarrow V'$, so $L^{(p_-)'}\left(0,T;L^2(\Omega)\right) \hookrightarrow L^{(p_-)'}(0,T;V')$, which implies that
$u\theta_t \in L^{(p_-)'}(0,T;V')$ and $\|u\theta_t\|_{L^{(p_-)'}(0,T;V')} \leq C(\theta)\|u\|_{W_{p(.)}(0,T)}$ □

Remark 3.1. Since $L^{(p_-)'}(0,T;V') = (L^{p_-}(0,T;V))'$ (since V is a separable reflexive space), and since $L^{p_-}(0,T;V) = L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right) \cap L^{p_-}\left(0,T;L^2(\Omega)\right) = E \cap F$, with $E \cap F$ being dense both in E and F, we have $L^{(p_-)'}(0,T;V') = E' + F' = L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right) + L^{(p_-)'}\left(0,T;L^2(\Omega)\right)$ and the norms of these spaces are equivalent.

We introduce the space $\widetilde{W}_{p(.)}(0,T)$ by

$$\widetilde{W}_{p(.)}(0,T) = \Big\{u \in L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right) \cap L^{\infty}\left(0,T;L^2(\Omega)\right); \nabla u \in \left(L^{p(.)}(Q)\right)^N, \\ u_t \in L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)\Big\}$$

Remark 3.2. $W^{-1,p'(.)}(\Omega) \hookrightarrow V'$, then $\widetilde{W}_{p(.)}(0,T)$ is continuously embedded in $W_{p(.)}(0,T)$.

Now, we give the definition and some properties of capacity.

Definition 3.2. If $U \subset Q$ is an open set, we define the parabolic capacity of U as

$$\mathrm{Cap}_{p(.)}(U) = \inf\left\{\|u\|_{W_{p(.)}(0,T)} : u \in W_{p(.)}(0,T), u \geq \chi_U \text{ almost everywhere in } Q\right\}. \tag{9}$$

Remark 3.3. We will use the convention that $\inf\emptyset = +\infty$ and for any borelian subset $B \subset Q$ the definition of capacity is extended by setting

$$\mathrm{Cap}_{p(.)}(B) = \inf\left\{\mathrm{Cap}_{p(.)}(U), U \text{ open subset of } Q, B \subset U\right\}. \tag{10}$$

Proposition 3.3. *The set function $E \mapsto Cap_{p(.)}(E)$ has the following properties.*
i) If $E_1 \subset E_2$, then

$$Cap_{p(.)}(E_1) \leq Cap_{p(.)}(E_2). \tag{11}$$

ii) For $E_i \subset Q$, $i \in \mathbb{N}$, we have

$$Cap_{p(.)}\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} Cap_{p(.)}(E_i). \tag{12}$$

Proof. *i)* Firstly, we consider the case where E_1 and E_2 are open sets of Q.
Since $E_1 \subset E_2$, we have

$$\left\{u \in W_{p(.)}(0,T), u \geq \chi_{E_1} \text{ a.e. } Q\right\} \supset \left\{u \in W_{p(.)}(0,T), u \geq \chi_{E_2} \text{ a.e. } Q\right\}.$$

Hence,

$$\begin{aligned} Cap_{p(.)}\left(E_1\right) &= \inf\left\{\|u\|_{W_{p(.)}(0,T)} : u \in W_{p(.)}\left(0,T\right), u \geq \chi_{E_1} \text{ a.e. } Q\right\} \\ &\leq \inf\left\{\|u\|_{W_{p(.)}(0,T)} : u \in W_{p(.)}\left(0,T\right), u \geq \chi_{E_2} \text{ a.e. } Q\right\} \\ &\leq Cap_{p(.)}\left(E_2\right). \end{aligned} \tag{13}$$

Now, we suppose that E_1 and E_2 are two borelians subsets of Q such that $E_1 \subset E_2$, then we have

$$\{U \text{ open set of } Q/E_2 \subset U\} \subset \{U \text{ open set of } Q/E_1 \subset U\}$$

Then, it follows that

$$\begin{aligned} Cap_{p(.)}\left(E_1\right) &= \inf\{U \text{ open set of } Q/E_1 \subset U\} \\ &\leq \inf\{U \text{ open set of } Q/E_2 \subset U\} \\ &\leq Cap_{p(.)}\left(E_2\right). \end{aligned} \tag{14}$$

ii) If $\sum_{i=1}^{\infty} Cap_{p(.)}\left(E_i\right) = +\infty$, then we have

$$Cap_{p(.)}\left(\bigcup_{i=1}^{\infty} E_i\right) = Cap_{p(.)}\left(\bigcup_{i=1}^{\infty} \{E_i/E_i \neq \emptyset\}\right) < +\infty = \sum_{i=1}^{\infty} Cap_{p(.)}\left(E_i\right). \tag{15}$$

Assuming that $\sum_{i=1}^{\infty} Cap_{p(.)}\left(E_i\right) < \infty$. Let U_i be open set containing E_i such that $Cap_{p(.)}\left(U_i\right) \leq Cap_{p(.)}\left(E_i\right) + \frac{\varepsilon}{2^i}$ and u_i be such that $u_i \geq \chi_{U_i}$ a.e. in Q with $\|u_i\|_{W_{p(.)}(0,T)} \leq Cap_{p(.)}\left(U_i\right) + \frac{\varepsilon}{2^i}$. Then,

$$\left\|\sum_{i=1}^{n} u_i\right\|_{W_{p(.)}(0,T)} \leq \sum_{i=1}^{n} \|u_i\|_{W_{p(.)}(0,T)} \leq \sum_{i=1}^{\infty} Cap_{p(.)}\left(E_i\right) + \varepsilon;$$

i.e. $\sum_{i=1}^{n} u_i$ converges strongly in $W_{p(.)}\left(0,T\right)$.

Let's now consider $u = \sum_{i=1}^{\infty} u_i$; we have $u \geq \chi_U$ a.e. in Q, where $U = \bigcup_{i=1}^{\infty} U_i$, so that, U being open,

$$Cap_{p(.)}\left(U\right) \leq \|u\|_{W_{p(.)}(0,T)} \leq \sum_{i=1}^{\infty} \|u_i\|_{W_{p(.)}(0,T)} \leq \sum_{i=1}^{\infty} Cap_{p(.)}\left(E_i\right) + \varepsilon. \tag{16}$$

Since $\bigcup_{i=1}^{\infty} E_i \subset U$, from (16) we get (12). □

The notion of capacity can be defined alternatively using compact sets of Q. Before that, we introduce the following density result(for the proof, we refer the reader to the proof of Theorem 2.11 in [6]).

Lemma 3.4. *Let* Ω *be a bounded subset of* $\mathbb{R}^N$ *and* $1 < p_- \leq p_+ < \infty$. *Then,* $C_c^\infty\left([0,T]\times\Omega\right)$ *is dense in* $W_{p(.)}\left(0,T\right)$.

Definition 3.3. Let K be a compact subset of Q. The capacity of K is defined as

$$cap\left(K\right) = \inf\left\{\|u\|_{W_{p(.)}(0,T)} : u \in C_c^\infty\left([0,T]\times\Omega\right), u \geq \chi_K\right\}.$$

The capacity of any open subset U of Q is then defined by

$$cap\left(U\right) = \sup\left\{cap\left(K\right), K\text{compact}, \ K \subset U\right\}$$

and the capacity of any Borelian set $B \subset Q$ by

$$cap\left(B\right) = \inf\left\{cap\left(U\right), U\text{open subset of } Q, \ B \subset U\right\}.$$

We have the following result.

Proposition 3.5. *i) The capacity cap satisfies the subadditivity property.*
ii) Let B *be a borelian subset of* Q. *Then,* $cap\left(B\right) = 0$ *if and only if* $Cap_{p(.)}\left(B\right) = 0$.

Proof. The proof is similar to the proofs of Proposition 2.13 and 2.14 in [6]. □

Now, we give a characterization of null capacity.

Theorem 3.6. *Let* B *be a borelian set in* Ω . *Let* $t_0 \in (0,T)$ *fixed. One has* $Cap_{p(.)}\left(\{t_0\}\times B\right) = 0$ *if and only if* $meas\left(B\right) = 0$.

Proof. Assume first that $Cap_{p(.)}\left(\{t_0\}\times B\right) = 0$ and let K be any compact set contained in B, so that $Cap_{p(.)}\left(\{t_0\}\times K\right) = 0$. Since, by Proposition 3.5, we also have that $cap\left(\{t_0\}\times B\right) = 0$, then, for all $\varepsilon > 0$, there exists a function $\psi_\varepsilon \in C_c^\infty\left([0,T]\times\Omega\right)$ such that $\|\psi_\varepsilon\|_{W_{p(.)}(0,T)} \leq \varepsilon$ and $\psi_\varepsilon\left(t_0\right) \geq 1$ on K. Since $W_{p(.)}\left(0,T\right)$ is embedded in $C\left([0,T];L^2\left(\Omega\right)\right)$, ones has then

$$\text{meas}\left(K\right) \leq \int_K \left|\psi_\varepsilon\left(t_0\right)\right|^2 dx \leq \|\psi_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C\left\|\psi_\varepsilon\right\|^2_{W_{p(.)}(0,T)} \leq C\varepsilon^2.$$

So, we deduce that $\text{meas}\left(K\right) \leq C\varepsilon^2$, and from the arbitrariness of ε, we get that $\text{meas}\left(K\right) = 0$. Since this is true for any compact subset contained in B, by regularity of the Lebesgue measure we conclude that $\text{meas}\left(B\right) = 0$.
Conversely, if $\text{meas}\left(B\right) = 0$, then there exists, for all $\varepsilon > 0$, an open set A_ε such that $B \subset A_\varepsilon$ and $\text{meas}\left(A_\varepsilon\right) < \varepsilon$.
Let us consider an $\varepsilon > 0$ fixed in what follows and, let K_n be a sequence of compact sets contained in A_ε such that $K_n \subset K_{n+1}$, for all $n \geq 1$ and $\bigcup_{n=1}^{\infty} K_n = A_\varepsilon$.
Let $\varphi_n \in C_c\left(A_\varepsilon\right)$ (the space of continuous functions with compact support in A_ε) be such that $0 \leq \varphi_n \leq 1$, $\varphi_n \equiv 1$ on K_n and $\varphi_n \leq \varphi_{n+1}$. Then, we consider for $t_0 \in [0,T]$, the problem

$$\begin{cases} (\psi_n)_t - div\left(\left|\nabla\psi_n\right|^{p(x)-2}\nabla\psi_n\right) = 0 & \text{in} \quad (t_0,T)\times\Omega \\ \psi_n\left(t_0\right) = \varphi_n & \text{in} \qquad \Omega \\ \psi_n = 0 & \text{on} \quad (t_0,T)\times\partial\Omega, \end{cases} \tag{17}$$

which admits (see [20]) a unique weak solution

$$\psi_n \in L^{p_-}\left(t_0,T;W_0^{1,p(.)}\left(\Omega\right)\right) \cap C\left([t_0,T];L^2\left(\Omega\right)\right)$$

and $(\psi_n)_t \in L^{(p_-)'}\left(t_0,T;W^{-1,p'(.)}(\Omega)\right)$ with $\nabla\psi_n \in \left(L^{p(.)}((t_0,T)\times\Omega)\right)^N$ such that for all $v \in C^1\left([t_0,T]\times\overline{\Omega}\right)$ with $v(.,T)=0$,

$$-\int_\Omega \varphi_n(x)\,v(t_0,x)\,dx - \int_{t_0}^T\int_\Omega \psi_n v_t dxdt + \int_{t_0}^T\int_\Omega |\nabla\psi_n|^{p(x)-2}\nabla\psi_n.\nabla v dxdt = 0 \tag{18}$$

holds true.

It's not difficult to see that $\psi_n \in L^{p_-}(t_0,T;V)$ and by Remark 3.1 we have $(\psi_n)_t \in L^{(p_-)'}(t_0,T;V')$ hence, $\psi_n \in W_{p(.)}(t_0,T)$. We know that $(\psi_n(s),v(s)) \in V^2$ for all $s \in [t_0,T]$ and $V \hookrightarrow L^2(\Omega) \hookrightarrow V'$, then thanks to [6] we have

$$\int_{t_0}^T\int_\Omega \psi_n v dxdt = \int_{t_0}^T \langle\psi_n,v\rangle_{L^2(\Omega),L^2(\Omega)}\,dt = \int_{t_0}^T \langle\psi_n,v\rangle_{V',V}\,dt. \tag{19}$$

Moreover, (ψ_n,v) satisfies the following integration by part formula

$$\begin{aligned}\int_{t_0}^T \langle v_t,\psi_n\rangle\,dt &= \langle\psi_n(T),v(T)\rangle_{L^2(\Omega),L^2(\Omega)} - \langle\psi_n(t_0),v(t_0)\rangle_{L^2(\Omega),L^2(\Omega)}\\ &\quad -\int_{t_0}^T \langle(\psi_n)_t,v\rangle_{L^2(\Omega),L^2(\Omega)}.\end{aligned} \tag{20}$$

Therefore, using (19), (20) and the fact that $v(.,T)=0$, we can rewrite (18) as follows.

$$\int_{t_0}^T (\psi_n)_t\,v dxdt + \int_{t_0}^T |\nabla\psi_n|^{p(x)-2}\nabla\psi_n.\nabla v dxdt = 0. \tag{21}$$

Since $C_c^\infty([t_0,T]\times\Omega)$ is dense in $W_{p(.)}(t_0,T)$, we can choose ψ_n as a test function in (21) to obtain

$$\int_{t_0}^T\int_\Omega \psi_n(\psi_n)_t + \int_{t_0}^T\int_\Omega |\nabla\psi_n|^{p(x)}\,dxdt = 0, \tag{22}$$

which is equivalent to

$$\frac{1}{2}\int_\Omega \psi_n(.,T)^2 + \int_{t_0}^T\int_\Omega |\nabla\psi_n|^{p(x)}\,dxdt = \frac{1}{2}\int_\Omega \varphi_n^2 dx. \tag{23}$$

So,

$$\int_{t_0}^T\int_\Omega |\nabla\psi_n|^{p(x)}\,dxdt \le \frac{1}{2}\int_\Omega \varphi_n^2 dx. \tag{24}$$

Therefore, using Proposition 2.1 we obtain

$$\begin{aligned}\|\nabla\psi\|_{\left(L^{p(.)}((t_0,T)\times\Omega)\right)} &\le \max\left\{\left(\int_{t_0}^T\!\!\int_\Omega |\nabla\psi_n|^{p(x)}\,dxdt\right)^{\frac{1}{p_-}},\left(\int_{t_0}^T\!\!\int_\Omega |\nabla\psi_n|^{p(x)}\,dxdt\right)^{\frac{1}{p_+}}\right\}\\ &\le \max\left\{\left(\frac{1}{2}\int_\Omega \varphi_n^2 dx\right)^{\frac{1}{p_-}},\left(\frac{1}{2}\int_\Omega \varphi_n^2 dx\right)^{\frac{1}{p_+}}\right\}\end{aligned} \tag{25}$$

and

$$\begin{aligned}\int_{t_0}^{T}\|\nabla\psi\|_{p(.)}^{p_-}\,dt &\leq \int_{t_0}^{T}\max\left\{\int_\Omega|\nabla\psi_n|^{p(x)}\,dx,\left(\int_\Omega|\nabla\psi_n|^{p(x)}\,dx\right)^{\frac{p_-}{p_+}}\right\}dt\\ &\leq \int_{t_0}^{T}\int_\Omega|\nabla\psi_n|^{p(x)}\,dxdt+T^{1-\frac{p_-}{p_+}}\left(\int_{t_0}^{T}\int_\Omega|\nabla\psi_n|^{p(x)}\,dxdt\right)^{\frac{p_-}{p_+}} \quad (26)\end{aligned}$$

then it follows that

$$\int_{t_0}^{T}\|\psi_n\|_{W_0^{1,p(.)}(\Omega)}^{p_-}\,dt\leq\frac{1}{2}\int_\Omega\varphi_n^2dx+T^{1-\frac{p_-}{p_+}}\left(\frac{1}{2}\int_\Omega\varphi_n^2dx\right)^{\frac{p_-}{p_+}}. \tag{27}$$

Hence,

$$\|\psi_n\|_{L^{p_-}\left(t_0,T;W_0^{1,p(.)}(\Omega)\right)}\leq\left(\frac{1}{2}\int_\Omega\varphi_n^2dx+T^{1-\frac{p_-}{p_+}}\left(\frac{1}{2}\int_\Omega\varphi_n^2dx\right)^{\frac{p_-}{p_+}}\right)^{\frac{1}{p_-}}. \tag{28}$$

In (21), we take $v=\psi_n\chi_{(t_0,t)}$ as a test function, where $\chi_{(t_0,t)}$ is defined as the characteristic function of (t_0,t), $t\in[t_0,T]$ then, using the integration by part formula, we get

$$\frac{1}{2}\int_\Omega\psi_n(.,t)^2\,dx+\int_{t_0}^{T}\int_\Omega|\nabla\psi_n|^{p(x)}\,dxdt=\frac{1}{2}\int_\Omega\varphi_n^2dx, \tag{29}$$

which implies that

$$\frac{1}{2}\int_\Omega\psi_n(.,t)^2\,dx\leq\frac{1}{2}\int_\Omega\varphi_n^2dx, \tag{30}$$

Consequently,

$$\|\psi_n\|_{L^\infty((t_0,t);L^2(\Omega))}\leq\left(\int_\Omega\varphi_n^2dx\right)^{\frac{1}{2}}. \tag{31}$$

Let $v\in L^{p_-}(0,T;V)$ such that $\|v\|_{L^{p_-}(0,T;V)}\leq 1$, for every $k\geq 1$,
$A_k=\{t\in[0,T]:\|v\|_V\geq k\}$ and $\mathcal{A}=\bigcup\limits_{k\geq 1}A_k$.

We have

$$\begin{aligned}\operatorname{meas}(\mathcal{A}) &= \frac{1}{k}\int_{\mathcal{A}}kdt\leq\frac{1}{k}\int_{\mathcal{A}}\|v\|_V\,dt\leq\frac{1}{k}\int_{\mathcal{A}}\|v\|_V^{p_-}\,dt\\ &\leq \frac{1}{k}\int_0^T\|v\|_V^{p_-}\,dt\leq\frac{1}{k}\|v\|_{L^{p_-}(0,T;V)}^{p_-}\leq\frac{1}{k}. \quad (32)\end{aligned}$$

Hence, we deduce by letting $k\to\infty$ that $\operatorname{meas}(\mathcal{A})=0$.
We use (22) and the Hölder type inequality to get

$$\begin{aligned}\left|\langle(\psi_n)_t,v\rangle_{L^{(p_-)'}(t_0,T;V'),L^{p_-}(t_0,T;V)}\right| &= \left|\int_{t_0}^{T}\langle(\psi_n)_t,v\rangle_{V',V}\,dt\right|\\ =\left|\int_{t_0}^{T}\int_\Omega(\psi_n)_t\,vdxdt\right| &\leq \int_{t_0}^{T}\int_\Omega\left||\nabla\psi_n|^{p(x)-2}\nabla\psi_n.\nabla vdxdt\right|\\ \leq 2\int_{t_0}^{T}\int_\Omega\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|\nabla v\|_{p(.)} &\leq \int_{t_0}^{T}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V. \quad (33)\end{aligned}$$

Since meas $(\mathcal{A})=0$, we deduce that

$$\begin{aligned}
&\int_{t_0}^{T}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V\,dt\\
&\quad= \int_{A_1}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V\,dt+\int_{[[t_0,T]\setminus A_1]}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V\,dt\\
&\quad\le \int_{\mathcal{A}}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V\,dt+\int_{[[t_0,T]\setminus A_1]}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V\,dt\\
&\quad\le \int_{[[t_0,T]\setminus A_1]}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V\,dt, \qquad (34)
\end{aligned}$$

which implies that

$$\begin{aligned}
\left|\langle(\psi_n)_t\,,v\rangle_{L^{(p_-)'}(t_0,T;V'),L^{p_-}(t_0,T;V)}\right| &\le \int_{[[t_0,T]\setminus A_1]}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V\,dt\\
&\le \int_{[[t_0,T]\setminus A_1]}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}dt \le \int_{t_0}^{T}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}dt \qquad (35)\\
&\le (T-t_0)^{1-\frac{1}{(p')_-}}\left(\int_{t_0}^{T}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}^{(p')_-}dt\right)^{\frac{1}{(p')_-}}.
\end{aligned}$$

Hence, we get

$$\|(\psi_n)_t\|_{L^{(p_-)'}(t_0,T;V')} \le T^{1-\frac{1}{(p')_-}}\left(\int_{t_0}^{T}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}^{(p')_-}dt\right)^{\frac{1}{(p')_-}}. \qquad (36)$$

Consequently, we use Proposition 2.1 to get

$$\begin{aligned}
&\int_{t_0}^{T}\left\||\nabla\psi|^{p(.)-1}\right\|_{p'(.)}^{(p')_-}dt \le \int_{t_0}^{T}\max\left\{\int_\Omega|\nabla\psi_n|^{p(x)}dx,\left(\int_\Omega|\nabla\psi_n|^{p(x)}dx\right)^{\frac{(p')_-}{(p')_+}}\right\}dt\\
&\le \int_{t_0}^{T}\int_\Omega|\nabla\psi_n|^{p(x)}dxdt+(T-t_0)^{1-\frac{(p')_-}{(p')_+}}\left(\int_{t_0}^{T}\int_\Omega|\nabla\psi_n|^{p(x)}dxdt\right)^{\frac{(p')_-}{(p')_+}}. \qquad (37)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\|(\psi_n)_t\|_{L^{(p_-)'}(t_0,T;V')}\\
&\le T^{1-\frac{1}{(p')_-}}\left(\int_{t_0}^{T}\!\!\int_\Omega|\nabla\psi_n|^{p(x)}dxdt+T^{1-\frac{(p')_-}{(p')_+}}\left(\int_{t_0}^{T}\!\!\int_\Omega|\nabla\psi_n|^{p(x)}dxdt\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}}\\
&\le T^{1-\frac{1}{(p')_-}}\left(\frac{1}{2}\int_\Omega\varphi_n^2dx+T^{1-\frac{(p')_-}{(p')_+}}\left(\frac{1}{2}\int_\Omega\varphi_n^2dx\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}}. \qquad (38)
\end{aligned}$$

Finally, combining (25), (26), (31) and (38), we conclude that

$$\begin{aligned}
&\|\nabla\psi_n\|_{L^{p(\cdot)}((t_0,T)\times\Omega)}+\|\psi_n\|_{L^{p_-}(t_0,T;W_0^{1,p(\cdot)}(\Omega))}+\|\psi_n\|_{L^\infty(t_0,t;L^2(\Omega))}+\|(\psi_n)_t\|_{L^{(p_-)'}(t_0,T;V')}\\
&\quad\le\left(\frac12\int_\Omega\varphi_n^2dx\right)^{\frac{1}{p_-}}+\left(\frac12\int_\Omega\varphi_n^2dx\right)^{\frac{1}{p_+}}+\left(\frac12\int_\Omega\varphi_n^2dx+T^{1-\frac{p_-}{p_+}}\left(\frac12\int_\Omega\varphi_n^2dx\right)^{\frac{p_-}{p_+}}\right)^{\frac{1}{p_-}}\\
&\quad+\left(\int_\Omega\varphi_n^2dx\right)^{\frac12}+T^{1-\frac{1}{(p')_-}}\left(\frac12\int_\Omega\varphi_n^2dx+T^{1-\frac{(p')_-}{(p')_+}}\left(\frac12\int_\Omega\varphi_n^2dx\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}}
\end{aligned}\qquad(39)$$

Let us now construct a function $\tilde{\psi_n}$ defined on $[0,T]$ by setting

$$\begin{cases}\tilde{\psi_n}=\psi_n & \text{in}\quad]t_0,T]\times\Omega\\ \tilde{\psi_n}=\psi_n\left(T-\dfrac{t\,(T-t_0)}{t_0}\right) & \text{in}\quad [0,t_0]\times\Omega.\end{cases}$$

By (39), we have

$$\begin{aligned}
&\left\|\nabla\tilde{\psi_n}\right\|_{(L^{p(\cdot)}((t_0,T)\times\Omega)}+\left\|\tilde{\psi_n}\right\|_{L^{p_-}(t_0,T;W_0^{1,p(\cdot)}(\Omega))}+\left\|\tilde{\psi_n}\right\|_{L^\infty(t_0,t;L^2(\Omega))}+\left\|\left(\tilde{\psi_n}\right)_t\right\|_{L^{(p_-)'}(t_0,T;V')}\\
&\le\left(\frac12\|\varphi_n\|^2_{L^2(\Omega)}\right)^{\frac{1}{p_-}}+\left(\|\varphi_n\|^2_{L^2(\Omega)}\right)^{\frac{1}{p_+}}+\left(\frac12\|\varphi_n\|^2_{L^2(\Omega)}+T^{1-\frac{p_-}{p_+}}\left(\frac12\|\varphi_n\|^2_{L^2(\Omega)}\right)^{\frac{p_-}{p_+}}\right)^{\frac{1}{p_-}}\\
&\quad+\|\varphi_n\|_{L^2(\Omega)}+T^{1-\frac{1}{(p')_-}}\left(\frac12\|\varphi_n\|^2_{L^2(\Omega)}+T^{1-\frac{(p')_-}{(p')_+}}\left(\frac12\|\varphi_n\|^2_{L^2(\Omega)}\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}}.
\end{aligned}\qquad(40)$$

Since $\varphi_n\in C_c(A_\varepsilon)$ and $0\le\varphi_n\le1$, we deduce that $\|\varphi_n\|^2_{L^2(\Omega)}\le meas\,(A_\varepsilon)\le\varepsilon$, then, it follows that

$$\begin{aligned}
&\left\|\nabla\tilde{\psi_n}\right\|_{\left(L^{p(\cdot)}((t_0,T)\times\Omega)\right)}+\left\|\tilde{\psi_n}\right\|_{L^{p_-}\left(t_0,T;W_0^{1,p(\cdot)}(\Omega)\right)}+\left\|\tilde{\psi_n}\right\|_{L^\infty(t_0,t;L^2(\Omega))}\\
&\quad+\left\|\left(\tilde{\psi_n}\right)_t\right\|_{L^{(p_-)'}(t_0,T;V')}\le\left(\frac12\varepsilon\right)^{\frac{1}{p_-}}+(\varepsilon)^{\frac{1}{p_+}}+\left(\frac12\varepsilon+T^{1-\frac{p_-}{p_+}}\left(\frac12\varepsilon\right)^{\frac{p_-}{p_+}}\right)^{\frac{1}{p_-}}+\varepsilon^{\frac12}\\
&\qquad+T^{1-\frac{1}{(p')_-}}\left(\frac12\varepsilon+T^{1-\frac{(p')_-}{(p')_+}}\left(\frac12\varepsilon\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}}.
\end{aligned}\qquad(41)$$

The fact that ψ_n belongs to $C\left([t_0,T],L^2(\Omega)\right)$, implies that $\psi_n\in C\left([t_0,T]\times\Omega\right)$, then it follows that $\tilde{\psi_n}\in C\left([t_0,T]\times\Omega\right)$. Therefore, the set $U_n=\left\{\tilde{\psi_n}>\frac12\right\}$ is open.

Since U_n is open and $2\tilde{\psi}_n > \chi_{U_n}$, we have

$$\begin{aligned} Cap_{p(.)}\left(U_n\right) &\leq 2\left\|\psi_n\right\|_{W_{p(.)}(0,T)} \qquad (42)\\ &\leq \left(\frac{1}{2}\varepsilon\right)^{\frac{1}{p_-}} + (\varepsilon)^{\frac{1}{p_+}} + \left(\frac{1}{2}\varepsilon + T^{1-\frac{p_-}{p_+}}\left(\frac{1}{2}\varepsilon\right)^{\frac{p_-}{p_+}}\right)^{\frac{1}{p_-}} + \varepsilon^{\frac{1}{2}}\\ &\quad + T^{1-\frac{1}{(p')_-}}\left(\frac{1}{2}\varepsilon + T^{1-\frac{(p')_-}{(p')_+}}\left(\frac{1}{2}\varepsilon\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}}. \qquad (43) \end{aligned}$$

Since the sequence φ_n is nondecreasing, we have that the sequence $\tilde{\psi}_n$ is nondecreasing as well, hence $U_n \subset U_{n+1}$, $Cap_{p(.)}\left(U_n\right)$ is also a nondecreasing sequence and bounded too. Let's show that

$$Cap_{p(.)}\left(U_\infty\right) = \lim_{n\to\infty} Cap_{p(.)}\left(U_n\right), \qquad (44)$$

where $U_\infty = \bigcup_{n=1}^{\infty} U_n$.

In fact, we have $U_n \subset U_\infty$, then

$$\lim_{n\to\infty} Cap_{p(.)}\left(U_n\right) \leq Cap_{p(.)}\left(U_\infty\right). \qquad (45)$$

Now, we take $\left(u_n\right)_{n\in\mathbb{N}} \subset W_{p(.)}\left(0,T\right)$ such that

$$u_n \geq \chi_{U_n} \text{ a.e. in } Q \text{ and } \left\|u_n\right\|_{W_{p(.)}(0,T)} \leq Cap_{p(.)}\left(U_n\right) + \frac{1}{n}.$$

Thanks to (42), $\left(u_n\right)_{n\in\mathbb{N}}$ is bounded in $W_{p(.)}\left(0,T\right)$, then we can extract a subsequence still denoted by $\left(u_n\right)_{n\in\mathbb{N}}$ such that $u_n \to u$ weakly in $W_{p(.)}\left(0,T\right)$ and a.e. in Q. Since U_n is nondecreasing and $\left(u_n\right)_{n\in\mathbb{N}}$ converges almost everywhere to u, we deduce that $u \geq \chi_{U_\infty}$ a.e. in Q, hence it follows that

$$Cap_{p(.)}\left(U_\infty\right) \leq \left\|u\right\|_{W_{p(.)}(0,T)} \leq \liminf_{n\to\infty} \left\|u_n\right\|_{W_{p(.)}(0,T)} \leq \lim_{n\to\infty} Cap_{p(.)}\left(U_n\right). \qquad (46)$$

Combining (45) and (46), we obtain (44).

Since $\varphi_n = 1$ on K_n for each n and $\{t_0\} \times A_\varepsilon \supset \{t_0\} \times B$ then, we conclude from (44) and (45) that

$$\begin{aligned} Cap_{p(.)}\left(\{t_0\} \times B\right) &\leq Cap_{p(.)}\left(U_\infty\right) = \lim_{n\infty} Cap_{p(.)}\left(U_n\right)\\ &\leq \left(\frac{1}{2}\varepsilon\right)^{\frac{1}{p_-}} + (\varepsilon)^{\frac{1}{p_+}} + \left(\frac{1}{2}\varepsilon + T^{1-\frac{p_-}{p_+}}\left(\frac{1}{2}\varepsilon\right)^{\frac{p_-}{p_+}}\right)^{\frac{1}{p_-}} + \varepsilon^{\frac{1}{2}}\\ &\quad + T^{1-\frac{1}{(p')_-}}\left(\frac{1}{2}\varepsilon + T^{1-\frac{(p')_-}{(p')_+}}\left(\frac{1}{2}\varepsilon\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}}. \qquad (47) \end{aligned}$$

Hence, letting $\varepsilon \to 0$ in (47), we deduce that $Cap_{p(.)}\left(\{t_0\} \times B\right) = 0$ □

3.2. Quasicontinuous function.

Definition 3.4. A claim is said to hold $Cap_{p(.)}$-quasi everywhere if it holds everywhere except on a set of zero $p(.)$-capacity. A function $u : Q \to \mathbb{R}$ is said to be $Cap_{p(.)}$-quasi continuous if for every $\varepsilon > 0$, there exists an open set U_ε with $Cap_{p(.)}(U_\varepsilon) < \varepsilon$ such that u restricted to $Q \setminus U_\varepsilon$ is continuous.

In this section, we prove that every element of $W_{p(.)}(0,T)$ admits cap-quasi continuous representative. We recall that the approach developed in elliptic case (see [4]) cannot extend in our situation since if $u \in W_{p(.)}(0,T)$, one may have $|u| \notin W_{p(.)}(0,T)$ (see [6]).

Lemma 3.7. *(i) Let u belongs to $W_{p(.)}(0,T)$; then there exists a function z in $\widetilde{W}_{p(.)}(0,T)$ such that $|u| < z$ and*

$$\|z\|_{\widetilde{W}_{p(.)}(0,T)} \leq C\left([u]_*^{\frac{1}{2}} + [u]_*^{\frac{1}{p_-}} + [u]_*^{\frac{1}{p_+}} + [u]_*^{\frac{1}{(p')_-}} + [u]_*^{\frac{1}{(p')_+}}\right), \tag{48}$$

where

$$\begin{aligned} [u]_* &= \rho_{p(.)}(|\nabla u|) + \|u_t\|^2_{L^{(p_-)'}(0,T;V')} + \|u\|^2_{L^\infty(0,t;L^2(\Omega))} + \|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')} \\ &\quad + \|u_t\|_{L^{(p_-)'}(0,T;V')} + \|u_t\|_{L^{(p_-)'}(0,T;V')}\|u\|_{L^\infty(0,t;L^2(\Omega))}. \end{aligned} \tag{49}$$

(ii) If u belongs to $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \cap L^\infty(Q)$ and u_t in $L^{(p_-)'}(0,T;W^{-1,p'(\cdot)}(\Omega)) + L^1(Q)$, then there exists $z \in \widetilde{W}_{p(.)}(0,T)$ such that $|u| < z$ and

$$[z] \leq C\left([u]_{**} + [u]_{**}^{\frac{1}{p_-}} + [u]_{**}^{\frac{1}{p_+}} + [u]_{**}^{\frac{1}{(p')_-}} + [u]_{**}^{\frac{1}{(p')_+}}\right), \tag{50}$$

where

$$\begin{aligned} [u]_{**} &= \rho_{p(.)}(|\nabla u|) + \|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_t\|^{(p_-)'}_{L^{(p_-)'}\left(0,T;W^{-1,p'(\cdot)}(\Omega)\right)+L^1(Q)} \\ &\quad + \|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(\cdot)}(\Omega)\right)+L^1(Q)} + \|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(\cdot)}(\Omega)\right)+L^1(Q)}\|u\|_{L^\infty(Q)} \end{aligned} \tag{51}$$

and

$$[z] = \|z\|^{p_-}_{L^{p_-}\left(0,T;W_0^{1,p(\cdot)}(\Omega)\right)} + \|z_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')} + \|z\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla z\|_{p(.)}. \tag{52}$$

Proof. We divide the proof in two steps.
Step 1. Let us consider the penalized problem

$$\begin{cases} (u_\varepsilon)_t - \Delta_{p(.)}u_\varepsilon = \dfrac{1}{\varepsilon}(u_\varepsilon - u)^- & \text{in } (0,T)\times\Omega \\ u_\varepsilon(0) = u^+(0) & \text{on } \Omega \\ u_\varepsilon = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{53}$$

According to [11], we can prove that this problem admits a nonnegative solution u_ε in $C([0,T];L^2(\Omega)) \cap L^{p_-}\left(0,T;W_0^{1,p(\cdot)}(\Omega)\right)$.

Taking $u_\varepsilon - u$ as a test function in (53) then, for every t in $[0,T]$ we have

$$\begin{aligned}\int_0^t \langle (u_\varepsilon - u)_t, u_\varepsilon - u\rangle\, ds + \int_0^t\int_\Omega \left|\nabla u_\varepsilon\right|^{p(x)} dxds &= \frac{1}{\varepsilon}\int_0^t\int_\Omega (u_\varepsilon - u)^-\,(u_\varepsilon - u)\, dxds \\ &+\int_0^t\int_\Omega \left|\nabla u_\varepsilon\right|^{p(x)-2}\nabla u_\varepsilon.\nabla u dxds - \int_0^t \langle u_t, u_\varepsilon - u\rangle\, dxds.\end{aligned}$$

By integration by parts formula and the fact that $(u_\varepsilon - u)^-\,(u_\varepsilon - u) \leq 0$, we deduce that

$$\begin{aligned}\frac{1}{2}\int_\Omega \left|u_\varepsilon - u\right|^2(t)\, dx + \int_0^t\int_\Omega \left|\nabla u_\varepsilon\right|^{p(x)} dxds &\leq \int_0^t\int_\Omega \left|\nabla u_\varepsilon\right|^{p(x)-2}\nabla u_\varepsilon.\nabla u dxds \\ &+\frac{1}{2}\int_\Omega \left|u_\varepsilon(0) - u(0)\right|^2 dx - \int_0^t \langle u_t, u_\varepsilon - u\rangle\, dxds;\end{aligned}$$

which implies that

$$\begin{aligned}\frac{1}{2}\int_\Omega \left|u_\varepsilon\right|^2(t)\, dx + \int_0^t\int_\Omega \left|\nabla u_\varepsilon\right|^{p(x)} dxds &\leq \int_0^t\int_\Omega \left|\nabla u_\varepsilon\right|^{p(x)-1}\left|\nabla u\right| dxds \\ &+\frac{1}{2}\int_\Omega \left|u(0)\right|^2 dx + \int_\Omega \left|u_\varepsilon(t)\right|\left|u(t)\right| dx - \int_0^t \langle u_t, u_\varepsilon - u\rangle\, ds.\end{aligned}$$

Now, we use the Young inequality to obtain

$$\int_0^t\int_\Omega \left|\nabla u_\varepsilon\right|^{p(x)-1}\left|\nabla u\right| dxdt \leq 2^{p^+}\int_Q \left|\nabla u\right|^{p(x)} dxdt + \frac{1}{2}\int_0^t\int_\Omega \left|\nabla u_\varepsilon\right|^{p(x)} dxds$$

and

$$\int_\Omega \left|u_\varepsilon(t)\right|\left|u(t)\right| dx \leq \frac{1}{4}\int_\Omega \left|u_\varepsilon(t)\right|^2 dx + 2\int_\Omega \left|u(t)\right|^2 dx.$$

Thus,

$$\begin{aligned}\frac{1}{4}\int_\Omega \left|u_\varepsilon\right|^2(t)\, dx + \frac{1}{2}\int_0^t\int_\Omega \left|\nabla u_\varepsilon\right|^{p(x)} dxds &\leq 2^{p_+}\int_Q \left|\nabla u\right|^{p(x)} dxds \\ &+\frac{5}{2}\left\|u\right\|^2_{L^\infty(0,T;L^2(\Omega))} - \int_0^t \langle u_t, u_\varepsilon - u\rangle\, ds.\end{aligned} \tag{54}$$

If we are in case i), u is in $W_{p(.)}\,(0,T)$ and we have

$$\begin{aligned}\left|\int_0^t \langle u_t, u_\varepsilon - u\rangle\, dt\right| &\leq \int_0^t \left\|u_t\right\|_{V'}\left\|u_\varepsilon - u\right\|_V dt \\ &\leq \int_0^t \left\|u_t\right\|_{V'}\left\|u_\varepsilon - u\right\|_{W_0^{1,p(.)}(\Omega)} dt + \int_0^t \left\|u_t\right\|_{V'}\left\|u_\varepsilon - u\right\|_{L^2(\Omega)} dt \\ &\leq \left\|u_t\right\|_{L^{(p_-)'}(0,t;V')}\left\|u_\varepsilon - u\right\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} + \left\|u_t\right\|_{L^1(0,t;V')}\left\|u_\varepsilon - u\right\|_{L^\infty(0,t;L^2(\Omega))} \\ &\leq \left\|u_t\right\|_{L^{(p_-)'}(0,T;V')}\left\|u_\varepsilon - u\right\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} + C\left\|u_t\right\|_{L^{(p_-)'}(0,T;V')}\left\|u\right\|_{L^\infty(0,t;L^2(\Omega))}.\end{aligned} \tag{55}$$

Thanks to Proposition 2.1 and Hölder inequality, we have

$$
\begin{aligned}
&\|u_\varepsilon - u\|^{p_-}_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} \\
&\quad \le \int_0^t \max\left\{\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dx, \left(\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dx\right)^{\frac{p_-}{p_+}}\right\} ds \qquad (56)\\
&\quad \le \int_0^t\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dxds + t^{1-(p_-/p_+)}\left(\int_0^t\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dxds\right)^{\frac{p_-}{p_+}}.
\end{aligned}
$$

Hence, if $\int_0^t\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dxds > 1$, we deduce that

$$
\begin{aligned}
&\|u_\varepsilon - u\|^{p_-}_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} \\
&\quad \le \int_0^t\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dxds + T^{1-(p_-/p_+)}\int_0^t\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dxds \\
&\quad \le \left(1+T^{1-(p_-/p_+)}\right)\int_0^t\int_\Omega (|\nabla u_\varepsilon| + |\nabla u|)^{p(x)}\,dxds \\
&\quad \le \left(1+T^{1-(p_-/p_+)}\right)\int_0^t\int_\Omega 2^{p_+-1}\left(|\nabla u_\varepsilon|^{p(x)} + |\nabla u|^{p(x)}\right) dxds \\
&\quad \le \left(1+T^{1-(p_-/p_+)}\right) 2^{p_+}\left(\int_0^t\int_\Omega |\nabla u_\varepsilon|^{p(x)}\,dxds + \int_0^t\int_\Omega |\nabla u|^{p(x)}\,dxds\right). \qquad (57)
\end{aligned}
$$

Since from the Young inequality, we have

$$
\begin{aligned}
&\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u_\varepsilon - u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} \qquad (58)\\
&= 2^{\frac{p_++2}{p_-}}\left(1+T^{1-(p_-/p_+)}\right)^{\frac{1}{p_-}}\|u_t\|_{L^{(p_-)'}(0,T;V')}\frac{\|u_\varepsilon - u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)}}{2^{\frac{p_++2}{p_-}}\left(1+T^{1-(p_-/p_+)}\right)^{\frac{1}{p_-}}} \\
&\le 2^{\frac{(p_-)'(p_++2)}{p_-}}\left(1+T^{1-(p_-/p_+)}\right)^{\frac{(p_-)'}{p_-}}\|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')} + \frac{\|u_\varepsilon - u\|^{p_-}_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)}}{2^{p_++2}\left(1+T^{1-(p_-/p_+)}\right)}.
\end{aligned}
$$

Then, if $\int_0^t\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dxds > 1$, by (57) and (58) we deduce that

$$
\begin{aligned}
&\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u_\varepsilon - u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} \qquad (59)\\
&\quad \le 2^{\frac{(p_-)'(p_++2)}{p_-}}\left(1+T^{1-(p_-/p_+)}\right)^{\frac{(p_-)'}{p_-}}\|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')} \\
&\quad +\frac{1}{4}\int_0^t\int_\Omega |\nabla u_\varepsilon|^{p(x)}\,dxds + \frac{1}{4}\int_Q |\nabla u_\varepsilon|^{p(x)}\,dxds.
\end{aligned}
$$

If $\int_0^t\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dxds \le 1$, from (56) we get

$$
\|u_\varepsilon - u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} \le \left(1+T^{1-(p_-/p_+)}\right)^{\frac{1}{p_-}}; \qquad (60)
$$

which implies that

$$\|u_t\|_{L^{(p_-)'}(0,T;V')} \|u_\varepsilon - u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} \leq \left(1+T^{1-(p_-/p_+)}\right)^{\frac{1}{p_-}} \|u_t\|_{L^{(p_-)'}(0,T;V')} . \tag{61}$$

Therefore, using (59) − (61) , we deduce that

$$\begin{aligned} &\|u_t\|_{L^{(p_-)'}(0,T;V')} \|u_\varepsilon - u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} \\ &\quad\leq 2^{\frac{(p_-)'(p_++2)}{p_-}} \left(1+T^{1-(p_-/p_+)}\right)^{\frac{(p_-)'}{p_-}} \|u_t\|_{L^{(p_-)'}(0,T;V')}^{(p_-)'} \qquad (62)\\ &\quad+\left(1+T^{1-(p_-/p_+)}\right)^{\frac{1}{p_-}} \|u_t\|_{L^{(p_-)'}(0,T;V')} + \frac{1}{4}\int_0^t\int_\Omega |\nabla u_\varepsilon|^{p(x)}\,dxds + \frac{1}{4}\int_Q |\nabla u_\varepsilon|^{p(x)}\,dxds. \end{aligned}$$

Note also that, from the Young inequality, we have

$$\begin{aligned} &C\|u_t\|_{L^{(p_-)'}(0,T;V')} \|u-u_\varepsilon\|_{L^\infty(0,t;L^2(\Omega))} \\ &\quad\leq C\|u_t\|_{L^{(p_-)'}(0,T;V')} \|u\|_{L^\infty(0,t;L^2(\Omega))} + 4C\|u_t\|_{L^{(p_-)'}(0,T;V')} + \frac{1}{4}\|u_\varepsilon\|_{L^\infty(0,t;L^2(\Omega))} \\ &\quad\leq C\|u_t\|_{L^{(p_-)'}(0,T;V')} \|u\|_{L^\infty(0,t;L^2(\Omega))} + 16C^2\|u_t\|^2_{L^{(p_-)'}(0,T;V')} + \frac{1}{16}\|u_\varepsilon\|^2_{L^\infty(0,t;L^2(\Omega))} \\ &\quad\leq C\|u_t\|_{L^{(p_-)'}(0,T;V')} \|u\|_{L^\infty(0,T;L^2(\Omega))} + 16C^2\|u_t\|^2_{L^{(p_-)'}(0,T;V')} + \frac{1}{8}\|u_\varepsilon\|^2_{L^\infty(0,t;L^2(\Omega))} . \end{aligned}$$

Consequently,

$$\begin{aligned} \left|\int_0^t \langle u_t, u_\varepsilon - u\rangle\,dt\right| \leq\; & 2^{\frac{(p_-)'(p_++2)}{p_-}} \left(1+T^{1-(p_-/p_+)'}\right)^{\frac{(p_-)'}{p_-}} \|u_t\|_{L^{(p_-)'}(0,T;V')}^{(p_-)'} \\ &+\left(1+T^{1-(p_-/p_+)'}\right)^{\frac{1}{p_-}} \|u_t\|_{L^{(p_-)'}(0,T;V')} + \frac{1}{4}\int_0^t\int_\Omega |\nabla u_\varepsilon|^{p(x)}\,dxds \\ &+\frac{1}{4}\int_Q |\nabla u_\varepsilon|^{p(x)}\,dxds + C\|u_t\|_{L^{(p_-)'}(0,T;V')} \|u\|_{L^\infty(0,t;L^2(\Omega))} \\ &+16C^2\|u_t\|^2_{L^{(p_-)'}(0,T;V')} + \frac{1}{8}\|u_\varepsilon\|^2_{L^\infty(0,t;L^2(\Omega))} . \end{aligned} \tag{63}$$

Combining(54) and (63), we obtain

$$\begin{aligned} &\frac{1}{4}\int_\Omega |u_\varepsilon|^2(t)\,dx - \frac{1}{8}\|u_\varepsilon\|^2_{L^\infty(0,t;L^2(\Omega))} + \frac{1}{4}\int_0^t\int_\Omega |\nabla u_\varepsilon|^{p(x)}\,dxdt \\ &\quad\leq C\left(\int_Q |\nabla u|^{p(x)}\,dxdt + \|u_t\|^2_{L^{(p_-)'}(0,T;V')} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')}\right. \\ &\qquad\left. + \|u_t\|_{L^{(p_-)'}(0,T;V')} + \|u_t\|_{L^{(p_-)'}(0,T;V')} \|u\|_{L^\infty(0,T;L^2(\Omega))}\right), \end{aligned} \tag{64}$$

which implies that

$$\begin{aligned} &\|u_\varepsilon\|^2_{L^\infty(0,t;L^2(\Omega))} + \|u_\varepsilon\|^{p_-}_{L^{p_-}0,T;W_0^{1,p(.)}(\Omega)} \\ &\quad\leq C\left(\int_Q |\nabla u|^{p(x)}\,dxdt + \|u_t\|^2_{L^{(p_-)'}(0,T;V')} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')}\right. \\ &\qquad\left. + \|u_t\|_{L^{(p_-)'}(0,T;V')} + \|u_t\|_{L^{(p_-)'}(0,T;V')} \|u\|_{L^\infty(0,T;L^2(\Omega))}\right). \end{aligned} \tag{65}$$

Now, we are in the case $ii)$ and we prove an L^{∞} estimate on u_{ε}. Let G_k be defined on $\mathbb{R}$ by $G_k(r) = (r-k)^+$, where $k = \|u\|_{L^{\infty}(\Omega)}$. We take $G_k(u_{\varepsilon}) = (u_{\varepsilon}-k)^+$ as a test function in (53), and using the fact that $G_k' = (G_k')^{p(.)}$, $u_{\varepsilon} \geq 0$, we obtain

$$\int_Q |\nabla G_k(u_{\varepsilon})|^{p(x)}\,dxdt = \int_Q G_k'(u_{\varepsilon})\,|\nabla u_{\varepsilon}|^{p(x)}\,dxdt \leq \int_Q \frac{1}{\varepsilon}(u_{\varepsilon}-u)^- G_k(u_{\varepsilon})\,dxdt$$

and since $(u_{\varepsilon}-u)\,G_k(u_{\varepsilon}) = 0$ for $k = \|u\|_{L^{\infty}(Q)}$, then it follows that

$$\|u_{\varepsilon}\|_{L^{\infty}(Q)} \leq \|u\|_{L^{\infty}(Q)}.$$

Thus, writing $u_t = u_t^1 + u_t^2$, with $u_t^1 \in L^{(p')_-}\left(0,T;W^{-1,p'(.)}(\Omega)\right)$ and $u_t^2 \in L^1(Q)$ such that $\|u_t^1\|_{L^{(p')_-}\left(0,T;W^{-1,p'(.)}(\Omega)\right)} + \|u_t^2\|_{L^1(Q)} \leq 2\,\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}$, one has

$$\begin{aligned}
\left|\int_0^t \langle u_t, u_{\varepsilon}-u\rangle\,ds\right| &\leq \int_0^t \|u_t^1\|_{W^{-1,p'(.)}(\Omega)}\|u-u_{\varepsilon}\|_{W_0^{1,p(.)}(\Omega)}dt + \|u_t^2\|_{L^1(Q)}\|u-u_{\varepsilon}\|_{L^{\infty}(Q)}\\
&\leq \|u_t^1\|_{L^{(p')_-}\left(0,t;W^{-1,p'(.)}(\Omega)\right)}\|u-u_{\varepsilon}\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} + 2\,\|u_t^2\|_{L^1(Q)}\,\|u\|_{L^{\infty}(Q)}\\
&\leq \|u_t^1\|_{L^{(p')_-}\left(0,T;W^{-1,p'(.)}(\Omega)\right)}\|u-u_{\varepsilon}\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} + 2\,\|u_t^2\|_{L^1(Q)}\,\|u\|_{L^{\infty}(Q)}\\
&\leq 2\,\|u_t\|_{L^{(p')_-}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}\|u-u_{\varepsilon}\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)}\\
&\quad +4\,\|u_t\|_{L^{(p')_-}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}\|u\|_{L^{\infty}(Q)}. \qquad (66)
\end{aligned}$$

From (62), we get

$$\begin{aligned}
&2\,\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}\|u_{\varepsilon}-u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)}\\
&\quad\leq 2^{\frac{(p_-)'(p_++2)}{p_-}}\left(1+T^{1-(p_-/p_+)}\right)^{\frac{(p_-)'}{p_-}}\|2u_t\|^{(p_-)'}_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}\\
&\qquad+\left(1+T^{1-(p_-/p_+)}\right)^{\frac{1}{p_-}}\|2u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}\\
&\qquad+\frac{1}{4}\int_0^t\int_{\Omega}|\nabla u_{\varepsilon}|^{p(x)}\,dxds + \frac{1}{4}\int_Q|\nabla u_{\varepsilon}|^{p(x)}\,dxds. \qquad (67)
\end{aligned}$$

Therefore, using (54) and (66)-(67), we obtain

$$\begin{aligned}
&\frac{1}{4}\int_{\Omega}|u_{\varepsilon}|^2(t)\,dx + \frac{1}{4}\int_0^t\int_{\Omega}|\nabla u_{\varepsilon}|^{p(x)}\,dxdt \qquad (68)\\
&\leq C\left(\int_Q|\nabla u|^{p(x)}\,dxdt + \|u\|^2_{L^{\infty}(0,T;L^2(\Omega))} + \|u_t\|^{(p_-)'}_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}\right.\\
&\left.\quad+\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)} + \|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}\|u\|_{L^{\infty}(Q)}\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|u_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))}+\|u_\varepsilon\|^{p_-}_{L^{p_-}\left(0,T;W_0^{1,p(\cdot)}(\Omega)\right)} \qquad (69)\\
&\quad\le C\left(\int_Q|\nabla u|^{p(x)}\,dxdt+\|u\|^2_{L^\infty(0,T;L^2(\Omega))}+\|u_t\|^{(p_-)'}_{L^{(p_-)'}\left(0,T;W^{-1,p'(\cdot)}(\Omega)\right)+L^1(Q)}\right.\\
&\quad\left.+\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(\cdot)}(\Omega)\right)+L^1(Q)}+\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(\cdot)}(\Omega)\right)+L^1(Q)}\|u\|_{L^\infty(Q)}\right).
\end{aligned}$$

Using estimates (65) or (68), we deduce that the sequence (u_ε) is bounded in $L^\infty(0,T;L^2(\Omega))$ and in $L^{p_-}\left(0,T;W_0^{1,p(\cdot)}(\Omega)\right)$. This implies the existence of a subsequence of (u_ε) converging to an element w weakly in $L^{p_-}\left(0,T;W_0^{1,p(\cdot)}(\Omega)\right)$ and weakly$-*$ in $L^\infty\left(0,T;L^2(\Omega)\right)$. As in [6], ones shows that if $\varepsilon<\eta$ then, $u_\varepsilon\ge u_\eta$. Therefore, we conclude that $(u_\varepsilon)_{\varepsilon>0}$ is a nonnegative decreasing bounded sequence in $L^1(Q)$. Consequently, from the monotone convergence theorem, u_ε converges to w in $L^1(Q)$ and almost everywhere in Q.
Taking $(u_\varepsilon-u)^-$ as a test function in (53), we obtain

$$\int_0^T\left\langle(u_\varepsilon)_t,(u_\varepsilon-u)^-\right\rangle dt+\int_0^T|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon.\nabla(u_\varepsilon-u)^-dxdt=\frac{1}{\varepsilon}\int_Q\left|(u_\varepsilon-u)^-\right|^2dxdt,$$

which implies that

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_Q\left|(u_\varepsilon-u)^-\right|^2dxdt+\frac12\int_Q\left|(u_\varepsilon-u)^-\right|^2(T)\,dx\\
&\quad=\int_0^T\left\langle(u_\varepsilon)_t,(u_\varepsilon-u)^-\right\rangle dt+\int_0^T|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon.\nabla(u_\varepsilon-u)^-\,dxdt.
\end{aligned}$$

Hence, by (65) in case $i)$ or (68) and L^∞-estimates in case $ii)$, we deduce that

$$\frac{1}{\varepsilon}\int_Q\left|(u_\varepsilon-u)^-\right|^2dxdt\le M,$$

which implies, by Fatou's lemma that $w\ge u$ and $w\ge u^+$ since $w\ge0$.
Step 2: In this step, one gives some estimates in $\widetilde{W}_{p(.)}(0,T)$. Thanks to [10], there exists a unique variational solution $z^\varepsilon\in L^\infty\left(0,T;L^2(\Omega)\right)\cap L^{p_-}\left(0,T;W_0^{1,p(\cdot)}(\Omega)\right)$ of the problem

$$\begin{cases}-z_t^\varepsilon-\Delta_{p(.)}z^\varepsilon=-2\Delta_{p(.)}u_\varepsilon & \text{in } (0,T)\times\Omega\\ z^\varepsilon(T)=u_\varepsilon(T) & \text{on } \Omega\\ z^\varepsilon=0 & \text{on } (0,T)\times\partial\Omega.\end{cases}\qquad(70)$$

Note that $-2\Delta_{p(.)}u_\varepsilon\ge(u_\varepsilon)_t-\Delta_{p(.)}u_\varepsilon$ in the distributional sense, which implies that $z^\varepsilon\ge u_\varepsilon$.
Taking z^ε as a test function in (70) and integrating between t and T and using the Young inequality, we obtain

$$\int_\Omega(z^\varepsilon(t))^2\,dx+\frac12\int_Q|\nabla z^\varepsilon|^{p(x)}\,dx\le\frac12\int_\Omega|u_\varepsilon|^2\,dx+\frac{4^{(p')_+}}{(p')_-}\int_Q|\nabla u_\varepsilon|^{p(x)}\,dxdt,$$

which implies that

$$\|z^{\varepsilon}\|^{2}_{L^{\infty}(0,T;L^{2}(\Omega))}+\|z^{\varepsilon}\|^{p_-}_{L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)}\leq C\left(\int_Q|\nabla u_{\varepsilon}|^{p(x)}dxdt+\|u_{\varepsilon}\|^{2}_{L^{\infty}(0,T;L^{2}(\Omega))}\right). \tag{71}$$

By Proposition 2.1, we have

$$\|\nabla z^{\varepsilon}\|_{(L^{p(.)}(Q))^{N}}\leq \max\left\{\left(\int_Q|\nabla z^{\varepsilon}|^{p(x)}\,dx\right)^{\frac{1}{p_-}},\left(\int_Q|\nabla z^{\varepsilon}|^{p(x)}\,dx\right)^{\frac{1}{p_+}}\right\}. \tag{72}$$

Hence, using (71) if we are in the case (i) i.e $u\in W_{p(.)}(0,T)$, then we deduce from $(64)-(65)$ the following estimate.

$$\begin{aligned}\|z^{\varepsilon}\|^{2}_{L^{\infty}(0,T;L^{2}(\Omega))}+\|z^{\varepsilon}\|^{p_-}_{L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)}&\leq C\left(\int_Q|\nabla u|^{p(x)}\,dxdt\right.\\&+\|u_t\|^{2}_{L^{(p_-)'}(0,T;V')}+\|u\|^{2}_{L^{\infty}(0,t;L^{2}(\Omega))}+\|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')}+\|u_t\|_{L^{(p_-)'}(0,T;V')}\\&\left.+\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u\|_{L^{\infty}(0,t;L^{2}(\Omega))}\right)\end{aligned} \tag{73}$$

and in the case (ii), we get from (69) the following estimate.

$$\begin{aligned}\|z^{\varepsilon}\|^{2}_{L^{\infty}(0,T;L^{2}(\Omega))}+\|z^{\varepsilon}\|^{p_-}_{L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)}&\leq C\left(\int_Q|\nabla u|^{p(x)}\,dxdt+\|u\|^{2}_{L^{\infty}(0,T;L^{2}(\Omega))}\right.\\&+\|u_t\|^{(p_-)'}_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^{1}(Q)}+\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^{1}(Q)}\\&\left.+\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^{1}(Q)}\|u\|_{L^{\infty}(Q)}\right).\end{aligned} \tag{74}$$

For reasons of simplicity one puts

$$\begin{aligned}[u]_{*}=\int_Q|\nabla u|^{p(x)}\,dxdt&+\|u_t\|^{2}_{L^{(p_-)'}(0,T;V')}+\|u\|^{2}_{L^{\infty}(0,t;L^{2}(\Omega))}+\|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')}\\&+\|u_t\|_{L^{(p_-)'}(0,T;V')}+\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u\|_{L^{\infty}(0,t;L^{2}(\Omega))}\end{aligned} \tag{75}$$

and

$$\begin{aligned}[u]_{**}=\int_Q|\nabla u|^{p(x)}\,dxdt&+\|u\|^{2}_{L^{\infty}(0,T;L^{2}(\Omega))}+\|u_t\|^{(p_-)'}_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^{1}(Q)}\\&+\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^{1}(Q)}+\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^{1}(Q)}\|u\|_{L^{\infty}(Q)}.\end{aligned} \tag{76}$$

We take $v\in L^{p_-}(0,T;V)$ as a test function in (70), to obtain

$$\begin{aligned}\left|\int_0^T\langle (z^{\varepsilon})_t\,,v\rangle\,dt\right|&\leq\left|\int_Q|\nabla z^{\varepsilon}|^{p(x)-2}\nabla z^{\varepsilon}.\nabla v dxdt\right|+\left|2\int_Q|\nabla u_{\varepsilon}|^{p(x-2)}\nabla u_{\varepsilon}.\nabla v dxdt\right|\\&\leq 2\int_0^T\left\||\nabla z^{\varepsilon}|^{p(x)-1}\right\|_{p'(.)}\|\nabla v\|_{p(.)}\,dt+4\int_0^T\left\||\nabla u_{\varepsilon}|^{p(x)-1}\right\|_{p'(.)}\|\nabla v\|_{p(.)}\,dt\\&\leq 4\int_0^T\left(\left\||\nabla z^{\varepsilon}|^{p(x)-1}\right\|_{p'(.)}+\left\||\nabla u_{\varepsilon}|^{p(x)-1}\right\|_{p'(.)}\right)\|v\|_{V}\,dt.\end{aligned} \tag{77}$$

Therefore, by the same method as in the proof of (38), it follows that

$$\begin{aligned}
&\left\|(z^{\varepsilon})_t\right\|_{L^{(p_-)'}(t_0,T;V')}\\
&\leq 4T^{1-\frac{1}{(p')_-}}\left(\int_0^T\int_\Omega|\nabla z^{\varepsilon}|^{p(x)}dxdt+T^{1-\frac{(p')_-}{(p')_+}}\left(\int_0^T\int_\Omega|\nabla z^{\varepsilon}|^{p(x)}dxdt\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}} \qquad (78)\\
&+4T^{1-\frac{1}{(p')_-}}\left(\int_0^T\int_\Omega|\nabla u_{\varepsilon}|^{p(x)}dxdt+T^{1-\frac{(p')_-}{(p')_+}}\left(\int_0^T\int_\Omega|\nabla u_{\varepsilon}|^{p(x)}dxdt\right)^{\frac{(p')_-}{(p)_+}}\right)^{\frac{1}{(p')_-}}.
\end{aligned}$$

We can rewrite (78) as follow.

$$\begin{aligned}
&\left\|(z^{\varepsilon})_t\right\|_{L^{(p_-)'}(t_0,T;V')}\\
&\leq C\left(\left(\int_0^T\int_\Omega|\nabla z^{\varepsilon}|^{p(x)}\,dxdt\right)^{\frac{1}{(p')_-}}+\left(\int_0^T\int_\Omega|\nabla z^{\varepsilon}|^{p(x)}\,dxdt\right)^{\frac{1}{(p')_+}}\right.\\
&\left.+\left(\int_0^T\int_\Omega|\nabla u_{\varepsilon}|^{p(x)}\,dxdt\right)^{\frac{1}{(p')_-}}+\left(\int_0^T\int_\Omega|\nabla u_{\varepsilon}|^{p(x)}\,dxdt\right)^{\frac{1}{(p')_+}}\right). \qquad (79)
\end{aligned}$$

Finally, in the case (i), i.e $u\in W_{p(.)}\left(0,T\right)$, we deduce that

$$\left\|(z^{\varepsilon})_t\right\|_{L^{(p_-)'}(t_0,T;V')}\leq C\left([u]_*^{\frac{1}{(p')_-}}+[u]_*^{\frac{1}{(p')_+}}\right). \qquad (80)$$

Hence combining $(72)-(73)$ and (80), we obtain

$$\left\|z^{\varepsilon}\right\|_{W_{p(.)}(0,T)}\leq C\left([u]_*^{\frac{1}{2}}+[u]_*^{\frac{1}{p_-}}+[u]_*^{\frac{1}{p_+}}+[u]_*^{\frac{1}{(p')_-}}+[u]_*^{\frac{1}{(p')_+}}\right) \qquad (81)$$

and since $\widetilde{W}_{p(.)}\left(0,T\right)\hookrightarrow W_{p(.)}\left(0,T\right)$, then

$$\left\|z^{\varepsilon}\right\|_{\widetilde{W}_{p(.)}(0,T)}\leq C\left([u]_*^{\frac{1}{2}}+[u]_*^{\frac{1}{p_-}}+[u]_*^{\frac{1}{p_+}}+[u]_*^{\frac{1}{(p')_-}}+[u]_*^{\frac{1}{(p')_+}}\right). \qquad (82)$$

For the second case, i.e $u\in L^{p_-}\left(0,T;W_0^{1,p(.)}\left(\Omega\right)\right)\cap L^{\infty}\left(Q\right)$, we have

$$\left\|(z^{\varepsilon})_t\right\|_{L^{(p_-)'}(t_0,T;V')}\leq C\left([u]_{**}^{\frac{1}{(p')_-}}+[u]_{**}^{\frac{1}{(p')_+}}\right). \qquad (83)$$

Then, from (72), (74) and (83), it follows that

$$\begin{aligned}
&[z^{\varepsilon}]=\left\|z^{\varepsilon}\right\|^2_{L^{\infty}(0,T;L^2(\Omega))}+\left\|z^{\varepsilon}\right\|^{p_-}_{L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)}+\left\|(z^{\varepsilon})\right\|_{L^{p_-}(0,T;V')}+\left\|\nabla z^{\varepsilon}\right\|_{L^{p(.)}(Q)}\\
&\leq C\left([u]_{**}+[u]_{**}^{\frac{1}{p_-}}+[u]^{\frac{1}{p_+}}\,[u]_{**}^{\frac{1}{(p')_-}}+[u]^{\frac{1}{(p')_+}}\right). \qquad (84)
\end{aligned}$$

According to (81), z^{ε} is bounded in $\widetilde{W}_{p(.)}\left(0,T\right)$. Hence, there exists a subsequence, still denoted by z^{ε} such that z^{ε} converges weakly to z in $L^{p_-}\left(0,T;W_0^{1,p(.)}\left(\Omega\right)\right)$

and weakly$*$ in $L^\infty\left(0,T;L^2(\Omega)\right)$, ∇z^ε converges weakly to ξ in $\left(L^{p'(.)}(Q)\right)^N$ and z_t^ε converges to $\bar{z}$ in $L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)$. Then, it follows that $z_t=\bar{z}$ and $\xi=\nabla z$. Therefore, $z\in\widetilde{W}_{p(.)}(0,T)$. Hence, from [16], we deduce that z^ε is compact in $L^1(Q)$. Consequently, $z^\varepsilon\to z$ a.e. in Q. Moreover, we have $z^\varepsilon\geq u_\varepsilon$. Then, letting $\varepsilon\to 0$, we get

$$z\geq w\geq u^+ \text{ a.e. in } Q. \tag{85}$$

Therefore, if $u\in W_{p(.)}(0,T)$, we deduce from (81) that

$$\|z^\varepsilon\|_{\widetilde{W}_{p(.)}(0,T)}\leq C\left([u]_*^{\frac{1}{2}}+[u]_*^{\frac{1}{p_-}}+[u]_*^{\frac{1}{p_+}}+[u]_*^{\frac{1}{(p')_-}}+[u]_*^{\frac{1}{(p')_+}}\right) \tag{86}$$

which implies that

$$\|z\|_{W_{p(.)}(0,T)}\leq C\left([u]_*^{\frac{1}{2}}+[u]_*^{\frac{1}{p_-}}+[u]_*^{\frac{1}{p_+}}+[u]_*^{\frac{1}{(p')_-}}+[u]_*^{\frac{1}{(p')_+}}\right) \tag{87}$$

and if $u\in L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)\cap L^\infty(Q)$, $u_t\in L^{p_-}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)$, we deduce from (84) that

$$[z]\leq C\left([u]_{**}+[u]_{**}^{\frac{1}{p_-}}+[u]_{**}^{\frac{1}{p_+}}+[u]_{**}^{\frac{1}{(p')_-}}+[u]_{**}^{\frac{1}{(p')_+}}\right). \tag{88}$$

Since we can obtain a similar result for the negative part u^-, we end the proof of the lemma by writing $u=u^++u^-$ □
As a consequence of the Lemma 3.7 we have the following.

Corollary 3.8. *For all* $u\in W_{p(.)}(0,T)$,

$$[u]_*\leq C\max\left\{\|u\|_{W_{p(.)}(0,T)}^{p_-},\|u\|_{W_{p(.)}(0,T)}^{(p_-)'}\right\}. \tag{89}$$

Moreover, there exists $z\in\widetilde{W}_{p(.)}(0,T)$ *such that* $|u|\leq z$ *and*

$$\|z\|_{\widetilde{W}_{p(.)}(0,T)}\leq C\max\left\{\|u\|_{W_{p(.)}(0,T)}^{\frac{p_-}{(p')_-}},\|u\|_{W_{p(.)}(0,T)}^{\frac{(p_-)'}{p_-}}\right\}. \tag{90}$$

Proof. Let's recall that

$$\begin{aligned}[u]_*=&\rho_{p(.)}(\nabla u)+\|u_t\|_{L^{(p_-)'}(0,T;V')}^2+\|u\|_{L^\infty(0,t;L^2(\Omega))}^2+\|u_t\|_{L^{(p_-)'}(0,T;V')}^{(p_-)'}\\&+\|u_t\|_{L^{(p_-)'}(0,T;V')}+\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u\|_{L^\infty(0,t;L^2(\Omega))}.\end{aligned}$$

We have

$$\|u_t\|_{L^{(p_-)'}(0,T;V')}\leq\|u\|_{W_{p(.)}(0,T)} \tag{91}$$

and by Proposition 2.1, we deduce that

$$\begin{aligned}\rho_{p(.)}(\nabla u)&\leq\max\left\{\|\nabla u\|_{L^{p(.)}(Q)}^{p_-},\|\nabla u\|_{L^{p(.)}(Q)}^{p_+}\right\}\\&\leq\|u\|_{W_{p(.)}(0,T)}^{p_-}+\|u\|_{W_{p(.)}(0,T)}^{p_+}.\end{aligned} \tag{92}$$

Using Proposition 3.2, we get

$$\|u\|_{L^\infty(0,T;L^2(\Omega))}\leq C\|u\|_{W_{p(.)}(0,T)}. \tag{93}$$

Hence, by (91)-(93), we obtain

$$\begin{aligned}[u]_* &\leq C\left(\|u\|_{W_{p(.)}(0,T)} + \|u\|^2_{W_{p(.)}(0,T)} + \|u\|^{p_-}_{W_{p(.)}(0,T)} + \|u\|^{p_+}_{W_{p(.)}(0,T)} + \|u\|^{(p_-)'}_{W_{p(.)}(0,T)}\right) \\ &\leq C\max\left\{\|u\|^{p_-}_{W_{p(.)}(0,T)}, \|u\|^{(p_-)'}_{W_{p(.)}(0,T)}\right\}. \qquad (94)\end{aligned}$$

Thanks to Lemma 3.7, there exists $z \in \widetilde{W}_{p(.)}(0,T)$ such that $|u| \leq z$ and

$$\|z\|_{W_{p(.)}(0,T)} \leq C\left([u]_*^{\frac{1}{2}} + [u]_*^{\frac{1}{p_-}} + [u]_*^{\frac{1}{p_+}} + [u]_*^{\frac{1}{(p')_-}} + [u]_*^{\frac{1}{(p')_+}}\right),$$

which implies that

$$\|z\|_{W_{p(.)}(0,T)} \leq C\max\left([u]_*^{\frac{1}{p_-}}, [u]_*^{\frac{1}{(p')_-}}\right).$$

Therefore, from (94) we obtain

$$\|z\|_{\widetilde{W}_{p(.)}(0,T)} \leq C\max\left\{\|u\|^{\frac{p_-}{(p')_-}}_{W_{p(.)}(0,T)}, \|u\|^{\frac{(p_-)'}{p_-}}_{W_{p(.)}(0,T)}\right\}. \qquad (95)$$

Then, we can prove the following result which gives the connection between the notions of capacity and continuity.

Proposition 3.9. *If u is cap-quasi continuous and belongs to $W_{p(.)}(0,T)$, then for all $t > 0$,*

$$cap_{p(.)}(\{|u| > t\}) \leq \frac{C}{t}max\left\{\|u\|^{\frac{p_-}{(p')_-}}_{W_{p(.)}(0,T)}, \|u\|^{\frac{(p_-)'}{p_-}}_{W_{p(.)}(0,T)}\right\}. \qquad (96)$$

Proof. We consider in the first step, the case where u belongs to $C_c([0,T] \times \Omega)$, this step is motivated by the fact that $C_c([0,T] \times \Omega)$ is dense in $W_{p(.)}(0,T)$. Thanks to Corollary 3.8, there exists $z \in \widetilde{W}_{p(.)}(0,T)$ such that $|u| \leq z$ holds true; then, since $\widetilde{W}_{p(.)}(0,T)$ is continuously embedding in $W_{p(.)}(0,T)$ and $\frac{z}{t} \geq 1$ on the set $\{|u| > t\}$, we have

$$cap_{p(.)}(\{|u| > t\}) \leq \left\|\frac{z}{t}\right\|_{W_{p(.)}(0,T)} \leq \frac{C}{t}\max\left\{\|u\|^{\frac{p_-}{(p')_-}}_{W_{p(.)}(0,T)}, \|u\|^{\frac{(p_-)'}{p_-}}_{W_{p(.)}(0,T)}\right\}.$$

For the second step, we suppose only that $u \in W_{p(.)}(0,T)$ and is $cap_{p(.)}$–quasi continuous. Let $\varepsilon > 0$ be fixed, then there exists an open set A_ε such that $cap_{p(.)}(A_\varepsilon) < \varepsilon$ and $u_{|(Q\setminus A_\varepsilon)}$ is continuous, which implies that $\{u_{|(Q\setminus A_\varepsilon)} > t\} \cap (Q \setminus A_\varepsilon)$ is an open set in $Q \setminus A_\varepsilon$. Then, there exists an open set $U \subset \mathbb{R}^N$ such that $\{u_{|(Q\setminus A_\varepsilon)} > t\} \cap (Q \setminus A_\varepsilon) = U \cap (Q \setminus A_\varepsilon)$. Consequently,

$$\{|u| > t\} \cup A_\varepsilon = \left(\{u_{|(Q\setminus A_\varepsilon)} > t\} \cap (Q \setminus A_\varepsilon)\right) \cup A_\varepsilon = (U \cup A_\varepsilon) \cap Q$$

is an open set.

Now, we consider the function z given by Corollary 3.8. Let $w \in W_{p(.)}(0,T)$ be such that $w\geq \chi_{A_\varepsilon}$ and $\|w\|_{W_{p(.)}(0,T)}\leq cap_{p(.)}(A_\varepsilon)+\varepsilon<2\varepsilon$. Since $w+\dfrac{z}{t}\geq 1$ a.e. in $\{|u|>t\}\cup A_\varepsilon$, we have

$$\begin{aligned} cap_{p(.)}(\{|u|>t\}) &\leq cap_{p(.)}(\{|u|>t\}\cup A_\varepsilon)\leq \left\|w+\frac{z}{t}\right\|_{W_{p(.)}(0,T)} \\ &\leq \|w\|_{W_{p(.)}(0,T)}+\frac{1}{t}\|z\|_{W_{p(.)}(0,T)}\leq 2\varepsilon+\frac{1}{t}\|z\|_{W_{p(.)}(0,T)}. \quad (97)\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, then, we deduce that

$$cap_{p(.)}(\{|u|>t\})\leq \frac{C}{t}\max\left\{\|u\|_{W_{p(.)}(0,T)}^{\frac{p_-}{(p')_-}},\|u\|_{W_{p(.)}(0,T)}^{\frac{(p_-)'}{p_-}}\right\}\square$$

As in elliptic case, we have the following result of quasicontinuity.

Lemma 3.10. *Any element v of $W_{p(.)}(0,T)$ has a cap-quasi continuous representative $\tilde{v}$ which is cap-quasi everywhere unique, in the sense that two cap-quasi continuous representatives of v are equal except on a set of null capacity.*

Proof. We adapt the proof given in [6]. Since $C_c([0,T]\times\Omega)$ is dense in $W_{p(.)}(0,T)$, there exists a sequence $(v^m)\subset C_c([0,T]\times\Omega)$ such that v^m converges to v in $W_{p(.)}(0,T)$, as $m\to\infty$. Moreover, we have

$$\sum_{m=1}^{\infty}2^m\max\left\{\left\|v^{m+1}-v^m\right\|_{W_{p(.)}(0,T)}^{\frac{p_-}{(p')_-}},\left\|v^{m+1}-v^m\right\|_{W_{p(.)}(0,T)}^{\frac{(p_-)'}{p_-}}\right\}<\infty.$$

We introduce the following subsets

$$\omega^m=\left\{\left|v^{m+1}-v^m\right|>2^{-m}\right\},\quad \Omega^r=\bigcup_{m\geq r}\omega^m.$$

Using the fact that $v^{m+1}-v^m$ is continuous and belongs to $W_{p(.)}(0,T)$, we apply Proposition 3.9 to obtain

$$cap_{p(.)}(\omega^m)\leq C2^m\max\left\{\left\|v^{m+1}-v^m\right\|_{W_{p(.)}(0,T)}^{\frac{p_-}{(p')_-}},\left\|v^{m+1}-v^m\right\|_{W_{p(.)}(0,T)}^{\frac{(p_-)'}{p_-}}\right\}.$$

By subadditivity, we get

$$cap_{p(.)}(\Omega^r)\leq C\sum_{m\geq r}2^m\max\left\{\left\|v^{m+1}-v^m\right\|_{W_{p(.)}(0,T)}^{\frac{p_-}{(p')_-}},\left\|v^{m+1}-v^m\right\|_{W_{p(.)}(0,T)}^{\frac{(p_-)'}{p_-}}\right\},$$

which implies that

$$\lim_{r\to\infty}cap_{p(.)}(\Omega^r)=0.$$

For any r, we have

$$\text{if}\quad (x,t)\notin\Omega^r,\quad\text{then}\quad \forall m\geq r,\left|v^{m+1}-v^m\right|(z)\leq 2^{-m}.$$

Hence, v^m converges uniformly on the complement of each Ω^r and pointwise in the complement of $\bigcap\limits_{r}^{\infty}\Omega_r$.

Moreover,

$$cap_{p(.)}\left(\bigcap_{r}^{\infty}\Omega^r\right)\leq cap_{p(.)}(\Omega^r)\to 0\quad\text{as}\quad r\quad\text{tends to infinity},$$

which prove that $cap_{p(.)}\left(\bigcap_{r}^{\infty}\Omega^{r}\right)=0.$

Therefore, the limit of v^{m} is defined cap-quasi everywhere and is cap-quasi continuous. Let us call $\tilde{v}$ this cap-quasi continuous representative of v and assume that there exists another representative z of v which is cap-quasi continuous and coincides with v almost everywhere in Q. Then we have, thanks to Proposition 3.9,

$$cap_{p(.)}\left(\left\{|z-\tilde{v}|>\frac{1}{k}\right\}\right)\leq Ck\max\left\{\|z-\tilde{v}\|_{W_{p(.)}(0,T)}^{\frac{p_{-}}{(p')_{-}}},\|z-\tilde{v}\|_{W_{p(.)}(0,T)}^{\frac{(p_{-})'}{p_{-}}}\right\},$$

since $\tilde{v}=z$ in $W_{p(.)}\left(0,T\right)$. This being true for any k, we obtain that $\tilde{v}=z$ cap-quasi everywhere, so that the cap-quasi continuous representative of v is unique up to sets of zero capacity □

In what follows, we need the following results.

Lemma 3.11. *Let* $\left(v_{n}\right)_{n\in\mathbb{N}}$ *be a sequence in* $W_{p(.)}\left(0,T\right)$ *which converges to* v *in* $W_{p(.)}\left(0,T\right)$, *then there exists a subsequence* $\left(\tilde{v}_{n_{k}}\right)_{k\in\mathbb{N}}$ *of* $\left(v_{n}\right)_{n\in\mathbb{N}}$ *which converges to* $\tilde{v}$ *cap-quasi everywhere.*

Proof. According to Proposition 3.9 and Lemma 3.10, the proof is similar to the proof of Lemma 2.2.1 in [6] □

4. Measures

In this part, we establish the relation between measures in Q and the notion of $p\left(.\right)$-parabolic capacity. We extend the results obtained in the case of constant exponent (see [6]) to the case of variable exponent. In the rest of the paper we denote by $\mathcal{M}_{b}\left(Q\right)$ the space of bounded measure in Q and $\mathcal{M}_{b}^{+}\left(Q\right)$ the subsets of nonnegative measures of $\mathcal{M}_{b}\left(Q\right)$. The duality between $\left(W_{p(.)}\left(0,T\right)\right)'$ and $W_{p(.)}\left(0,T\right)$ is denoted by $\langle\langle .,.\rangle\rangle$, $\left(W_{p(.)}\left(0,T\right)\right)'\cap\mathcal{M}_{b}\left(Q\right)$ is the set of element $\gamma\in\left(W_{p(.)}\left(0,T\right)\right)'$ such that there exists $c>0$ satisfying, for all $\varphi\in\mathcal{C}_{c}^{\infty}\left(Q\right)$, $|\langle\langle\gamma,\varphi\rangle\rangle|\leq c\left\|\varphi\right\|_{L^{\infty}(Q)}$. Every $\gamma\in\left(W_{p(.)}\left(0,T\right)\right)'\cap\mathcal{M}_{b}\left(Q\right)$ is identified by unique linear application $\varphi\in\mathcal{C}_{c}^{\infty}\left(Q\right)\mapsto\int_{Q}\varphi d\gamma^{meas}$ where γ^{meas} belongs to $\mathcal{M}_{b}\left(Q\right)$. The set of $\gamma\in\left(W_{p(.)}\left(0,T\right)\right)'\cap\mathcal{M}_{b}\left(Q\right)$ such that $\gamma^{meas}\in\mathcal{M}_{b}^{+}\left(Q\right)$ is denoted by $\left(W_{p(.)}\left(0,T\right)\right)'\cap\mathcal{M}_{b}^{+}\left(Q\right)$.

Definition 4.1. We define

$$\mathcal{M}_{0}\left(Q\right)=\left\{\mu\in\mathcal{M}_{b}\left(Q\right):\mu\left(E\right)=0\quad\text{for every}\quad E\subset Q\quad\text{such that}\quad cap_{p(.)}\left(E\right)=0\right\}.$$

The nonnegative measures in $\mathcal{M}_{0}\left(Q\right)$ will be said to belongs to $\mathcal{M}_{0}^{+}\left(Q\right)$.

Proposition 4.1. *Let* μ *belongs to* $\mathcal{M}_{0}^{+}\left(Q\right)$. *Then, there exists* $\gamma\in\left(W_{p(.)}\left(0,T\right)\right)'\cap\mathcal{M}_{b}^{+}\left(Q\right)$ *and a nonnegative function* $f\in L^{1}\left(Q,\gamma^{meas}\right)$ *such that* $\mu=f\gamma^{meas}$.

Proof. Let $u\in W_{p(.)}\left(0,T\right)$. Since by Lemma 3.7, u admits a cap-quasi continuous representative denoted $\tilde{u}$ which is cap-quasi everywhere unique, then we can define the following functional $F:W_{p(.)}\left(0,T\right)\rightarrow\mathbb{R}$ by $F\left(u\right)=\int_{Q}\max\left\{\tilde{u},0\right\}d\mu.$

The function F is convex and lower semicontinuous on $W_{p(.)}(0,T)$ (the lower semicontinuity follows from Fatou's Lemma and Lemma 3.11). Since $W_{p(.)}(0,T)$ is separable, the function F is the supremum of a countable family of continuous affine functions. Hence, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\left(W_{p(.)}(0,T)\right)'$ and a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $F(u)=\sup_{n\in\mathbb{N}}\{\langle\langle\lambda_n,u\rangle\rangle+a_n\}$.
We have $F(0)=0$, which implies that $a_n\le 0$. Then, it follows that

$$F(u)\le \sup_{n\in\mathbb{N}}\{\langle\langle\lambda_n,u\rangle\rangle\}. \tag{98}$$

Since for every $t>0$ and for every $u\in W_{p(.)}(0,T)$, we have

$$t\langle\langle\lambda_n,u\rangle\rangle+a_n\le F(tu)=tF(u) \tag{99}$$

then, we get $\langle\langle\lambda_n,u\rangle\rangle\le F(u)$; hence, by (98) we deduce that

$$F(u)=\sup_{n\in\mathbb{N}}\{\langle\langle\lambda_n,u\rangle\rangle\}. \tag{100}$$

Now, we are going to show that λ_n belongs to $\left(W_{p(.)}(0,T)\right)'$. Using (100) and the definition of F, we obtain

$$\langle\langle\lambda_n,\varphi\rangle\rangle\le\int_Q \max\{\varphi,0\}\,d\mu\le\|\mu\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)}, \tag{101}$$

for all $\varphi\in C_c^\infty(Q)$. Since the inequality (101) remains true for $-\varphi$, we deduce that $|\langle\langle\lambda_n,\varphi\rangle\rangle|\le\|\mu\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)}$, hence $\lambda_n\in\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_b(Q)$.
For all $\varphi\in C_c^\infty(Q)$ such that $\varphi\ge 0$, we have

$$-\langle\langle\lambda_n,\varphi\rangle\rangle=\langle\langle\lambda_n,-\varphi\rangle\rangle\le F(-\varphi)=0$$

which implies that

$$0\le\langle\langle\lambda_n,\varphi\rangle\rangle=\int_Q\varphi d\lambda_n^{meas}.$$

Then, it follows that λ_n^{meas} belongs to $\mathcal{M}_b^+(Q)$, that is equivalent to say that $\lambda_n\in\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_b^+(Q)$. By (100), for any nonnegative $\varphi\in C_c^\infty(Q)$ we have

$$\int_Q\varphi d\lambda_n^{meas}=\langle\langle\lambda_n,\varphi\rangle\rangle\le\int_Q\varphi d\mu,$$

then

$$\lambda_n^{meas}\le\mu, \tag{102}$$

moreover, we can write $\|\lambda_n^{meas}\|_{\mathcal{M}_b(Q)}\le\|\mu\|_{\mathcal{M}_b(Q)}$.
We define $\gamma\in\left(W_{p(.)}(0,T)\right)'$ by

$$\gamma=\sum_{n=1}^{\infty}\frac{\lambda_n}{2^n(\|\lambda_n\|_{\left(W_{p(.)}(0,T)\right)'}+1)}. \tag{103}$$

The serie γ is absolutely convergent in $\left(W_{p(.)}(0,T)\right)'$, moreover for all $\varphi\in C_c^\infty(Q)$, we have

$$\begin{aligned}|\langle\langle\gamma,\varphi\rangle\rangle| &= \left|\sum_{n=1}^{\infty}\frac{\langle\langle\lambda_n,\varphi\rangle\rangle}{2^n(\|\lambda_n\|_{\left(W_{p(.)}(0,T)\right)'}+1)}\right|\\ &\le \sum_{n=1}^{\infty}\frac{\|\lambda_n^{meas}\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)}}{2^n}\le\|\mu\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)},\end{aligned}$$

which implies that $\gamma \in \left(W_{p(.)}\,(0,T)\right)' \cap \mathcal{M}_b\,(Q)\,.$
Thanks to (103), for all $\varphi \in C_c^{\infty}\,(Q)$, we have

$$\begin{aligned}\int_Q \varphi d\gamma^{meas} &= \langle\langle \gamma,\varphi\rangle\rangle = \sum_{n=1}^{\infty} \frac{\langle\langle \lambda_n,\varphi\rangle\rangle}{2^n(\|\lambda_n\|_{\left(W_{p(.)}(0,T)\right)'}+1)} \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n(\|\lambda_n\|_{\left(W_{p(.)}(0,T)\right)'}+1)} \int_Q \varphi d\lambda_n^{meas},\end{aligned}$$

hence,

$$\gamma^{meas} = \frac{\lambda_n^{meas}}{2^n(\|\lambda_n\|_{\left(W_{p(.)}(0,T)\right)'}+1)} \tag{104}$$

and since $\lambda_n^{meas} \geq 0$, γ^{meas} is a nonnegative measure. For every $n \in \mathbb{N}$, the measure λ^{meas} is absolutely continuous with respect to γ^{meas} thus, there exists a nonnegative function $f_n \in L^1\,(Q, d\gamma^{meas})$ such that $\lambda_n^{meas} = f_n\gamma^{meas}$. Then, from (100) we get

$$\int_Q \varphi d\mu = \sup\nolimits_{n\in\mathbb{N}} \int_Q f_n\varphi d\gamma^{meas}, \tag{105}$$

for any nonnegative $\varphi \in C_c^{\infty}\,(Q)$. Since by (102), we have $f_n\gamma^{meas} = \lambda_n^{meas} \leq \mu$, then

$$\int_B f_n d\gamma^{meas} \leq \mu\,(B)\,, \tag{106}$$

for any borelian subset B in Q and every $n \in \mathbb{N}$. So we can write

$$\int_B \sup\{f_1, f_2, \cdots, f_k\}\, d\gamma^{meas} \leq \mu\,(B)\,, \tag{107}$$

for any borelian subset B in Q and any $k \geq 1$. Letting k tends to infinity we deduce by the monotone convergence theorem

$$\int_B f d\gamma^{meas} \leq \mu\,(B)\,, \tag{108}$$

where $f = \sup_{n\in\mathbb{N}}\{f_n\}$, hence by (104), we obtain

$$\int_B \varphi d\mu = \sup\nolimits_{n\in\mathbb{N}} \int_Q f_n\varphi d\gamma^{meas} \leq \int_Q f\varphi d\gamma^{meas} \leq \int_Q \varphi d\mu, \tag{109}$$

for every nonnegative function $\varphi \in C_c^{\infty}\,(Q)$ which implies that $\mu = f\gamma^{meas}$ and from the fact that $\mu\,(Q) < +\infty$, we get $f \in L^1\,(Q, d\gamma^{meas})$ □

Lemma 4.2. *Let* $g \in \left(W_{p(.)}\,(0,T)\right)'$. *Then, there exists* $g_1 \in L^{(p_-)'}\left(0,T; W^{-1,p'(.)}\,(\Omega)\right)$, $g_2 \in L^{p_-}\,(0,T;V)$, $F \in \left(L^{p'(.)}\,(Q)\right)^N$ *and* $g_3 \in L^{(p_-)'}\left(0,T;L^2\,(\Omega)\right)$ *such that*

$$\langle\langle g,u\rangle\rangle = \int_0^T \langle g_1,u\rangle\, dt + \int_0^T \langle u_t, g_2\rangle + \int_Q F.\nabla u dxdt \int_Q g_3 u dxdt, \quad \forall u \in W_{p(.)}\,(0,T)\,.$$

Moreover, we can choose (g_1, g_2, F, g_3) *such that*

$$\begin{aligned}\|g_1\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)} + \|g_2\|_{L^{p_-}(0,T;V)} + |||F|||_{L^{p'(.)}(Q)} + \|g_3\|_{L^{(p_-)'}(0,T;L^2(\Omega))} \\ \leq C\,\|g\|_{\left(W_{p(.)}(0,T)\right)'}\,.\end{aligned} \tag{110}$$

Proof. We introduce the following functional space

$$E = L^{p_-}(0,T;V) \times \left(L^{p(.)}(Q)\right)^N \times L^{(p_-)'}(0,T;V')$$

endowed with the norm

$$\|(v_1,v_2,v_3)\|_E = \|v_1\|_{L^{p_-}(0,T;V)} + \||v_2|\|_{L^{p(.)}(Q)} + \|v_3\|_{L^{(p_-)'}(0,T;V')}$$

and we consider the map $T : W_{p(.)}(0,T) \to E$ by $T(u) = (u, \nabla u, u_t)$.
Since

$$\|T(u)\|_E = \|(u_t, \nabla u, u)\|_E = \|u\|_{W_{p(.)}(0,T)}. \tag{111}$$

Then T is isometric from $W_{p(.)}(0,T)$ to E.
Setting $G = T\left(W_{p(.)}(0,T)\right)$, then T^{-1} is defined from G to $W_{p(.)}(0,T)$. Now, we take $g \in \left(W_{p(.)}(0,T)\right)'$ and we introduce the functional $\Phi : G \to \mathbb{R}$ by $\Phi(v_1,v_2,v_3) = \left\langle\left\langle g, T^{-1}(v_1,v_2,v_3)\right\rangle\right\rangle$.
Since Φ is a continuous linear form on G then by Hahn-Banach theorem, it can be extended to a continuous linear form on E still denoted by Φ with $\|\Phi\|_{E'} = \|g\|_{\left(W_{p(.)}(0,T)\right)'}$.
Consequently, there exists $h_1 \in (L^{p_-}(0,T;V))'$, $F = (f_1, f_2, \cdots, f_N) \in \left(L^{p'(.)}(Q)\right)^N$ and $h_2 \in \left(L^{(p_-)'}(0,T;V')\right)'$ such that

$$\begin{aligned}\Phi(v_1,v_2,v_3) \quad = \quad & \langle h_1, v_1\rangle_{(L^{p_-}(0,T;V))', L^{p_-}(0,T;V)} + \langle F, v_2\rangle_{\left(L^{p'(.)}(Q)\right)^N, \left(L^{p(.)}(Q)\right)^N} \\ & + \langle h_2, v_3\rangle_{\left(L^{(p_-)'}(0,T;V')\right)', L^{(p_-)'}(0,T;V')}.\end{aligned} \tag{112}$$

Moreover, we have

$$\|h_1\|_{(L^{p_-}(0,T;V))'} + \||F|\|_{L^{p'(.)}(Q)} + \|h_2\|_{\left(L^{(p_-)'}(0,T;V')\right)'} \le \|\Phi\|_{E'}. \tag{113}$$

Thanks to Remark 3.1, we have

$$(L^{p_-}(0,T;V))' = L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right) + L^{(p_-)'}\left(0,T;L^2(\Omega)\right)$$

(with equivalent norms). Then, there exists $g_1 \in L^{(p_-)'}(0,T;W^{-1,p'(.)}(\Omega))$ and $g_3 \in L^{(p_-)'}(0,T;L^2(\Omega))$ such that

$$\langle h_1, v_1\rangle_{(L^{p_-}(0,T;V))', L^{p_-}(0,T;V)} = \int_0^T \langle g_1, v_1\rangle\, dt + \int_Q g_3 v_1 dxdt. \tag{114}$$

Since $\left(L^{(p_-)'}(0,T;V')\right)' = L^{p_-}(0,T;V)$, there exists $g_2 \in L^{p_-}(0,T;V)$ such that

$$\langle h_2, v_2\rangle_{\left(L^{(p_-)'}(0,T;V')\right)', L^{(p_-)'}(0,T;V')} = \int_0^T \langle v_2, g_2\rangle\, dt. \tag{115}$$

Therefore, we have

$$\Phi(v_1,v_2,v_3) = \int_0^T \langle g_1, v_1\rangle\, dt + \int_0^T \langle v_2, g_2\rangle\, dt + \int_Q F\nabla u dxdt + \int_Q g_3 v_1 dxdt$$

with

$$\begin{aligned}
&\|g_1\|_{L^{(p_+)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)} + \|g_3\|_{L^{(p_+)'}(0,T;L^2(\Omega))} + \|F\|_{\left(L^{p'(.)}(Q)\right)^N} + \|g_2\|_{L^{p_-}(0,T;V)} \\
&\le C\left(\|h_1\|_{L^{(p_-)'}(0,T;V')} + \||F|\|_{L^{p'(.)}(Q)} + \|h_2\|_{\left(L^{(p_-)'}(0,T;V')\right)'}\right) \\
&\le C\,\|g\|_{\left(W_{p(.)}(0,T)\right)'}\,. \qquad (116)
\end{aligned}$$

Then it follows that for all $u \in W_{p(.)}(0,T)$, we have

$$\begin{aligned}
\langle\langle g,u\rangle\rangle &= \left\langle\left\langle g, T^{-1}(T(u))\right\rangle\right\rangle = \Phi(T(u)) \\
&= \int_0^T \langle g_1,u\rangle\, dt + \int_0^T \langle u_t, g_2\rangle\, dt + \int_Q F\nabla u\; dxdt + \int_Q g_3 u\; dxdt \qquad (117)
\end{aligned}$$

Since for all $\theta \in C_c^\infty(Q)$, the multiplication $\varphi \mapsto \theta\varphi$ is linear continuous from $W_{p(.)}(0,T)$ to $W_{p(.)}(0,T)$, we can define the multiplication of an element $\nu \in (W_{p(.)}(0,T))'$ by θ thanks to a duality method : $\theta\nu \in \left(W_{p(.)}(0,T)\right)'$ is defined by $\langle\theta\nu,\varphi\rangle = \langle\nu,\theta\varphi\rangle$. Then, the following result can be proved similarly to that in [6].

Lemma 4.3. *Let* $\nu \in \left(W_{p(.)}(0,T)\right)' \cap \mathcal{M}_b(Q)$ *and* $\theta \in C_c^\infty(Q)$. *We take* ρ_n *as a sequence of symmetric (i.e* $\rho_n(.,-) = \rho_n(.)$*) regularizing kernels in* $\mathbb{R}\times\mathbb{R}^N$ *and* $\mu = \theta\nu \in \left(W_{p(.)}(0,T)\right)'$. *Then,* $\mu \in \left(W_{p(.)}(0,T)\right)' \cap \mathcal{M}_b(Q)$, $\mu^{meas} = \theta\nu^{meas}$, μ^{meas} *has a compact support in* Q *and*

$$\|\mu^{meas} * \rho_n\|_{L^1(Q)} \le \|\nu^{meas}\|_{\mathcal{M}_b(Q)} \quad \text{and} \quad \mu^{meas} * \rho_n \to \mu \quad \text{in} \quad \left(W_{p(.)}(0,T)\right)'. \qquad (118)$$

Proof. Since $\theta \in C_c^\infty(Q)$ and $\nu \in \left(W_{p(.)}(0,T)\right)'$, then $\mu = \theta\nu \in \left(W_{p(.)}(0,T)\right)'$. Moreover, for all $\varphi \in C_c^\infty(Q)$, we have $|\langle\langle\mu,\varphi\rangle\rangle| = |\langle\langle\nu,\theta\nu\rangle\rangle| \le C\|\theta\varphi\|_{L^\infty(Q)} \le C\|\theta\|_{L^\infty(Q)}\|\varphi\|_{L^\infty(Q)}$, which implies that $\mu \in \left(W_{p(.)}(0,T)\right)' \cap \mathcal{M}_b(Q)$.
For all $\varphi \in C_c^\infty(Q)$, we have

$$\int_Q \varphi d\mu^{meas} = \langle\langle\mu,\varphi\rangle\rangle = \langle\langle\nu,\theta\varphi\rangle\rangle = \int_Q \theta\varphi d\nu^{meas},$$

hence $\mu^{meas} = \theta\nu^{meas}$ and μ^{meas} has compact support. Therefore, $\mu^{meas} * \rho_n$ is well defined and belongs to $C_c^\infty(Q)$ for n large enough. Moreover, we have $\|\mu^{meas} * \rho_n\|_{L^1(Q)} \le \|\mu^{meas}\|_{\mathcal{M}_b(Q)}$.
Since $\nu \in (W_{p(.)}(0,T))'$, then by the Lemma 4.2, there exists $(g_1,g_2,F,g_3) \in L^{(p_-)'}(0,T;W^{-1,p'(.)}(\Omega)) \times L^{p_-}(0,T;V) \times (L^{p'}(Q))^N \times L^{(p_-)'}(0,T;L^2(\Omega))$ such that

$$\begin{aligned}
\langle\langle\mu,\varphi\rangle\rangle &= \langle\langle\nu,\theta\varphi\rangle\rangle \\
&= \int_0^T \langle g_1,\theta\varphi\rangle\, dt + \int_0^T \left\langle(\theta\varphi)_t, g_2\right\rangle dt + \int_Q F.\nabla(\theta\varphi)\, dxdt + \int_Q g_3\theta\varphi dt \\
&= \int_0^T \langle g_1,\theta\varphi\rangle\, dt + \int_0^T \langle\varphi_t,\theta g_2\rangle\, dt + \int_0^T \langle\theta_t\varphi, g_2\rangle\, dt \\
&\quad + \int_Q F.\nabla(\theta\varphi)\, dxdt + \int_Q g_3\theta\varphi dt,
\end{aligned}$$

for all $\varphi \in W_{p(.)}(0,T)$. Since by the proof of the second part of Proposition 3.2, the term $\theta_t\varphi$ belongs to $L^{(p_-)'}\left(0,T;L^2(\Omega)\right)$, then we have

$$\int_0^T \langle \theta_t\varphi, g_2\rangle\, dt = \int_Q \theta_t\varphi g_2 dxdt.$$

We have $g_1 \in L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)$, then there exists $G_1 \in \left(L^{p'(.)}(Q)\right)^N$ such that $g_1 = div\,(G_1)$, so that

$$\int_0^T \langle \theta g_1, \varphi\rangle\, dt = \int_0^T \langle div\,(G_1), \varphi\rangle\, dt - \int_0^T \langle G_1\nabla\theta, \varphi\rangle\, dt.$$

Moreover, we have

$$\int_Q F.\nabla(\theta\varphi)\, dxdt = \int_Q F.\nabla\theta\varphi dxdt + \int_Q \theta F.\nabla\varphi dxdt.$$

Thus, for all $\varphi \in W_{p(.)}(0,T)$, one has

$$\begin{aligned}\langle\langle \mu, \varphi\rangle\rangle &= \int_0^T \langle div\,(\theta G_1), \varphi\rangle\, dt + \int_0^T \langle \varphi_t, \theta g_2\rangle\, dt + \int_Q F.\nabla\theta\varphi dxdt \\ &+ \int_Q \theta F.\nabla\varphi dxdt + \int_Q g_3\theta\varphi dt - \int_Q G_1\nabla\theta\varphi dxdt + \int_Q \theta_t\varphi g_2 dxdt. \quad (119)\end{aligned}$$

For n large enough, $\operatorname{supp}(\theta)\cup\operatorname{supp}(\rho_n)$ is included in a fixed compact $K \subset Q$. Then it follows that $\operatorname{supp}(\mu^{meas} * \rho_n) = \operatorname{supp}(\theta\nu^{meas} * \rho_n)$ is also contained in K. Now, we take $\xi \in C_c^\infty(Q)$ be such that $\xi \equiv 1$ on a neighborhood of K; then for n large enough, $\operatorname{supp}(\xi)\cup\operatorname{supp}(\rho_n)$ is a compact subset of Q. Since $C_c^\infty(Q) \hookrightarrow \left(W_{p(.)}(0,T)\right)'$, for all $\varphi \in W_{p(.)}(0,T)$, we have

$$\langle\langle \mu^{meas} * \rho_n, \varphi\rangle\rangle = \int_Q \varphi\mu^{meas} * \rho_n dxdt.$$

Hence, for all $\varphi \in C_c^\infty([0,T]\times\Omega)$, we have

$$\langle\langle \mu^{meas} * \rho_n, \varphi\rangle\rangle = \int_Q \xi\varphi\mu^{meas} * \rho_n dxdt = \int_Q (\xi\varphi) * \rho_n d\mu^{meas}.$$

We suppose that n is large enough, then $\operatorname{supp}((\xi\varphi) * \rho_n)$ is a compact subset of Q and since $(\xi\varphi) * \rho_n$ belongs to $C_c^\infty(Q)$, then by (119), we get

$$\begin{aligned}&\langle\langle \mu^{meas} * \rho_n, \varphi\rangle\rangle = \langle\langle \mu, (\xi\varphi) * \rho_n\rangle\rangle \\ &= \int_0^T \langle div\,(\theta G_1), (\xi\varphi) * \rho_n\rangle\, dt + \int_0^T \langle ((\xi\varphi) * \rho_n)_t, \theta g_2\rangle\, dt \\ &\quad + \int_Q F.\nabla\theta(\xi\varphi) * \rho_n dxdt + \int_Q \theta F.\nabla(\xi\varphi) * \rho_n dxdt + \int_Q g_3\theta(\xi\varphi) * \rho_n dt \\ &\quad - \int_Q G_1\nabla\theta(\xi\varphi) * \rho_n dxdt + \int_Q \theta_t g_2(\xi\varphi) * \rho_n dxdt.\end{aligned}$$

According to the support of θ and ξ we can write

$$\begin{aligned}\langle\langle\mu^{meas}*\rho_n,\varphi\rangle\rangle &= \int_0^T \langle div\left((\theta G_1)*\rho_n\right),\xi\varphi\rangle\,dt+\int_0^T \langle(\xi\varphi)_t,(\theta g_2)*\rho_n\rangle\,dt\\ &+\int_Q (F.\nabla\theta)*\rho_n\xi\varphi dxdt+\int_Q \theta F.\nabla(\xi\varphi)*\rho_n dxdt+\int_Q(\theta g_3)*\rho_n\xi\varphi dxdt\\ &-\int_Q(G_1\nabla\theta)\,\xi\varphi dxdt+\int_Q(\theta_t g_2)*\rho_n\xi\varphi dxdt.\end{aligned}$$

Now, using the fact that $\xi\equiv 1$ on a neighborhood of $\operatorname{supp}(\theta)\cup\operatorname{supp}(\rho_n)$, we obtain

$$\begin{aligned}\langle\langle\mu^{meas}*\rho_n,\varphi\rangle\rangle &= \int_0^T \langle div\left((\theta G_1)*\rho_n\right),\varphi\rangle\,dt+\int_0^T \langle\varphi_t,(\theta g_2)*\rho_n\rangle\,dt \quad (120)\\ &+\int_Q (F.\nabla\theta)*\rho_n\varphi dxdt+\int_Q \theta F.\nabla\varphi*\rho_n dxdt+\int_Q(\theta g_3)*\rho_n\varphi dt\\ &-\int_Q(G_1\nabla\theta)*\rho_n\varphi dxdt+\int_Q(\theta_t g_2)*\rho_n\varphi dxdt,\end{aligned}$$

for all $\varphi\in C_c^\infty([0,T]\times\Omega)$, but since this space is dense in $W_{p(.)}(0,T)$ and both sides are continuous with respect to the norm of $W_{p(.)}(0,T)$, equality (120) remains true for $\varphi\in W_{p(.)}(0,T)$.

We have $(\theta G_1)*\rho_n\to\theta G_1$ in $\left(L^{p'(.)}(Q)\right)^N$, $(\theta g_2)*\rho_n\to\theta g_2$ in $L^{p_-}(0,T;V)$, $(F.\nabla\theta)*\rho_n\to F.\nabla\theta$ in $L^{p'(.)}(Q)$, $\nabla\varphi*\rho_n\to\nabla\varphi$ in $\left(L^{p(.)}(Q)\right)^N$, $(\theta g_3)*\rho_n\to\theta g_3$ in $L^{(p_+)'}\left(0,T;L^2(\Omega)\right)$, $(G_1\nabla\theta)*\rho_n\to G_1\nabla\theta$ in $L^{p'(.)}(Q)$ and $(\theta_t g_2)*\rho_n\to\theta_t g_2$ in $L^{p_-}\left(0,T;L^2(\Omega)\right)$, then subtracting (119) and (120), we obtain

$$\begin{aligned}|\langle\langle\mu^{meas}*\rho_n,\varphi\rangle\rangle| &= \left|\int_0^T \langle div\left((\theta G_1)*\rho_n-\theta G_1\right),\varphi\rangle\,dt\right.\\ &+\int_0^T\langle\varphi_t,(\theta g_2)*\rho_n-\theta g_2\rangle\,dt+\int_Q\left((\theta g_3)*\rho_n-\theta g_3\right)\varphi dxdt\\ &+\int_Q\left((G_1\nabla\theta)-(G_1\nabla\theta)*\rho_n\right)\varphi dxdt+\int_Q\left((\theta_t g_2)*\rho_n-\theta_t g_2\right)\varphi dxdt\\ &\left.+\int_Q\left((F.\nabla\theta)*\rho_n-F.\nabla\theta\right)\varphi dxdt+\int_Q\theta F.\left(\nabla\varphi*\rho_n-\nabla\varphi\right)dxdt\right|\\ \le\Big(&\|(\theta G_1)*\rho_n-\theta G_1\|_{\left(L^{p'(.)}(Q)\right)^N}\|\nabla\varphi\|_{\left(L^{p(.)}(Q)\right)^N}+\|(\theta g_2)*\rho_n-\theta g_2\|_{L^{p_-}(0,T;V)}\\ &\times\|\varphi_t\|_{L^{(p_-)'}(0,T;V')}+\|(\theta g_3)*\rho_n-\theta g_3\|_{L^{(p_+)'}(0,T;L^2(\Omega))}\|\varphi\|_{L^{p_+}(0,T;L^2(\Omega))}\\ &+\|G_1\nabla\theta-(G_1\nabla\theta)*\rho_n\|_{L^{p'(.)}(Q)}\|\varphi\|_{L^{p(.)}(Q)}+\|(\theta_t g_2)*\rho_n-\theta_t g_2\|_{L^{p_-}(0,T;L^2(\Omega))}\\ &\times\|\varphi\|_{L^{(p_-)'}(0,T;L^2(\Omega))}+\|(F.\nabla\theta)*\rho_n-F.\nabla\theta\|_{L^{p'(.)}(Q)}\|\varphi\|_{L^{p(.)}(Q)}\\ &+\|\theta F\|_{\left(L^{p'(.)}(Q)\right)^N}\|\nabla\varphi*\rho_n-\nabla\varphi\|_{\left(L^{p(.)}(Q)\right)^N}\Big)\end{aligned}$$

$$\leq C\Big(\|(\theta G_1)*\rho_n-\theta G_1\|_{\left(L^{p'(.)}(Q)\right)^N}+\|(\theta g_2)*\rho_n-\theta g_2\|_{L^{p_-}(0,T;V)}$$
$$+\|(\theta g_3)*\rho_n-\theta g_3\|_{L^{(p_+)'}(0,T;L^2(\Omega))}+\|(F.\nabla\theta)*\rho_n-F.\nabla\theta\|_{L^{p'(.)}(Q)}$$
$$+\|G_1\nabla\theta-(G_1\nabla\theta)*\rho_n\|_{L^{p'(.)}(Q)}+\|(\theta_t g_2)*\rho_n-\theta_t g_2\|_{L^{p_-}(0,T;L^2(\Omega))}\Big)\|\varphi\|_{W_{p(.)}(0,T)}$$
$$+\|\theta F\|_{\left(L^{p'(.)}(Q)\right)^N}\|\nabla\varphi*\rho_n-\nabla\varphi\|_{\left(L^{p(.)}(Q)\right)^N},$$

which implies that $\mu^{meas}*\rho_n$ converges to μ in $W_{p(.)}(0,T)$ □

Theorem 4.4. *Let $\mu\in\mathcal{M}_0(Q)$ then there exists $g\in\left(W_{p(.)}(0,T)\right)'$ and $h\in L^1(Q)$ such that $\mu=g+h$ in the sense that*

$$\int_Q\varphi d\mu=\langle\langle g,\varphi\rangle\rangle+\int_Q h\varphi dxdt, \tag{121}$$

for all $\varphi\in C_c^\infty([0,T]\times\Omega)$.

Proof. Since μ belongs to $\mathcal{M}_0(Q)$, then by Hahn Banach decomposition of μ we have $\mu^+,\mu^-\in\mathcal{M}_0(Q)$, so we can assume that $\mu\in\mathcal{M}_0^+(Q)$. Hence, from the Proposition 4.1, there exists $\gamma\in\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_0^+(Q)$ and nonnegative Borel function $f\in L^1(Q,d\gamma^{meas})$ such that

$$\mu(B)=\int_B fd\gamma^{meas}\quad\text{for all Borel set } B\quad\text{in}\quad Q.$$

Since γ^{meas} is a regular measure and $C_c^\infty(Q)$ is dense in $L^1(Q,d\gamma^{meas})$, then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(Q)$ such that f_n converges strongly to f in $L^1(Q,d\gamma^{meas})$. Moreover, we have $\sum_{n=0}^{\infty}\|f_n-f_{n-1}\|_{L^1(Q,d\gamma^{meas})}<\infty$.

Defining ν_n by $\nu_n=(f_n-f_{n-1})\gamma\in\left(W_{p(.)}(0,T)\right)'$, then by Lemma 4.3 we get $\nu\in\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_b(Q)$ and $\sum_{n=0}^{\infty}\nu_n^{meas}=\sum_{n=0}^{\infty}(f_n-f_{n-1})\gamma^{meas}$ strongly converges to μ in $\mathcal{M}_b(Q)$. Therefore, we can consider μ as compactly supported measure. Using the Lemma 4.3, we deduce that $\rho_l*\nu_n^{meas}$ strongly converges to ν_n in $\left(W_{p(.)}(0,T)\right)'$, hence we can extract a subsequence still denoted by l such that $\|\rho_l*\nu_n^{meas}-\nu_n\|_{\left(W_{p(.)}(0,T)\right)'}\leq\frac{1}{2^n}$.

Let us rewrite now $\sum_{k=0}^{n}\nu_k^{meas}$ as follows

$$\sum_{k=0}^{n}\nu_k^{meas}=\sum_{k=0}^{n}\rho_{l_k}*\nu_k^{meas}+\sum_{k=0}^{n}\left(\nu_k^{meas}-\rho_{l_k}*\nu_k^{meas}\right). \tag{122}$$

In the following, we denote respectively by m_n, h_n the first and second term in (122) and we define the sequence g_n by $g_n=\sum_{k=0}^{n}\left(\nu_k-\rho_{l_k}*\nu_k^{meas}\right)$, so m_n is a measure with compact support, h_n is a function in $C_c^\infty(Q)$ and g_n belongs to $\left(W_{p(.)}(0,T)\right)'$.

We have $g_n = \sum_{k=0}^{n} \left(\nu_k^{meas} - \rho_{l_k} * \nu_k^{meas}\right)$. Taking θ_n in $C_c^\infty(Q)$ be such that $\theta \equiv 1$ on a neighborhood of $(\operatorname{supp}(f_0) \cup \cdots \cup \operatorname{supp}(f_n)) \cap \operatorname{supp}\left(\sum_{k=0}^{n} \rho_{l_k} * \nu_k^{meas}\right)$, then we can write $g_n = \theta_n g_n$.
Since all terms in (122) has compact support, we can use $\varphi \in C_c^\infty([0,T] \times \Omega)$ as test function in (122) to obtain

$$\int_Q \varphi dm_n = \int_Q h_n \varphi dxdt + \langle\langle g_n, \varphi \rangle\rangle \tag{123}$$

since

$$\int_Q \varphi dg_n^{meas} = \int_Q \theta_n \varphi dg_n^{meas} = \langle\langle g_n, \theta_n \varphi \rangle\rangle = \langle\langle g_n, \varphi \rangle\rangle .$$

We have

$$\|h\|_{L^1(Q)} \leq \sum_{k=0}^{\infty} \|\rho_{l_k} * \nu_k^{meas}\|_{L^1(Q)} \leq \sum_{k=0}^{\infty} \|\nu_k^{meas}\|_{\mathcal{M}_b(Q)} < \infty,$$

which implies the existence of a subsequence of $(h_n)_{n\in\mathbb{N}}$ converging to an element h in $L^1(Q)$. We have

$$\|g_n\| \leq \sum_{k=0}^{\infty} \|\nu_k - \rho_{l_k} * \nu_n^{meas}\|_{\left(W_{p(.)}(0,T)\right)'} \leq \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty,$$

hence $(h_n)_{n\in\mathbb{N}}$ converges strongly to an element g in $\left(W_{p(.)}(0,T)\right)'$. Then it follows that

$$\langle\langle g_n, \varphi \rangle\rangle + \int_Q h_n \varphi dxdt \to \langle\langle g, \varphi \rangle\rangle + \int_Q h \varphi dxdt, \tag{124}$$

for every $\varphi \in C_c^\infty([0,T] \times \Omega)$.
Now, we prove that $\int_Q \varphi dm_n$ converges to $\int_Q \varphi d\mu$. For that, we recall the following linear and continuous injection

$$\begin{cases} \mathcal{M}_b(Q) & \to \left(C(\bar{Q})\right)' \\ \quad m & \mapsto \tilde{m} \quad \text{defined by} \quad \tilde{m}(f) = \int_Q f dm. \end{cases}$$

We know that m_n strongly converges to μ in $\mathcal{M}_b(Q)$, $\tilde{m}_n$ strongly converges to $\tilde{m}$ and since $\varphi \in C\left(\bar{Q}\right)$, we have

$$\int_Q \varphi dm_n = \tilde{m}_n(\varphi) \to \tilde{m}(\varphi) = \int_Q \varphi d\mu. \tag{125}$$

Combining $(123) - (125)$, we get (121) □
As consequences of Theorem 4.4 and Lemma 4.2, we have the following decomposition theorem which is the main result of this part.

Theorem 4.5. *Let $\mu \in \mathcal{M}_0(Q)$ then there exists (f, F, g_1, g_2) such that $f \in L^1(Q)$, $F \in \left(L^{p'(.)}(Q)\right)^N$, $g_1 \in L^{(p_-)'}\left(0,T; W^{-1,p'(.)}(\Omega)\right)$,$g_2 \in L^{p_-}(0,T;V)$ such that*

$$\int_Q \varphi d\mu = \int_Q f \varphi dxdt + \int_Q F.\nabla u dxdt + \int_0^T \langle g_1, \varphi \rangle dt - \int_0^T \langle \varphi_t, g_2 \rangle dt,$$

$\forall \varphi \in C_c^\infty([0,T]\times\Omega)$. *Such a triplet* (f, F, g_1, g_2) *will be called a decomposition of* μ.

Notice that the decomposition of $\mu \in \mathcal{M}_0(Q)$ given by the previous theorem is not unique, however as in [6] the following result can be proved.

Lemma 4.6. *Let* $\mu \in \mathcal{M}_0(Q)$ *and let* (f, F, g_1, g_2), $\left(\tilde{f}, \tilde{F}, \tilde{g}_1, \tilde{g}_2\right)$ *be two different decompositions of* μ *according to Theorem* 4.5. *Then we have*

$$\int_0^T \left\langle (g_2-\tilde{g}_2)_t, \varphi\right\rangle dt = \int_Q \left(\tilde{f}-f\right)\varphi dxdt + \int_Q \left(\tilde{F}-F\right).\nabla\varphi dxdt + \int_0^T \left\langle \tilde{g}_1 - g_1, \varphi\right\rangle dt \tag{126}$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$. *Moreover,* $g_2-\tilde{g}_2 \in C\left([0,T]; L^1(Q)\right)$ *and* $(g_2-\tilde{g}_2)(0) = 0$.

Proof. We have

$$\int_Q \varphi d\mu = \int_Q f\varphi dxdt + \int_Q F.\nabla\varphi dxdt + \int_0^T \langle g_1, \varphi\rangle dt - \int_0^T \langle \varphi_t, g_2\rangle dt \tag{127}$$

and

$$\int_Q \varphi d\mu = \int_Q \tilde{f}\varphi dxdt + \int_Q \tilde{F}.\nabla\varphi dxdt + \int_0^T \langle \tilde{g}_1, \varphi\rangle dt - \int_0^T \langle \varphi_t, \tilde{g}_2\rangle dt, \tag{128}$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$, then subtracting (126) and (127), we get

$$\int_Q \left(\tilde{f}-f\right)\varphi dxdt + \int_Q \left(\tilde{F}-F\right).\nabla\varphi dxdt + \int_0^T \langle \tilde{g}_1 - g_1, \varphi\rangle dt = -\int_0^T \langle \varphi_t, g_2-\tilde{g}_2\rangle dt, \tag{129}$$

which is equivalent to say that

$$\int_Q \left(\tilde{f}-f\right)\varphi dxdt + \int_Q \left(\tilde{F}-F\right).\nabla\varphi dxdt + \int_0^T \langle \tilde{g}_1 - g_1, \varphi\rangle dt = \int_0^T \left\langle (g_2-\tilde{g}_2)_t, \varphi\right\rangle dt, \tag{130}$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$.
Since $g_2 - \tilde{g}_2 \in L^{p-}\left(0, T; W_0^{1,p(.)}(\Omega)\right)$, applying Theorem 1.1 in [13], we deduce that $g_2 - \tilde{g}_2 \in C\left([0,T]; L^1(\Omega)\right)$.
Since, by the integration by part formula, we have

$$\int_0^T \langle \varphi_t, g_2-\tilde{g}_2\rangle dt + \int_0^T \left\langle (g_2-\tilde{g}_2)_t, \varphi\right\rangle dt = \int_\Omega \varphi(0)(g_2-\tilde{g}_2)(0)dx,$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$, such that $\varphi(T) = 0$, then, from (129), we obtain

$$\int_\Omega \varphi(0)(g_2-\tilde{g}_2)(0)dx = 0.$$

Choosing $\varphi = (T-t)\psi$ with $\psi \in C_c^\infty(\Omega)$, we get

$$T\int_\Omega (g_2-\tilde{g}_2)(0)\psi dx = 0 \quad \text{for all} \quad \psi \in C_c^\infty(\Omega),$$

which implies that $(g_2-\tilde{g}_2)(0) = 0$□

References

[1] D.G. Aronson, Removable singularities for linear parabolic equations, *Arch. Rat. Mech. Anal.* **17** (1964), 79–84.

[2] M. Bendahmane, P. Wittbold, A. Zimmermann, Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1 data, *J. Diff. Equ.* **249** (2010), 1483–1515.

[3] L. Boccardo, T. Gallouët, L. Orsina, Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **13** (1996), 539–551.

[4] L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents,* Springer, 2010.

[5] J. Droniou, *Intégration et Espaces de Sobolev à valeurs vectorielles*, Lecture notes, Université de Provence, Marseille, 2001.

[6] J. Droniou, A. Porretta, A. Prignet, Parabolic Capacity and soft measures for nonlinear equations, *Potential Analysis* **19**, no. 2 (2003), 99–161.

[7] D.E. Edmunds, L.A. Peletier, Removable singularities of solutions of quasilinear parabolic equations, *J. London Math. Soc.* **2** (1970), no. 2, 273–283.

[8] X. Fan, D. Zhao, On the spaces $L^{p}(x)\,(\Omega)$ and $W^{m,p(x)}\,(\Omega)$, *J. Math. Anal. Appl.* **263** (2001), 424–446.

[9] E. Lanconelli, Sul problema di Dirichlet per l'equazione del calore, *Ann. Mat. Pura Appl.* **97** (1973), 83–114.

[10] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaire*, Dunod et Gauthier-Villars, (1969).

[11] J.-L. Lions, Sur certaines équations paraboliques non linéaires, *Bull. Soc. math. France* **93** (1969), 155–175.

[12] I. Nyanquini, S. Ouaro, S. Soma, Entropy solution to nonlinear multivalued elliptic problem with variable exponents and measure data, *Ann. Univ. of Craiova Math. Comp. Sci. Ser.* **40** (2013), no. 2, 174–198.

[13] A. Porretta, Existence results for nonlinear parabolic equations via strong convergence of truncations, *Ann. Mat. Pura ed Appl.* (IV) **177** (1999), 143–172.

[14] V. Rădulescu, Nonlinear elliptic equations with variable exponent: old and new, *Nonlinear Analysis TMA* **121** (2015), 336–369.

[15] V. Rădulescu, D. Repovš, *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*, CRC Press, Taylor and Francis Group, Boca Raton FL, 2015.

[16] J. Simon, Compact sets in $L^p(0,T;B)$, *Ann. Mat. Pura Appl.* **146** (1987), no. 4, 65–96.

[17] L. Wang, Y. Fan, W. Ge, Existence and multiplicity of solutions for a Neumann problem involving the $p\,(x)-$ Laplace operator, *Nonlinear Anal.* **71** (2009), 4259–4270.

[18] N. A. Watson, Thermal capacity *Proc. London Math. Soc.* **37** (1978), 342–362.

[19] J. Yao, Solutions for Neumann boundary value problems involving $p\,(x)-$ Laplace operator, *Nonlinear Anal.* **68** (2008), 1271–1283.

[20] C. Zhang, S. Zhou, Renormalized and entropy solutions for nonlinear parabolic equations with variable exponents and L^1 data, *J. Diff. Equations* **248** (2010), 1376–1400.

[21] D. Zhao, W.J. Qiang, X.L. Fan, On generalized Orlicz spaces $L^{p(x)}$, *J. Gansu Sci.* **9** (1997), no. 2, 1–7.

(Stanislas OUARO) Laboratoire de Mathématiques et Informatique (LAMI), UFR
Sciences Exactes et Appliquées, Université Ouaga 1 Pr Joseph KI-ZERBO
03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso
E-mail address: `souaro@univ-ouaga.bf, ouaro@yahoo.fr`

(Urbain TRAORE) Laboratoire de Mathématiques et Informatique (LAMI), UFR
Sciences Exactes et Appliquées, Université Ouaga 1 Pr Joseph KI-ZERBO
03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso
E-mail address: `urbain.traore@yahoo.fr`

Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 64–77
ISSN: 1223-6934

A Quasi-Uniformity On BCC-algebras

S. Mehrshad and N. Kouhestani

Abstract. We introduce a quasi-uniformity $\mathcal{U}$ on a BCC-algebra X by a family of ideals of X. If $T(\mathcal{U})$ is the topology induced by $\mathcal{U}$, we study some conditions under which $(X, T(\mathcal{U}))$ becomes a (semi)topological BCC-algebra. Also, we show that bicompletion of the quasi-uniformity $\mathcal{U}$ can be considered a $T(\mathcal{U}^\star)$-topological BCC-algebra which contains X as a sub-dense space.

2010 Mathematics Subject Classification. 06F35, 22A26 .
Key words and phrases. BCC-algebra, (semi)topological BCC-algebra, filter, Quasi-uniforme space, Bicompletion.

---

1. Introduction

In 1966, Y. Imai and K. Iséki in [13] introduced a class of algebras of type $(2, 0)$ called BCK-algebras which generalizes on one hand the notion of algebra of sets whit the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of impliction algebra. K. Iséki posed an interesting problem whether the class of BCK-algebras form a variety. In connection with this problem Y. Komori in [14] introduced a notion of BCC-algebras which is a generalization of notion BCK-algebras and proved that class of all BCC-algebras is not a variety. W.A. Dudek in [9] redefined the notion of BCC-algebras by using a dual form of the ordinary definition. Further study of BCC-algebras was continued [3, 6, 7, 8]. In 1937, André Weil in [17] introuduced the concept of a uniform space as a generalization of the concept of a metric space in which many non-topological invariants can be defined. The study of quasi-uniformities started in 1948 with Nachbin's investigations on uniform preordered spaces. In 1960, Á. Csaszar introduced quasi-uniform spaces and showed that every topological space is quasi-uniformizable. This result established an interesting analogy between metrizable spaces and topological spaces. quasi-uniform structures were also studied in algebraic structures. See for example [15]. In this paper, in section 3, we use of ideals of a BCC-algebra X to define a quasi-uniformity $\mathcal{U}$ on X. We show that $(X,\mathcal{U})$ is precompact but it is not T_1 and T_2. We prove that for each cardinal number α there is a T_0 quasi-uniform BCC-algebra. In section 4, by using of regular ideals we make the uniformity $\mathcal{U}^*$ on X and show that $(X, T(\mathcal{U}^*))$ is compact semi topological BCC-algebra, where $T(\mathcal{U}^*)$ is induced topology by $\mathcal{U}^*$ on X. Finally, we obtain $\mathcal{U}^*$- Cauchy filters and then construct a bicompletion BCC-algebra $(\widetilde{X},\widetilde{\mathcal{U}})$ of $(X,\mathcal{U})$ and prove that $(\widetilde{X}, T(\widetilde{\mathcal{U}}))$ is a topological BCC-algebra which has X as a sub-dense-BCC-algebra.

Received June 4, 2015.

2. Preliminary

2.1. Topological Space. Recall that a set A with a family $\mathcal{T}$ of its subsets is called a *topological space*, denoted by $(A, \mathcal{T})$, if $\mathcal{T}$ is closed under finite intersections and arbitrary unions. The members of $\mathcal{U}$ are called *open sets* of A and the complement of $A \in \mathcal{U}$, that is $A \setminus U$, is said to be a *closed set*. If B is a subset of A, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$ (or $cl_u B$). A subfamily $\{U_\alpha : \alpha \in I\}$ of $\mathcal{T}$ is said to be a *base* of $\mathcal{T}$ if for each $x \in U \in \mathcal{T}$ there exists an $\alpha \in I$ such that $x \in U_\alpha \subseteq U$, or equivalently, each U in $\mathcal{T}$ is the union of members of $\{U_\alpha\}$. A subset P of A is said to be a *neighborhood* of $x \in A$, if there exists an open set U such that $x \in U \subseteq P$. Let $\mathcal{U}_x$ denote the totality of all neighborhoods of x in A. Then a subfamily $\mathcal{V}_x$ of $\mathcal{U}_x$ is said to form a *fundamental system* of neighborhoods of x, if for each U_x in $\mathcal{U}_x$, there exists a V_x in $\mathcal{V}_x$ such that $V_x \subseteq U_x$. Topological space $(A, \mathcal{T})$ is said to be *compact*, if each open covering of A is reducible to a finite open covering, *locally compact*, if for each $x \in A$ there exist an open neighborhood U of x and a compact subset K such that $x \in U \subseteq K$. Also $(A, \mathcal{T})$ is said to be *disconnected* if there are two nonempty, disjoint, open subsets $U, V \subseteq A$ such that $A = U \cup V$, and connected otherwise, *totally disconnected* if each nonempty connected subset of A has one point only, *locally connected* if each open neighborhood of every point x contains a connected open neighborhood of x. The maximal connected subset containing a point of A is called the *component* of that point [2].

2.2. Quasi-Uniform Space. Let A be a non-empty set and $\emptyset \neq \mathcal{F} \subseteq P(A)$. Then $\mathcal{F}$ is called a *filter* on $P(A)$, if for each $F_1, F_2 \in \mathcal{F}$:
(i) $F_1 \in \mathcal{F}$ and $F_1 \subseteq F$ imply $F \in \mathcal{F}$,
(ii) $F_1 \cap F_2 \in \mathcal{F}$,
(iii) $\emptyset \notin \mathcal{F}$.
A subset $\mathcal{B}$ of a filter $\mathcal{F}$ on A is a *base* of $\mathcal{F}$ iff, every set of $\mathcal{F}$ contains a set of $\mathcal{B}$. If $\mathcal{F}$ is a family of nonempty subsets of A, then we denote generated filter by $\mathcal{F}$ with $fil(\mathcal{F})$.

A *quasi-uniformity* on a set A is a filter Q on $P(X \times X)$ such that
(i) $\triangle = \{(x, x) \in A \times A : x \in A\} \subseteq q$, for each $q \in Q$,
(ii) For each $q \in Q$, there is a $p \in Q$ such that $p \circ p \subseteq q$ where

$$p \circ p = \{(x, y) \in A \times A : \exists z \in A \ s.t \ (x, z), (z, y) \in p\}.$$

The pair (A, Q) is called a *quasi-uniform space*. If Q is a quasi-uniformity on a set A, then $q^{-1} = \{q^{-1} : q \in Q\}$ is also a quasi-uniformity on A called the *conjugate* of Q. It is well-known that if a quasi-uniformity satisfies condition: $q \in Q$ implies $q^{-1} \in Q$, then Q is a *uniformity*. Also Q is a uniformity on A provided

$$\forall q \in Q \ \exists p \in Q \ s.t \ p^{-1} \circ p \subseteq q.$$

Furthermore, $Q^* = Q \vee Q^{-1}$ is a uniformity on A. A subfamily $\mathcal{C}$ of quasi-uniformity Q is said to be a base for Q iff, each $q \in Q$ contains some member of $\mathcal{C}$. The topology $T(Q) = \{G \subseteq X : \forall x \in G \ \exists q \in Q \ s.t \ q(x) \subseteq G\}$ is called the topology induced by the quasi-uniformity Q [11].

Proposition 2.1. [11] *Let $\mathcal{C}$ be a family of subset of $X \times X$ such that*
(i) *$\triangle \subseteq B$, for each $B \in \mathcal{C}$;*

(ii) for $B_1, B_2 \in \mathcal{C}$, *there is a* $B_3 \in \mathcal{C}$ *such that* $B_3 \subseteq B_1 \cap B_2$*;*
(iii) for each $B \in \mathcal{C}$, *there is a* $C \in \mathcal{C}$ *such that* $C \circ C \subseteq B$.
Then there is the unique quasi-uniformity $\mathcal{U} = \{U \subseteq X \times X : \exists B \in \mathcal{C} : \ B \subseteq U\}$ *on* X *for which* $\mathcal{C}$ *is a base.*

Definition 2.1. [11] (*i*) A filter $\mathcal{G}$ on quasi-uniform space (A, Q) is called $Q^\star$-*Cauchy* filter if for each $U \in Q$, there is a $G \in \mathcal{G}$ such that $G \times G \subseteq U$.
(*ii*) A quasi-uniform space (A, Q) is called *bicomplete* if each $Q^\star$-Cauchy filter converges with respect to the topology $T(Q^\star)$.
(*iii*) A *bicompletion* of a quasi-uniform space (A, Q) is a bicomplete quasi-uniform space $(Y, \mathcal{V})$ that has a $T(\mathcal{V}^\star)$-dense subspace quasi-unimorphic to (A, Q).
(*iv*) A $Q^\star$-Cauchy filter on a quasi-uniform space (A, Q) is *minimal* provided that it contains no $Q^\star$-Cauchy filter other than itself.

Lemma 2.2. [11] *Let* $\mathcal{G}$ *be a* $Q^\star$*-Cauchy filter on a quasi-uniform space* (A, Q)*. Then, there is exactly one minimal* $Q^\star$*-Cauchy filter coarser than* $\mathcal{G}$*. Furthermore, if* $\mathcal{B}$ *is a base for* $\mathcal{G}$*, then* $\{q(B) : B \in \mathcal{B}$ *and* q *is a symetric member of* $Q^\star\}$ *is a base for the minimal* $Q^\star$*-Cauchy filter coarser than* $\mathcal{G}$*.*

Lemma 2.3. [11] *Let* (A, Q) *be a* T_0 *quasi-uniform space and* $\widetilde{A}$ *be the set of all minimal* Q^**-Cauchy filters on it. For each* $q \in Q$*, let*

$$\widetilde{q} = \{(\mathcal{G}, \mathcal{H}) \in \widetilde{A} \times \widetilde{A} : \exists G \in \mathcal{G} \text{ and } H \in \mathcal{H} \text{ s.t } G \times H \subseteq q\},$$

and $\widetilde{Q} = fil\{\widetilde{q} : q \in Q\}$*. Then the following statements hold:*
(i) $(\widetilde{A}, \widetilde{Q})$ *is a* T_0 *bicomplete quasi-uniform space and* (A, Q) *is a quasi-uniformly embedded as a* $T((\widetilde{Q^\star}))$*-dense subspace of* $(\widetilde{A}, \widetilde{Q})$ *by the map* $i : X \to \widetilde{A}$ *such that, for each* $x \in A$*,* $i(x)$ *is the* $T(Q^\star)$*-neighborhood filter at* x*. Furthermore, the uniformities* $(\widetilde{Q})^\star$ *and* $(\widetilde{Q^\star})$ *coincide.*
(ii) Any T_0 *bicomplete of* (A, Q) *is a quasi-unimorphic to* $(\widetilde{A}, \widetilde{Q})$*.*

In Lemma 2.3, (A, Q) is T_0 if $(x, y) \in \bigcap_{B \in \mathcal{C}} B$ and $(y, x) \in \bigcap_{B \in \mathcal{C}} B$ imply $x = y$, for each $x, y \in A$. Also (A, Q) is T_0 quasi-uniform space if and only if $(A, T(Q))$ is a T_0 topological space.

2.3. BCC- Algebra.

A BCC-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms, for all $x, y, z \in X$:
(1) $((x * y) * (z * y)) * (x * z) = 0$,
(2) $0 * x = 0$,
(3) $x * 0 = x$,
(4)$x * y = 0$ and $y * x = 0$ imply $x = y$.

A non empty subset S of BCC-algebra X is called subalgebra of X if it is closed under BCC-operation. For a BCC-algebra X, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$. On any BCC-algebra X one can define the natural order $\leq$ putting

$$x \leq y \Leftrightarrow x * y = 0$$

it is not difficult to verify that this order is partial and 0 is its smallest element.
In BCC-algebra X, following hold: for any $x, y, z \in X$
(5) $(x * y) * (z * y) \leq x * z$,
(6) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$,

(7) $x \wedge y \leq x, y$
(8) $x * y \leq x$
(9) $(x * y) * z \leq x * (y * z)$
(10) $x * x = 0$,
(11) $(x * y) * x = 0$. [8]

Definition 2.2. [4] Let X be a BCC-algebra and $\emptyset \neq I \subseteq X$. I is called an ideal of X if it satisfies the following conditions:
(12) $0 \in I$,
(13) $x * y \in I$ and $y \in I$ imply $x \in I$.

If I is an ideal in BCC-algebra of X, then I is a subalgebra. Moreover, if $x \in I$ and $y \leq x$, then $y \in I$. An ideal I is said to be *regular ideal* if the relation

$$x \equiv^I y \Longleftrightarrow x * y, y * x \in I$$

is a congruence relation. In this case we denote $x/I = \{y : x \equiv^I y\}$ and $X/I = \{x/I : x \in X\}$. X/I is a BCC-algebra by $x/I * y/I = (x * y)/I$.

3. A quasi-uniformity in BCC-algebras

In this section we let X be a BCC-algebra and η be an arbitrary family of ideals of X which is closed under intersection.

Definition 3.1. Let $\mathcal{T}$ be a topology on a BCC-algebra X. Then:
(i) $*$ is continuous in (first)second variable if $x * y \in U \in \mathcal{T}$, then there is a (V) $W \in \mathcal{T}$ such that $(x \in V)$ $y \in W$ and $(V * x \subseteq U)$ $x * W \subseteq U$. In this case, we also say $(X, *, \mathcal{T})$ is (right) left topological BCC-algebra.
(ii) $(X, *, \mathcal{T})$ is semitopological BCC-algebra if it is left and right topological BCC-algebra, i.e. if $x * y \in U \in \mathcal{T}$, then there are $V, W \in \mathcal{T}$ such that $x \in V, y \in W$ and $x * W \subseteq U$ and $V * y \subseteq U$.
(iii) $(X, *, \mathcal{T})$ is topological BCC-algebra if $*$ is continuous , i.e. if $x * y \subseteq U \in \mathcal{T}$, then there are two neighborhoods V, W of x, y, respectively, such that $V * W \subseteq U$.

Definition 3.2. A *quasi-uniform BCC-algebra* is a BCC-algebra endowed with a quasi-uniformity.

Theorem 3.1. *Let X be a BCC-algebra. The set $\mathcal{C} = \{I_L : I \in \eta\}$ is a base for a quasi-uniformity $\mathcal{U}$ on X, where $I_L = \{(x, y) \in X \times X : y * x \in I\}$.*

Proof. Let $I \in \eta$. Then $\triangle \subseteq I$, because for any $x \in X$, $x * x = 0 \in I$. Now we prove that $I_L \circ I_L \subseteq I_L$. Let $(x, y) \in I_L \circ I_L$. Then there exists $z \in X$ such that $(x, z) \in I_L$ and $(z, y) \in I_L$. Hence $z*x$ and $y*z$ are in I. Since $((y*x)*(z*x))*(y*z) = 0 \in I$ and $y*z \in I$, $(y*x)*(z*x) \in I$. Again since $z*x \in I$, we get that $y*x \in I$. This implies that $(x, y) \in I_L$ and so $I_L \circ I_L \subseteq I_L$. Since η is closed under intersection for each $I, J \in \eta$, $I_L \cap J_L = (I \cap J)_L \in \mathcal{C}$. Thus, $\mathcal{C}$ satisfies in conditions $(i), (ii), (iii)$ from Proposition 2.1. Hence $\mathcal{C}$ is a base for the quasi-uniformity $\{U \in X \times X : \exists I \in \eta \ s.t \ I_L \subseteq U\}$. □

Notation. From now on, $\mathcal{U}$ is the unifomity in Theorem 3.1 and $T(\mathcal{U}) = \{G \subseteq X : \forall x \in G \ \exists I \in \eta \ s.t \ I_L(x) \subseteq G\}$ is induced topology by it.

Example 3.1. Let $X = \{0, 1, 2, 3\}$ be a BCC-algebra with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	1
3	3	3	3	0

Then obviously $I_1 = \{0\}$,$I_2 = \{0, 1, 2\}$ and $I_3 = X$ are ideals of X. Clearly,

$$(I_1)_L = \triangle \cup \{(1, 0), (2, 0), (3, 0), (2, 1)\},$$

$$(I_2)_L = \triangle \cup \{(1, 0), (2, 0), (3, 0), (2, 1), (0, 1), (0, 2)\}$$

and $(I_3)_L = X \times X$. Therefore, by Theorem 3.1, $\mathcal{B} = \{(I_i)_L : i = 1, 2, 3\}$ is a base of the quasi-uniformity $\mathcal{U} = \{U \subseteq X \times X : \exists i \in \{1, 2, 3\}\ s.t\ (I_i)_L \subseteq U\}$ on X. Moreover $(I_1)_L(0) = \{0\}$, $(I_1)_L(1) = \{0, 1\}$ and $(I_1)_L(3) = (I_2)_L(3) = \{0, 3\}$. Also,

$$(I_2)_L(0) = (I_2)_L(1) = (I_1)_L(2) = (I_2)_L(2) = \{0, 1, 2\},$$

$$(I_3)_L(0) = (I_3)_L(1) = (I_3)_L(2) = (I_3)_L(3) = X,$$

Therefore $T(\mathcal{U}) = \{U \subseteq X \times X : \forall x \in U\ \exists i \in \{1, 2, 3\}\ s.t\ (I_i)_L(x) \subseteq U\}$.

Recall subset I of BCC-algera X is called BCC-ideal if $0 \in I$ and $(x*y)*z \in I$,$y \in I$ imply $x * z \in I$. In a BCC-algebra any BCC-ideal is an ideal. [7]

Lemma 3.2. *For any $I \in \eta$ and $x \in X$, define $I_L(x) = \{y \in X : y * x \in I\}$. Then following holds:*
(i) $0 \in I_L(x)$,
(ii) if $x \leq y$, *then* $I_L(x) \subseteq I_L(y)$,
(iii) if $y \in I_L(x)$, *then* $I_L(y) \subseteq I_L(x)$,
(iv) if $x \in I$, *then* $I_L(x) = I$,
(v) if $y \in I$, *then* $I_L(x * y) \subseteq I_L(x)$ *for each* $x \in X$,
(vi) if I *is a BCC-ideal and* $x \in I$, *then for any* $y \in X$, $I_L(x * y) \subseteq I_L(y)$.

Proof. *(i)* Since $0 = 0 * x \in I$, $0 \in I_L(x)$.
(ii) Let $z \in I_L(x)$. Then $z * x \in I$. Since $x \leq y$, by (2), $z * y \leq z * x$. Hence $z * y \in I$, which implies that $z \in I_L(y)$.
(iii) Let $z \in I_L(y)$. Then $z * y \in I$. Since $y \in I_L(x)$, $y * x \in I$. Now from $((z * x) * (y * x)) * (z * y) = 0$ we conclude that $z * x \in I$ and so $z \in I_L(x)$.
(iv) Since $x \in I$,

$$y \in I_L(x) \Leftrightarrow (x, y) \in I_L \Leftrightarrow y * x \in I \Leftrightarrow y \in I.$$

(v) Let $z \in I_L(x * y)$. Then $z * (x * y) \in I$. By (9), $(z * x) * y \leq z * (x * y)$. Therefore $(z * x) * y \in I$. Since $y \in I$, $z * x \in I$. Hence $z \in I_L(x)$.
(vi) Let $z \in I_L(x*y)$. Then $(z*x)*y \in I$. Since $x \in I$ and I is a BCC-ideal, $z*y \in I$. Hence $z \in I_L(y)$. □

Theorem 3.3. *$T(\mathcal{U})$ is the smallest topology on X which includes η and $(X, *, T(\mathcal{U}))$ is a right topological BCC-algebra.*

Proof. By Lemma 3.2 *(iii)*, it is easy to prove that $I_L(x) \in T(\mathcal{U})$, for each $x \in X$ and $I \in \eta$. Now let $x, y \in X$ and $x * y \in G \in T(\mathcal{U})$. Then there exists $I \in \eta$ such that $I_L(x * y) \subseteq G$. Let $z \in I_L(x)$. Since $z * x \in I$ and $((z * y) * (x * y)) * (z * x) = 0 \in I$, $(z*y)*(x*y)$ is in I and so $z*y \in I_L(x*y)$. Hence $I_L(x)*y \subseteq I_L(x*y)$. This implies that $*$ is contiuous in first variable. Now suppose $\mathcal{T}$ is a topology on X such that $*$ is continuous in first variable and $\eta \subseteq \mathcal{T}$. We show that $T(\mathcal{U}) \subseteq \mathcal{T}$. For this, given $x \in G \in T(\mathcal{U}^{\star})$. Then there exists $I \in \eta$ such that $I_L(x) \subseteq G$. Since $x * x = 0 \in I \in \mathcal{T}$, there exists $V \in \mathcal{T}$ such that $x \in V$ and $V * x \subseteq I$. If $z \in V$, then $z * x \in I$ and so $z \in I_L(x)$. Hence $x \in V \subseteq I_L(x) \subseteq G$. Thus $T(\mathcal{U}) \subseteq \mathcal{T}$. □

Recall a non zero element $a \in X$ is called an *atom* of a BCC-algebra if $x \leq a$ implies $x = 0$ or $x = a$. It is easy to see if $a \neq b$ are atoms, then $a * b = a$. [6]

Proposition 3.4. *If all non zero elements of BCC-algebra X are atoms, then:*
(i) for each $I \in \eta$ and $x \in X$, $I_L(x) = I$,
(ii) $(X, *, T(\mathcal{U}))$ is a topological BCC-algebra,
(iii) $(X,\mathcal{U})$ is a uniform space,

Proof. (i) The proof is obvious.
(ii) Let $x, y \in X$ and $x * y \in G \in T(\mathcal{U})$. Then there exists $I \in \eta$ such that $I_L(x * y) = I \subseteq G$. Now

$$x * y \in I_L(x) * I_L(y) = I * I \subseteq I \subseteq G.$$

(iii) Let $U \in \mathcal{U}$. Then there exists, $I \in \eta$ such that $I_L \subseteq U$. We claim that $I_L^{-1} \circ I_L \subseteq U$. Let $(x, y) \in I_L^{-1} \circ I_L$. For some a $z \in X$ we have $(x, z) \in I_L^{-1}$ and $(z, y) \in I_L$. Hence $x * z \in I$ and $y * z \in I$. Since x, y are atoms, $x, y \in I$. Therefore, $(x, y) \in I_L \subseteq U$. □

Recall that a quasi-uniform space (A, Q) is said to be *precompact* if for each $q \in Q$ there exist $x_1, x_2, ..., x_n \in A$ such that $A = \cup_{i=1}^{n} q(x_i)$. [11]

Proposition 3.5. *Let X be a BCC-algebra. The following conditions are equivalent:*
(i) the topological space $(X, T(\mathcal{U}))$ is compact,
(ii) the quasi-uniform space $(X,\mathcal{U})$ is precompact,
(iii) there exists $S = \{x_1, x_2, ..., x_n\} \subseteq X$ such that for all $a \in X$ and $I \in \eta$, $a * x_i \in I$, for some $x_i \in S$.

Proof. $(i) \Rightarrow (ii)$ it is clear.
$(ii) \Rightarrow (iii)$ Let $I \in \eta$. Since $(X,\mathcal{U})$ is precompact, there exist $x_1, x_2, ..., x_n \in X$ such that $X = \cup_{i=1}^{n} I_L(x_i)$. If $a \in X$, then there exists x_i such that $a \in I_L(x_i)$. Therefore $a * x_i \in I$.
$(iii) \Rightarrow (i)$ Let $X = \cup_{\alpha \in \Omega} G_\alpha$, where each G_α is an open set of X. Then for any $x_i \in S$ there exists $\alpha_i \in \Omega$ such that $x_i \in G_{\alpha_i}$. Since G_{α_i} is an open set, there exists $I \in \eta$ such that $I_L(x_i) \subseteq G_{\alpha_i}$, For any $a \in X$ by hypothesis $a * x_i \in I$ for some $x_i \in S$. Hence $a \in I_L(x_i) \subseteq G_{\alpha_i}$. Therefore, $X = \cup_{i=1}^{n} I_L(x_i) \subseteq \cup_{i=1}^{n} G_{\alpha_i}$. So $(X, T(\mathcal{U}))$ is compact. □

Proposition 3.6. *Let $\eta = \{I\}$. Then:*
(i) if I^c is a finite set, then topological space $(X, T(\mathcal{U}))$ is compact,
(ii) the set I is compact in topological space $(X, T(\mathcal{U}))$,
(iii) for any $x \in X$, $I_L(x)$ is compact set in topological space $(X, T(\mathcal{U}))$.

Proof. (i) Let $\{G_\alpha : \alpha \in \Omega\}$ be an open cover of X and $I^c = \{x_1, x_2, ..., x_n\}$. Then there exist $\alpha_0, \alpha_1, ..., \alpha_n \in \Omega$ such that $0 \in G_{\alpha_0}, x_1 \in G_{\alpha_1}, x_2 \in G_{\alpha_2}, ...x_n \in G_{\alpha_n}$. By (3), $I = I_L(0) \subseteq G_{\alpha_0}$, so $X = I \cup I^c \subseteq G_{\alpha_0} \cup G_{\alpha_1} ... \cup G_{\alpha_n}$.
(ii) Let $I \subseteq \bigcup_{\alpha \in \Omega} G_\alpha$, where each G_α is an open set of X. Since $0 \in I$, there is $\alpha \in \Omega$

such that $0 \in G_\alpha$. Then $I = I_L(0) \subseteq G_\alpha$. Hence I is a compact set in topological space $(X, T(\mathcal{U}))$.
(iii) Suppose $x \in X$ and $\{G_\alpha : \alpha \in \Omega\}$ an open cover of $I_L(x)$. Since $x \in I_L(x)$, there exists $\alpha \in \Omega$ such that $x \in G_\alpha$. Hence $I_L(x) \subseteq G_\alpha$. □

Let (A, Q) be a quasi-uniform space and $\mathcal{C}$ be a base for it. Recall (A, Q) is said to be T_1 quasi-uniform space if $\triangle = \bigcap_{B \in \mathcal{C}} B$ and T_2 quasi-uniform space if $\triangle = \bigcap_{B \in \mathcal{C}} B^{-1} o B$. [11]

Proposition 3.7. *quasi-uniform space* $(X, \mathcal{U})$ *is* T_0 *space iff,* $\{0\} \in \eta$. *But it is not* T_1 *and* T_2 *space.*

Proof. Let $(x, y), (y, x) \in \bigcap_{I \in \eta} I_L$. Hence $x * y \in I, y * x \in I$, for all $I \in \eta$. Since $\{0\} \in \eta$, $x * y = y * x = 0$. By (4), $x = y$. Hence $(X, \mathcal{U})$ is T_0 space. Conversely, let $(X, \mathcal{U})$ be T_0. Let $x \in \bigcap_{I \in \eta} I$. Then for each $I \in \eta$, $x * 0 = x$ and $0 * x = 0$, both, are in I. So $(x, 0), (0, x) \in \bigcap_{I \in \eta} I_L$. Since $(X, \mathcal{U})$ is T_0, $x = 0$. Hence $\bigcap_{I \in \eta} I = \{0\}$. Since η is closed under intersection, $\{0\} \in \eta$.
For any $y \in X$, $(y, 0) \in \bigcap_{I \in \eta} I_L$. Hence $\bigcap_{U \in \mathcal{U}} U \neq \triangle$ which implies that $(X, \mathcal{U})$ is not T_1 and T_2. □

Proposition 3.8. *Let for any* $a \in X$, $l_a : X \to X$ *by* $l_a(x) = a * x$ *be an open map. Then* $(X, T(\mathcal{U}))$ *is a* T_0 *space.*

Proof. Let $x, y \in X$ and $x \neq y$. By *(iv)* of Lemma 3.2, I is in $T(\mathcal{U})$, so $x * I$ and $y * I$ are open neighborhoods of x, y, respectively. We claim that $y \notin x * I$ or $x \notin y * I$. If $y \in x * I$ and $x \in y * I$, then there exist $z_1, z_2 \in I$ such that $x = y * z_1$ and $y = x * z_2$. By (8), $x \leq y$ and $y \leq x$. So $x * y = y * x = 0$. By(4), $x = y$. This is a contradiction. □

Proposition 3.9. *The following conditions are equivalent:*
(i) $(X, T(\mathcal{U}))$ *is a* T_0 *space,*
(ii) for every $0 \neq x \in X$ *there is* $I \in \eta$ *such that* $x \notin I$,
(iii) for each $0 \neq x \in X$ *there exists* $U \in T(\mathcal{U})$ *such that* $x \notin U$.

Proof. $(i \Rightarrow ii)$ Let $0 \neq x \in X$. Since $(X, T(\mathcal{U}))$ is T_0, there is an open neighborhood G of 0 such that $x \notin G$. As $0 \in G$, there is $I \in \eta$ such that $0 \in I \subseteq G$. Clearly $x \notin I$.
$(ii \Rightarrow iii)$ Because for each $I \in \eta$, I belongs $T(\mathcal{U})$, the proof is obvious.
$(iii \Rightarrow i)$ Let $x, y \in X$ and $x \neq y$. Then $x * y \neq 0$ or $y * x \neq 0$. Without the lost of generality, suppose $x * y \neq 0$. By hypothesis there exists $G \in T(\mathcal{U})$ such that $x * y \notin G$. Since $0 \in G$, there exists $I \in \eta$ such that $I = I_L(0) \subseteq G$. Since $(X, *, T(\mathcal{U}))$ is right topological BCC-algebra and $0 * x = 0$, there is $J \in \eta$ such that $J_L(0) * x \subseteq I$. Let $K = I \cap J$. We claim that $x \notin K_L(y)$. If $x \in K_L(y)$, then $x * y \in K \subseteq I \subseteq G$. This is a contradiction. Hence $(X, T(\mathcal{U}))$ is a T_0 space. Conversely, Let $0 \neq x \in X$. Since $(X, T(\mathcal{U}))$ is a T_0 space and each open set in $(X, T(\mathcal{U}))$ contains 0, there exists $U \in T(\mathcal{U})$ such that $x \notin U$. □

Let (A, Q) and (A^*, R) be quasi-uniform spaces.The map $f : (A, Q) \to (A^*, R)$ is called quasi-uniform continuous if for each $r \in R$ there exists $q \in Q$ such that $(x, y) \in q$ implies $(f(x), f(y)) \in r$. [11],[16]

Proposition 3.10. *Let* X *be a BCC-algebra and* $a \in X$. *The mapping* $r_a : (X, \mathcal{U}) \to (X, \mathcal{U})$ *given by* $r_a(x) = x * a$ *for all* $x \in X$ *is quasi-uniform continuous.*

Proof. Let $U \in \mathcal{U}$. Then there exists $I \in \eta$ such that $I_L \subseteq U$. Let $(x, y) \in I_L$. Since $y * x \in I$ and $(y * a) * (x * a) \leq (y * x)$, we get that $(y * a) * (x * a) \in I$ and so

$$(r_a(x), r_a(y)) = ((x * a), (y * a)) \in I_L \subseteq U. \qquad \square$$

Theorem 3.11. *For each $n \geq 4$, there exists a quasi uniform BCC-algebra of order n.*

Proof. Let $(X, *, 0)$ be a BCC-algebra and η be a family of ideals in X which is closed under intersection. By Theorem 3.1, there is a uniformity $\mathcal{U}$ on X. Suppose $a \notin X$ and $X' = X \cup \{a\}$. Then X' is a BCC-algebra by

$$x \otimes y = \begin{cases} x * y & \text{if } x, y \in X \\ a & \text{if } x = a, y = 0 \\ 0 & \text{if } x = a, y \neq 0 \\ x & \text{if } x \in X, y = a \end{cases} \tag{1}$$

We prove that for all $I \in \eta$, $I' = I \cup \{a\}$ is an ideal of X'. Clearly, $0 \in I'$. Let $x \otimes y \in I'$ and $y \in I'$. If $x, y \neq a$, then $x * y \in I$. Since I is an ideal in X and $y \in I$, we get that $x \in I \subseteq I'$. If $x = a$, clearly $x \in I'$. If $x \in X$ and $y = a$, then $x = x \otimes y \in I'$. Thus $\eta' = \{I' : I \in \eta\}$ is a family of ideals in X' which is closed under intersection. By Theorem 3.1, there is a uniformity $\mathcal{U}'$ on X'.
By Example 3.1, there is a quasi-uniform BCC-algebra of order 4. If $(X, *, 0, \mathcal{U})$ is a quasi-uniform BCC-algebra of order n, then by the above paragraph there is a quasi-uniform BCC-algebra of order $n + 1$. $\square$

Corollary 3.12. *For each $n \geq 4$, there is a right topological BCC-algebra of order n.*

Proof. By Theorems 3.11 and 3.3, the proof is obvious. $\square$

Theorem 3.13. *For each $n \geq 4$, there is a T_0 quasi-uniform BCC-algebra of order n.*

Proof. Let $(X, *, 0)$ be a BCC-algebra and $a \notin X$. Then $X^{'} = X \cup \{a\}$ is a BCC-algebra by

$$x \otimes y = \begin{cases} x * y & \text{if } x, y \in X \\ 0 & \text{if } x \in X, y = a \\ a & \text{if } x = a, y \in X \end{cases} \tag{2}$$

First we show that every ideal in X is an ideal in X'. Let I be an ideal in X, $x \otimes y \in I$ and $y \in I$. $x \neq a$ because $a * y = a \notin I$. Since $x * y \in I$, $x, y \in X$ and I is an ideal in X, we get that $x \in I$. Hence if η is a family of ideals in X which is closed under intersection it is in X' so. By Theorem 3.1, there are quasi-uniformities $\mathcal{U}$, $\mathcal{U}^{'}$ on X, $X^{'}$, respectively. By Proposition 3.7, $(X, \mathcal{U})$ is a T_0 quasi-uniform space iff $\{0\} \in \eta$ iff $(X^{'}, \mathcal{U}^{'})$ is T_0 quasi-uniform space.
Now by Example 3.1, $(X, \mathcal{U})$ is a T_0 quasi-uniform BCC-algebra of order 4. Let $(X, *, 0, \mathcal{U})$ be a T_0 quasi-uniform BCC-algebra of order n. Then by the above paragraph, we can find a quasi-uniform BCC-algebra $(X^{'}, \mathcal{U}^{'})$ of order n+1. $\square$

Theorem 3.14. *Let α be an infinite cardinal number. Then there is a T_0 quasi-uniform BCC-algebra of order α.*

Proof. Let X be a set with cardinal number α. Consider $X_0 = \{x_0 = 0, x_1, x_2, ...\}$ a countable subset of X. Define

$$x_i * x_j = \begin{cases} 0 & \text{if } i = j \\ x_i & \text{if } i \neq j. \end{cases} \tag{3}$$

Then $(X_0, *, 0)$ is a BCC-algebra. Let η be a collection of ideals in X_0 which is closed under intersection and contains $\{0\}$. Then by Theorem 3.1 and Proposition 3.7, there is a quasi-uniformity $\mathcal{U}_0$ on X_0 such that $(X_0, \mathcal{U}_0)$ is a T_0 quasi-uniform BCC-algebra. Now, define the binary operation $\otimes$ on X by

$$x \otimes y = \begin{cases} x * y & \text{if } x, y \in X_0 \\ 0 & \text{if } x \in X_0, y \notin X_0 \\ x & \text{if } x \notin X_0, y \in X_0 \\ 0 & \text{if } x = y \notin X_0 \\ x & \text{if } x \neq y \;\; x, y \notin X_0. \end{cases} \tag{4}$$

It is routine to check that X is a BCC-algebra of order α. Let $I \in \eta$ and $x, y \in X$ such that $x \otimes y \in I$ and $y \in I$. Then $y \in X_0$. If $x \in X_0$, then since I is an ideal in X_0 and $x * y = x \otimes y \in I$, we get that $x \in I$. If $x \notin X_0$, then $x = x \otimes y \in I$. This proves that η is a collection of ideals in X which is closed under intersection and contains $\{0\}$. Hence by Theorem 3.1 and Proposition 3.7, there is a T_0 quasi-uniformity $\mathcal{U}$ on X. □

Corollary 3.15. *If α is a cardinal number, then there is a T_0 right topological BCC-algebra.*

4. The bicompletion of topological BCC-algebra

In this section, we let X be a *BCC*-algebra and η be an arbitrary family of regular ideals of X which is closed under intersection and prove that for T_0 quasi-uniform BCC-algebra $(X,\mathcal{U})$, the bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$ admits the structure of a topological BCC-algebra such that X is a $T(\widetilde{\mathcal{U}})^{\star}$-dense sub BCC-algebra of $\widetilde{X}$.

Proposition 4.1. *Let I be a regular ideal of BCC-algebra X. Define $I_L^{-1} = \{(x, y) \in X \times X : (y, x) \in I_L\}$ and $I_L^{\star} = I_L \cap I_L^{-1}$. Then following holds:*
(i) $I_L^{-1} = \{(x, y) \in X \times X : x * y \in I\}$,
(ii) $I_L^{-1}(x) = \{y \in X : x * y \in I\}$,
(iii) $I_L^{-1}(0) = X$,
(iv) $I_L^{\star} = \{(x, y) \in X \times X : x \equiv^I y\}$,
(v) $I_L^{\star}(x) = \{y \in X : x \equiv^I y\} = x/I$,
(vi) *if* $x \in I$, *then* $I_L^{\star}(x) = I$,
(vii) $I_L^{\star}(I_L^{\star}(0)) = I_L^{\star}(0)$,
(viii) $I_L^{\star}(G * H) = I_L^{\star}(G) * I_L^{\star}(H)$.

Proof. The proofs (i),(ii),(iv),(v) and $(viii)$ are easy. To prove (iii), let $x \in X$. Since $0 * x = 0 \in I$, by (ii) , $x \in I_L^{-1}(0)$. So $X \subseteq I_L^{-1}(0)$.
(vi)

$$z \in I_L^{\star}(x) \Leftrightarrow z \equiv^I x \Leftrightarrow x * z \in I, z * x \in I \Leftrightarrow z \in I.$$

(vii) By (iv) we have

$$I_L^{\star}(I_L^{\star}(0)) = I_L^{\star}(I) = \{y \in X : \exists x \in I \; s.t \; y \equiv^I x\} = \{y \in X : y \in I\} = I = I_L^{\star}(0). \quad \square$$

Theorem 4.2. *There is a uniformity $\mathcal{U}^\star$ on X such that $(X, T(\mathcal{U}^\star))$ is a completely regular topological BCC-algebras, where $T(\mathcal{U}^\star)$ is the induced topology by $\mathcal{U}^\star$ on X.*

Proof. Let $\mathcal{B} = \{I_L^\star : I \in \eta\}$. As the proof of Theorem 3.1, we can show that $\mathcal{B}$ is a base for the quasi-uniformity $\mathcal{U}^\star = \{U \subseteq X \times X : \exists I \in \eta \ s.t \ I_L^\star \subseteq U\}$. We prove $\mathcal{U}^\star$ is a uniformity. For this we must show $U^{-1} \in \mathcal{U}^\star$, for all $U \in \mathcal{U}^\star$. Let $U \in \mathcal{U}^\star$. Then $I_L^\star \subseteq U$ for some $I \in \eta$. Since $I_L^\star = (I_L^\star)^{-1}$, $(I_L^\star)^{-1} \subseteq U$ and so $I_L^\star \subseteq U^{-1}$. This implies that $U^{-1} \in \mathcal{U}^\star$. Now suppose $T(\mathcal{U}^\star) = \{G \subseteq X : \forall x \in G \ \exists I \in \eta \ s.t \ I_L^\star(x) \subseteq U\}$ is the induced topology by $\mathcal{U}^\star$ on X. We will prove that $*$ is continuous. For this, suppose $x * y \in G \in T(\mathcal{U}^\star)$. Then there exists $I \in \eta$ such that $I_L^\star(x * y) \subseteq G$. Let $z \in I_L^\star(x) * I_L^\star(y)$. Then $z = \alpha * \beta$, for some $\alpha \in I_L^\star(x)$ and $\beta \in I_L^\star(y)$. Since $\alpha \equiv^I x$ and $\beta \equiv^I y$ and $\equiv^I$ is congruence relation, $x * y \equiv^I \alpha * \beta = z$. This implies that $z \in I_L^\star(x * y)$ and so $I_L^\star(x) * I_L^\star(y) \subseteq I_L^\star(x * y)$. Finally, since $T(\mathcal{U}^\star)$ is the induced topology by uniformity $\mathcal{U}^\star$, it is completely regular on X. □

Example 4.1. Let $(X, *, 0)$ be as BCC-algebra in example 3.1. It is easy to see that I_1, I_2 and I_3 are regular ideals of X. Hence $(I_1)_L^\star = \triangle$,

$$(I_2)_L^\star = \triangle \cup \{(0,1), (1,0), (0,2), (2,0), (1,2), (2,1)\}$$

and $(I_3)_L^\star = X \times X$. Therefore, $\mathcal{U}^\star = \{U \subseteq X \times X : \exists i \in \{1,2,3\} \ s.t \ (I_i)_L^\star \subseteq U\}$.

Example 4.2. Let $X = [0, \infty)$. Then X is a BCC-algebra with the following operation

$$x * y = \begin{cases} 0 & \text{if } x \leqslant y \\ x & \text{if } x > y. \end{cases} \tag{5}$$

Let $I_n = [0, n]$, for each $n \geq 1$. We show that I_n is a regular ideal. Let $(x * y) * z \in I_n$ and $y \in I_n$. If $y < x$, then $x * z = (x * y) * z \in I_n$. If $y \geq x$, then $x \in I_n$. Since $x * z$ is x or 0, we get that $x * z \in I_n$. Thus, I_n is a BCC-ideal and so is a regular ideal. Moreover,

$${I_n}_L^\star = \{(x, y) \in X \times X : x * y, y * x \leq n\} = \{(x, y) \in X \times X : x, y \in I_n\} = I_n \times I_n.$$

Now let $\eta = \{I_n : n \geq 1\}$. Then η is a family of regular ideals which is closed under intersection. By Theorem 4.2, $\mathcal{U}^\star = \{U \subseteq X \times X : \exists n \geq 1 \ s.t \ I_n \times I_n \subseteq U\}$.

A topological space A is *connected* if and only if it has only A and $\emptyset$ as closed and open subsets.

Proposition 4.3. *The space $(X, T(\mathcal{U}^\star))$ is connected if and only if $\eta = \{X\}$.*

Proof. Let $X \neq I \in \eta$ and $x \notin I$. It is clear that $I_L^\star(x) \in T(\mathcal{U}^\star)$. We show that $I_L^\star(x)$ is closed in this space. Let $y \in \overline{I_L^\star(x)}$. Then there is a $z \in I_L^\star(y) \cap I_L^\star(x)$. Hence $y \equiv^I z \equiv^I x$ which implies that $y \in I_L^\star(x)$. Obviously, $I_L^*(x)$ is nonempty. If $I_L^*(x) = X$, then 0 is in it and so $x \equiv^I 0$ which implies that $x \in I$, a contradiction. Thus, $I_L^*(x)$ is a nonempty proper subset of X which is closed and open. Hence this space is not connected. Conversely, let $\eta = \{X\}$. Then $T(\mathcal{U}^\star) = \{\emptyset, X\}$. Hence $(X, T(\mathcal{U}^\star))$ is connected. □

Recall quasi-uniform space (A, Q) is totally bounded, if for each $q \in Q$ there exist sets $S_1, S_2, ..., S_n$ such that $A = \bigcup_{i=1}^n S_i$ and for each $1 \leq i \leq n$, $S_i \times S_i \subseteq q$.[11],[16]

Proposition 4.4. *The following conditions are equivalent:*
(i) for each $I \in \eta$, X/I is finite,
(ii) $(X,\mathcal{U})$ is totally bounded,
(iii) $(X, T(\mathcal{U}^{\star}))$ is compact.

Proof. $(i \Rightarrow ii)$ Let for each $I \in \eta$, X/I be finite. We prove that $(X,\mathcal{U})$ is totally bounded. Let $I \in \eta$. Since X/I is finite, there are $x_1, x_2, ..., x_n \in X$ such that $X = \bigcup_{i=1}^{n} x_i/I$. For each $1 \leq i \leq n$, $x_i/I \times x_i/I \subseteq I_L$ because if $(x,y) \in x_i/I \times x_i/I$, then $x \equiv^I x_i \equiv^I y$ and so $(x,y) \in I_L$. This proves that $(X,\mathcal{U})$ is totally bounded.
$(ii \Rightarrow iii)$ Let $(X,\mathcal{U})$ be totally bounded and $I \in \eta$. There exist sets $S_1, S_2, ..., S_n$, such that $\bigcup_{i=1}^{n} S_i = X$ and for each $1 \leq i \leq n$, $S_i \times S_i \subseteq I_L$. Let $1 \leq i \leq n$ and $x, y \in S_i$. Since (x,y) and (y,x) are in I_L, we get $x \equiv^I y$. This proves that $S_i \subseteq x_i/I$, for some $x_i \in S_i$. Now to prove that $(X, T(\mathcal{U}^{\star}))$ is compact let $X = \bigcup_{\alpha \in \Omega} G_\alpha$, where each G_α is in $T(\mathcal{U}^{\star})$. Then there are $G_{\alpha_1}, ..., G_{\alpha_n}$ such that $x_i \in G_{\alpha_i}$ for each $1 \leq i \leq n$. Now suppose $x \in X$, then $x \in x_i/I$, for some $1 \leq i \leq n$ and so $x \in I_L^{\star}(x_i) \subseteq G_{\alpha_i}$.Therefore $X \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$, which shows that $(X, T(\mathcal{U}^{\star}))$ is compact.
$(iii \Rightarrow i)$ Let $I \in \eta$. Since $\{I_L^{\star}(x) : x \in X\}$ is an open cover of X in $T(\mathcal{U}^{\star})$, there are $x_1, x_2, ..., x_n \in X$ such that $X \subseteq \bigcup_{i=1}^{n} I_L^{\star}(x_i)$. Now it is easy to see that $X/I = \{x_1/I, ..., x_n/I\}$. □

Theorem 4.5. *Let $(X, *, \mathcal{T})$ be a semi topological BCC-algebra. If $\eta \subseteq \mathcal{T}$, then $T(\mathcal{U}^{*}) \subseteq \mathcal{T}$.*

Proof. Let $(X, *, \mathcal{T})$ be a semitopological BCC-algebra which includes η. Given $x \in G \in T(\mathcal{U}^{\star})$. Then there exists $I \in \eta$ such that $I_L^{\star}(x) \subseteq G$. Since $x * x = 0 \in I \in \mathcal{T}$, there exists $U \in \mathcal{T}$ such that $x \in U$ and $x * U, U * x \subseteq I$. If $z \in U$, then $x * z, z * x \in I$ and so $z \in I_L^{\star}(x)$. Hence $x \in U \subseteq I_L^{\star}(x) \subseteq G$. Thus $T(\mathcal{U}^{\star}) \subseteq \mathcal{T}$. □

Lemma 4.6. *Let $\mathcal{B}$ be a base for $\mathcal{U}^{\star}$-Cauchy filter $\mathcal{G}$ on quasi-uniform BCC-algebra $(X,\mathcal{U})$. Then the set $\{I_L^{\star}(B) : I \in \eta\ ,\ B \in \mathcal{B}\}$ is a base for a uniqe minimal $\mathcal{U}^{\star}$-Cauchy filter coarser than $\mathcal{G}$.*

Proof. By Lemma 2.2, the set $\{U(B) : B \in \mathcal{B}\ ,\ U \in \mathcal{U}^{\star}\}$ is a base for the unique minimal $\mathcal{U}^{\star}$-Cauchy filter $\mathcal{G}_0$ coareser than $\mathcal{G}$. Let $U \in \mathcal{U}^{\star}$ and $B \in \mathcal{B}$. Then for some $I \in \eta\ ,\ I_L^{\star} \subseteq U$. So $I_L^{\star}(B) \subseteq U(B)$. Now it is easy to prove that the set $\{I_L^{\star}(B) : I \in \eta\ ,\ B \in \mathcal{B}\}$ is a base for $\mathcal{G}_0$. □

Lemma 4.7. *η is a base for a minimal $\mathcal{U}^{\star}$-Cauchy filter $\mathcal{I}$ on quasi-uniform BCC-algebra $(X,\mathcal{U})$.*

Proof. Let $\mathcal{C} = \{S \subseteq X : \exists I \in \eta\ s.t\ I \subseteq S\}$. It is easy to prove that $\mathcal{C}$ is a filter with base η. To prove that $\mathcal{C}$ is a $\mathcal{U}^{*}$-Cauchy filter , let $U \in \mathcal{U}$. There is a $I \in \eta$ such that $I_L \subseteq U$. If $x, y \in I_L^{\star}(0)$, then $x \equiv^I y$ and so $(x,y) \in I_L^{\star} \subseteq I_L \subseteq U$. This proves that $I_L^{\star}(0) \times I_L^{\star}(0) \subseteq U$. By Proposition 4.1$(vi)$, $I \times I \subseteq U$. Hence $\mathcal{C}$ is a $\mathcal{U}^{\star}$-Cauchy filter. By Lemma 2.2, the set $\{I_L^{\star}(I_L^{\star}(0)) : I \in \eta\}$ is a base for the uniqe minimal $\mathcal{U}^{\star}$-Cauchy filter $\mathcal{I}$ coareser than $\mathcal{C}$. But by Proposition 4.1 (vii), $I_L^{\star}(I_L^{\star}(0)) = I_L^{\star}(0) = I$. Therefore, η is a base for $\mathcal{I} = \mathcal{C}$. □

Lemma 4.8. *Let $\mathcal{G}$ and $\mathcal{H}$ be $\mathcal{U}^{\star}$-Cauchy filters on X. Then $\mathcal{G} * \mathcal{H} = \{G * H : G \in \mathcal{G},\ H \in \mathcal{H}\}$ is a $\mathcal{U}^{\star}$-Cauchy filter base on X.*

Proof. Let $I \in \eta$. Since $\mathcal{G}$ and $\mathcal{H}$ are $\mathcal{U}^\star$-Cauchy filters, there are $G \in \mathcal{G}$ and $H \in \mathcal{H}$ such that $G \times G \subseteq I_L$ and $H \times H \subseteq I_L$. We show that $G * H \times G * H \subseteq I_L$. Let $g_1, g_2 \in G$ and $h_1, h_2 \in H$. Then, $(g_1, g_2), (g_2, g_1), (h_1, h_2), (h_2, h_1)$ are in I_L. So $g_1 \equiv^I g_2$ and $h_1 \equiv^I h_2$. Since $\equiv^I$ is congruence, $g_1 * h_1 \equiv^I g_2 * h_2$, which implies that $(g_1 * h_1, g_2 * h_2) \in I_L^\star$. □

Theorem 4.9. *There is a quasi-uniform space* $(\widetilde{X},\widetilde{\mathcal{U}})$ *of minimal* $\mathcal{U}^\star$*-Cauchy filers of quasi-uniform BCC-algebra* $(X,\mathcal{U})$ *that admits a BCC-algebra structure.*

Proof. Let $\widetilde{X}$ be the family of all minimal $\mathcal{U}^\star$-Cauchy filters of quasi-uniform BCC-algebra $(X,\mathcal{U})$. Let for each $U \in \mathcal{U}$,

$$\widetilde{U} = \{(\mathcal{G},\mathcal{H}) \in \widetilde{X} \times \widetilde{X} : \exists G \in \mathcal{G}\ ,\ H \in \mathcal{H}\ s.t\ G \times H \subseteq U\}.$$

If $\widetilde{\mathcal{U}} = fil\{\widetilde{U} : U \in \mathcal{U}\}$, then $(\widetilde{X},\widetilde{\mathcal{U}})$ is a quasi-uniform space of minimal $\mathcal{U}^\star$-Cauchy filters of $(X,\mathcal{U})$. Let $\mathcal{G}, \mathcal{H} \in \widetilde{X}$. Since $\mathcal{G}, \mathcal{H}$ are minimal $\mathcal{U}^\star$-Cauchy filters on X, then by Lemma 4.8, $\mathcal{G} * \mathcal{H}$ is $\mathcal{U}^\star$-Cauchy filter base on X. We define $\mathcal{G}\tilde{*}\mathcal{H}$ as the minimal $\mathcal{U}^\star$-Cauchy filter contained $\mathcal{G} * \mathcal{H}$. By Lemma 2.2, the set $\{I_L^*(G * H) : G \in \mathcal{G},\ H \in \mathcal{H},\ I \in \eta\}$ is a base of $\mathcal{G}\tilde{*}\mathcal{H}$. But by Proposition 4.1 $(viii)$, $I_L^*(G*H) = I_L^*(G)*I_L^*(H)$, so the set $\{I_L^*(G) * I_L^*(H) : G \in \mathcal{G},\ H \in \mathcal{H},\ I \in \eta\}$ is a base of it. Now we will prove that $(\widetilde{X}, \tilde{*})$ is a BCC-algebra. For this, we have to prove that:
(i) $((\mathcal{G}\tilde{*}\mathcal{H})\tilde{*}(\mathcal{K}\tilde{*}\mathcal{H}))\tilde{*}(\mathcal{G}\tilde{*}\mathcal{K}) = \mathcal{I}$
(ii) $I\tilde{*}\mathcal{G} = \mathcal{I}$
(iii) $\mathcal{G}\tilde{*}\mathcal{I} = \mathcal{G}$
(iv) $\mathcal{G}\tilde{*}\mathcal{H} = \mathcal{H}\tilde{*}\mathcal{G} = \mathcal{I} \Rightarrow \mathcal{G} = \mathcal{H}$
where $\mathcal{G}, \mathcal{H}, \mathcal{K} \in \widetilde{X}$, and $\mathcal{I}$ is minimal $\mathcal{U}^\star$-Cauchy filter in Lemma 4.7.
(i) Let $\mathcal{G}, \mathcal{H}, \mathcal{K} \in \widetilde{X}$. By Lemma 4.6, the set S_1 defined by

$$\{I_{1L}^\star(I_{2L}^\star(I_{3L}^\star(G_1*H_1)*I_{4L}^\star(K_1*H_2))*I_{5L}^\star(G_2*K_2)) : I_i \in \eta,\ G_i \in \mathcal{G},\ H_i \in \mathcal{H},\ K_i \in \mathcal{K}\}$$

is the base of minimal $\mathcal{U}^\star$-Cauchy filter $((\mathcal{G}\tilde{*}\mathcal{H})\tilde{*}(\mathcal{K}\tilde{*}\mathcal{H}))\tilde{*}(\mathcal{G}\tilde{*}\mathcal{K})$ and by Lemma 4.7, η is the base of minimal $\mathcal{U}^\star$-Cauchy filter $\mathcal{I}$. Let $I_{1L}^\star(I_{2L}^\star(I_{3L}^\star(G_1 * H_1) * I_{4L}^\star(K_1 * H_2)) * I_{5L}^\star(G_2 * K_2)) \in S_1$. Put $I = \bigcap_{j=1}^4 I_{jL}$, $G = G_1 \cap G_2$, $H = H_1 \cap H_2$ and $K = K_1 \cap K_2$. Then

$$I_L^\star(I_L^\star(I_L^\star(G * H) * I_L^\star(K * H)) * I_L^\star(G_* K))$$

is a subset of

$$I_{1L}^\star(I_{2L}^\star(I_{3L}^\star(G_1 * H_1) * I_{4L}^\star(K_1 * H_2)) * I_{5L}^\star(G_2 * K_2)) \in S_1.$$

Now since $((g * h) * (k * h)) * (g * k) = 0$, for each $g \in G,\ h \in H$ and $k \in K$, it is easy to prove that

$$I_L^\star(0) \subseteq I_L^\star(I_L^\star(I_L^\star(G * H) * I_L^\star(K * H)) * I_L^\star(G_* K)).$$

Hence $\mathcal{I} \subseteq ((\mathcal{G}\tilde{*}\mathcal{H})\tilde{*}(\mathcal{K}\tilde{*}\mathcal{H}))\tilde{*}(\mathcal{G}\tilde{*}\mathcal{K})$. Minimality $((\mathcal{G}\tilde{*}\mathcal{H})\tilde{*}(\mathcal{K}\tilde{*}\mathcal{H}))\tilde{*}(\mathcal{G}\tilde{*}\mathcal{K})$ implies that

$$\mathcal{I} = ((\mathcal{G}\tilde{*}\mathcal{H})\tilde{*}(\mathcal{K}\tilde{*}\mathcal{H}))\tilde{*}(\mathcal{G}\tilde{*}\mathcal{K}).$$

(ii) The sets $S_1 = \{I_L^\star(I_L^\star(0) * G) : I \in \eta\ , G \in \mathcal{G}\}$ and $\eta = \{I_L^\star(0) : I \in \eta\}$ are bases of minimal $\mathcal{U}^\star$-Cauchy filters $\mathcal{I}\tilde{*}\mathcal{G}$ and $\mathcal{I}$, respectively. But for each $I \in \eta$ and $G \in \mathcal{G}$, by Proposition 4.1 $(viii)$,

$$I_L^\star(I_L^\star(0) * G) = I_L^\star(I_L^\star(0)) * I_L^\star(G) = I_L^\star(0) * I_L^\star(G) = I_L^\star(0 * G) = I_L^\star(0).$$

So $S_1 = \eta$ and $\mathcal{I} = \mathcal{I}\tilde{*}\mathcal{G}$.
(iii) The sets $\{I^\star_L(G * I^\star_L(0)) : G \in \mathcal{G},\ I \in \eta\}$ and $\{I^\star_L(G) : G \in \mathcal{G}\}$ are the bases of $\mathcal{G}\tilde{*}\mathcal{I}$ and $\mathcal{G}$. For each $I \in \eta$ and $G \in \mathcal{G}$, by Proposition 4.1 $(viii)$,

$$I^\star_L(G * I^\star_L(0)) = I^\star_L(G) * I^\star_L(I^\star_L(0)) = I^\star_L(G) * I^\star_L(0) = I^\star_L(G * 0) = I^\star_L(G).$$

So $S_1 = S_2$ and hence $\mathcal{G} = \mathcal{G}\tilde{*}\mathcal{I}$.
(iv) The sets $S_1 = \{I^\star_L(G) : I \in \eta\ ,\ G \in \mathcal{G}\}$, $S_2 = \{I^\star_L(H) : I \in \eta\ ,\ H \in \mathcal{H}\}$, $S_3 = \{I^\star_L(G*H) : I \in \eta\ ,\ G \in \mathcal{G},\ H \in \mathcal{H}\}$, $S_4 = \{I^\star_L(H*G) : I \in \eta\ ,\ G \in \mathcal{G},\ H \in \mathcal{H}\}$ and $\eta = \{I^\star_L(0) : I \in \eta\}$ are the bases of $\mathcal{G}$, $\mathcal{H}$, $\mathcal{G}\tilde{*}\mathcal{H}$, $\mathcal{H}\tilde{*}\mathcal{G}$ and $\mathcal{I}$ respectively. Let $I^\star_L(G') \in\ S_1$. Since $\mathcal{G}\tilde{*}\mathcal{H} = \mathcal{H}\tilde{*}\mathcal{G} = \mathcal{I}$, $J^\star_L(G_0 * H_0) = K^\star_L(H_1 * G_1) = I^\star_L(0) = I$ for some $J, K \in \eta$. Let $G = G' \cap G_0 \cap G_1$ and $H = H_0 \cap H_1$. Now for each $g \in G$ and $h \in H$,

$$g * h \in J^\star_L(g) * J^\star_L(h) = J^\star_L(g * h) \subseteq J^\star_L(G * H) \subseteq J^\star_L(G_0 * H_0) = I.$$

Hence $g * h \in I$. With the similar argument we have $h * g \in I$. So $I^\star_L(g) = I^\star_L(h)$. Therefore, $I^\star_L(H) = I^\star_L(G) \subseteq I^\star_L(G')$. Hence $I^\star_L(G') \in \mathcal{H}$. So $\mathcal{G} \subseteq \mathcal{H}$. By minimality, $\mathcal{H} = \mathcal{G}$. ☐

Theorem 4.10. *If quasi-uniform BCC-algebra* $(X,\mathcal{U})$ *is a* T_0*,Then*
(i) $(\widetilde{X},\widetilde{\mathcal{U}})$ *is the bicompletion of* $(X,\mathcal{U})$.
(ii) X *is a sub BCC-algebra of* $\widetilde{X}$.
(iii) $(\widetilde{X}, T(\widetilde{\mathcal{U}^\star}))$ *is a topological BCC-algebra.*

Proof. (i) By Lemma 2.2 and Lemma 2.3 , $(\widetilde{X},\widetilde{\mathcal{U}})$ is the unique T_0 bicompletion quasi-uniform of $(X,\mathcal{U})$ and the mapping $i : X \to \widetilde{X}$ defined by

$$i(x) = \{W \subseteq X : W\ is\ a\ T(\widetilde{\mathcal{U}^\star}) - neighborhood\ of\ x\}$$

is a quasi-uniform embedded and $cl_{T(\widetilde{\mathcal{U}^\star})} i(X) = \widetilde{X}$.
(ii) Let $x, y \in X$. We shall prove that $i(x)\tilde{*}i(y) = i(x * y)$. By Lemma 2.3, the set

$$S = \{I^\star_L(W_x * W_y) : I \in \eta\ ,\ W_x\ W_y\ are\ T(\widetilde{\mathcal{U}^\star}) - neighborhoods\ x, y\}$$

is base for $i(x)\tilde{*}i(y)$. Since $I^\star_L(x * y) \subseteq I^\star_L(W_x\tilde{*}W_y)$ and $I^\star_L(x * y) \in i(x * y)$, we deduced that filter $i(x)\tilde{*}i(y)$ is contained in the filter $i(x * y)$. Since they are minimal $\mathcal{U}^\star$-Cauchy filters, $i(x)\tilde{*}i(y) = i(x * y)$. Hence X is a sub-BCC-algebra of $\widetilde{X}$.
(iii) By Lemma 2.3, $(\widetilde{\mathcal{U}})^\star = \widetilde{\mathcal{U}^\star}$. Hence

$$T(\widetilde{\mathcal{U}^\star}) = \{S \subseteq \widetilde{X} : \forall \mathcal{G} \in S\ \exists I \in \eta\ s.t\ \widetilde{I^\star_L}(\mathcal{G}) \subseteq S\}.$$

We prove that $(\widetilde{X}, T(\widetilde{\mathcal{U}^\star}))$ is a topological BCC-algebra. Let $\mathcal{G}\tilde{*}\mathcal{H} \in \widetilde{I^\star_L}(\mathcal{G}\tilde{*}\mathcal{H})$. We show that $\widetilde{I^\star_L}(\mathcal{G})\tilde{*}\widetilde{I^\star_L}(\mathcal{H}) \subseteq \widetilde{I^\star_L}(\mathcal{G}\tilde{*}\mathcal{H})$. Let $\mathcal{G}_1 \in \widetilde{I^\star_L}(\mathcal{G})$ and $\mathcal{H}_1 \in \widetilde{I^\star_L}(\mathcal{H})$. Then, there are $G \in \mathcal{G}, G_1 \in \mathcal{G}_1, H \in \mathcal{H}$ and $H_1 \in \mathcal{H}_1$ such that $G \times G_1 \subseteq I^\star_L$ and $H \times H_1 \subseteq I^\star_L$. By Lemma 2.3, $I^\star_L(G * H) \in \mathcal{G}\tilde{*}\mathcal{H}$ and $I^\star_L(G_1 * H_1) \in \mathcal{G}_1\tilde{*}\mathcal{H}_1$. We have to prove that $\mathcal{G}_1\tilde{*}\mathcal{H}_1 \in \widetilde{I^\star_L}(\mathcal{G}\tilde{*}\mathcal{H})$. For this, it is enough to show that $I^\star_L(G * H) \times I^\star_L(G_1 * H_1) \subseteq I^\star_L$. Let $y \in I^\star_L(G * H)$ and $y_1 \in I^\star_L(G_1 * H_1)$. Then, $y \equiv^I g * h$ and $y_1 \equiv^I g_1 * h_1$ for some $g \in G, g_1 \in G_1, h \in H, h_1 \in H_1$. Since $(g, g_1), (h, h_1)$ are in $I^\star_L$, we get $g * h \equiv^I g_1 * h_1$. Hence $(y, y_1) \in I^\star_L$. ☐

5. Conclusion

In this paper on a BCC-algebra of X we introduced the quasi-uniformity $\mathcal{U}$ induced by a family η of BCC-ideals of X. We studied some properties of topological space $(X, T(\mathcal{U}))$. Next researches can study the following assertions:
(1) separation axioms on $(X, T(\mathcal{U}))$ and $(X, T(\mathcal{U}^*))$,
(2) quasi-uniform continuouty of the operation of X in quasi-uniform space $(X,\mathcal{U})$,
(3) quasi-uniform continuous homomorphisms on $(X,\mathcal{U})$,
(4) quasi-uniform quotient BCC-algebras.

References

[1] R. A. Borzooei, G. R. Rezaei, N. Kouhestani, On (semi)topological BL-algebra, *Iranian Journal of Mathematical Sciences and Informatics* **6** (2011), no. 1, 59–77.
[2] N. Bourbaki, *Elements of mathematics general topology*, Addison-Wesley Publishing Company, 1966.
[3] W.A. Dudek, A new characterization of ideals in BCC-algebras, *Novi Sad J. Math.* **29** (1999), no. 1, 1–6.
[4] J. Hao, Ideal theory of BCC-algebras, *Sci. Math. Japo.* **3** (1998), 373–381.
[5] W.A. Dudek, Initial segments in BCC-algebras, *Mathematica Moravica* **4** (2000), 27–34.
[6] W.A. Dudek, X.Zhang, On atoms in BCC-algebras, *Discussiones Mathematicae ser. Algebra and Stochastic Methods* **15** (1995), 81–85.
[7] W.A. Dudek, On ideal and congruences in BCC-algebras, *Czecho. Math. J.* **48(123)** (1998), 21–29.
[8] W.A. Dudek, On proper BCC-algebras, *Bull. Inst. Math. Acad. Sinica* **20** (1992), 137–150.
[9] W.A. Dudek, On BCC-algebras, *Logique et Analyse* **129-130** (1990),103–111.
[10] W.A. Dudek, Subalgebras in finite *BCC*-algebras, *Bull. Inst. Math. Acad. Sinica* **28** (2000), 201–206.
[11] P. Fletcher, W.F. Lindgren, *Quasi-uniform Spaces*, Lecture Notes in Pure and Applied Mathematics 77, Marcel Dekker, New York, 1982.
[12] M. Haveshki, E. Eslami, A. Borumand Saeid, *A topology induced by uniformity on BL-algebras*, Math. Log. Quart. **53** (2007), no. 2, 162–169.
[13] Y. Imai, K. Iséki, On axioms system of propositional calculi XIV, *Poc. Japan Acad.* **42** (1966), 19–22.
[14] Y. Komori, The variety generated by BCC-algebras is finitely based, *Reports Fac. Sci, Shizuoka Univ.* **17** (1983), 13–16.
[15] H.P.A. Künzi, J. Marin, S. Romaguera, Quasi-Uniformities on Topological semigroups and Bicompletion, *Semigroups Forum* **62** (2001), 403–422.
[16] M.G. Murdeshwar, S.A. Naimpally, *Quasi-Uniform Topological Spaces*, Noordhoff, Groningen, 1966.
[17] A. Weil, *sur les espaces a structuure uniforme et sur la topologibgeneral*, Gauthier-Villars, Paris, 1973.

(S. Mehrshad) Department of Mathematics, University of Zabol, Zabol, Iran
E-mail address: `smehrshad@uoz.ac.ir`

(N. Kouhestani) Fuzzy Systems Research Center, University of Sistan and Baluchestan, Zahedan, Iran
E-mail address: `Kouhestani@math.usb.ac.ir`

Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 78–86
ISSN: 1223-6934

New approach to the calculation of fractal dimension of the lungs

K. Lamrini Uahabi and M. Atounti

Abstract. In the present work, using a new approach, we will calculate the fractal dimension of the lungs, through a precise technique of mathematical simulation. Note that the main objective of this study is to contribute to further confirm the idea of the fractal structure of the lungs; structure permitted by the phenomenon of self-similarity.

2010 Mathematics Subject Classification. 28A80, 92B05.
Key words and phrases. Fractals, fractal dimension, modeling, lungs, self-similarity, gas exchange surface.

1. Introduction

The lungs are essential organs of respiration. Located in the rib cage, they provide the necessary gas exchange between air and blood. Man has two lungs, a right lung comprises three lobes and the left lung has two lobes and a location on its inner face which corresponds to the place of the heart. The lungs are enveloped by the pleura, thin membrane which prevents the friction against the chest wall. The pleura is constituted by two sheets separated from one another by a small amount of serous fluid that allows the lungs to move during breathing. Below the lungs, there is the diaphragm, which separates the rib cage and the abdomen which is a very important muscle in breathing.

The air that reaches the lungs passes through the pharynx, larynx and trachea extending into the lungs by two bronchi, one right and one left, which in turn are divided into bronchi increasingly small, like branches of a tree.

The smallest bronchi, namely the bronchioles lead to tiny bags that are filled with air, the pulmonary alveoli. The lungs of an adult are provided with an average of 600 millions alveoli, which represent a gas exchange surface of about 140 m^2 [12].

Breathing helps both to bring the blood oxygen which live cells and ridding the blood of carbon dioxide contained therein.

On the inspiration (see Figure 1), the diaphragm contracts and descends, the chest expands, the outside air, full of oxygen enters the lungs and fills the cells (flow). It is through tiny vessels that form the capillary network that will make exchanges between alveoli and blood: the oxygen of the air contained in the alveoli passes into the blood to reach the red blood cells and heart which sends it throughout the body.

Upon expiration, the diaphragm rises, thorax decreases in volume and air, charged with carbon dioxide is expelled to the trachea (ebb). In fact, the carbon dioxide of

---

Received November 11, 2015.

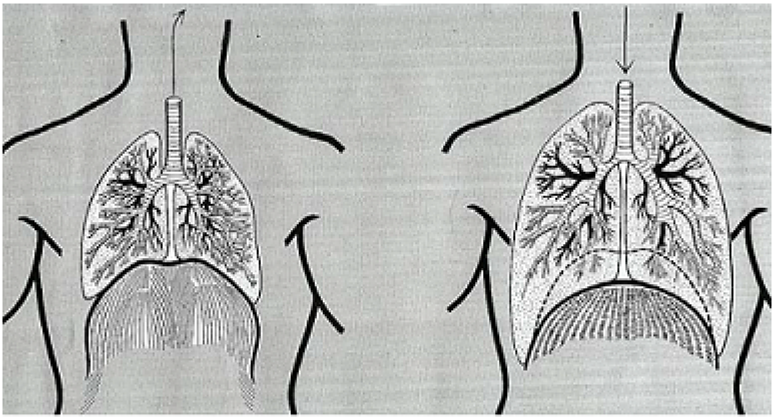

FIGURE 1. Variations in lung volume during inspiration-expiration.

blood takes the opposite route passing from the blood to the alveoli and thence to the outside air.

A particularity in the lungs; the oxygen-rich blood sent from the heart to all the tissues of the body passes through the arteries (the red blood), and the blood responsible for carbon dioxide returns to the heart through the veins (the blue blood). In the lungs it is the opposite: the carbon dioxide-rich blood goes from the heart to the lungs through the pulmonary artery, the oxygen-rich blood from the lungs goes to the heart through the pulmonary vein.

Lung actually allows gas exchange by the combination of several factors such as tissue elasticity, fluid mechanics, the diffusion through the membranes, etc. The complexity of these phenomena entails the coordination of sometimes contradictory effects in appearance [14]. We don't necessarily know all the experimentally measured quantities desired, and we cannot repeat the experiments to infinity to refute or confirm certain assumptions. Hence it would be very wise to model [5]. It is therefore obvious that mathematical modeling of these phenomena would be very useful for an accurate approach. This is exactly the great interest of fractal geometry. Indeed, the Euclidean geometry is powerless in solving such problems; it is applicable only in the case of smooth and regular shapes. Thus, a point has a dimension equal to zero, a line has a dimension one, a plane has a dimension two and a volume has a dimension equal to three. In contrast, fractal geometry, meanwhile, deals with non-integer dimensions [1], [19], for example ranging between one and two and between two and three, etc. The fractal dimension is actually the size of irregular curves [7], [8]. And it is this specificity that opens huge horizons in vast areas and in the medical field in particular.

What means the word "fractal"? The fractals are mathematical objects sufficiently broken and irregular. Another property of the term "fractal" is self-similarity: the fractal objects have the same information at different scales, i.e. similar and identical reproductions at scales smaller and smaller.

The main tool in the study of fractals is the notion of fractal dimension, in its many forms. Given the diversity and complexity of these fractals, there are several definitions of fractal dimensions and which do not always coincide. Among others, the Hausdorff dimension, the box dimension and the dimension of self-similar sets. They are used mainly to measure "the degree of irregularity" or "filling rate" for example, in the plane of a fractal curve. It will stick in the present paper to the self-similarity dimension.

The pulmonary anatomy has a branching tree structure. This tree, shown in Figure 2, has a self-similar phenomenon in large structures like small. The small sizes of trees are similar to those of the larger size. Therefore, the lung has a fractal structure, allowing optimization (compromise) and thus a mathematical modeling of the lungs and the respiratory function [15], through fractal analysis [11].

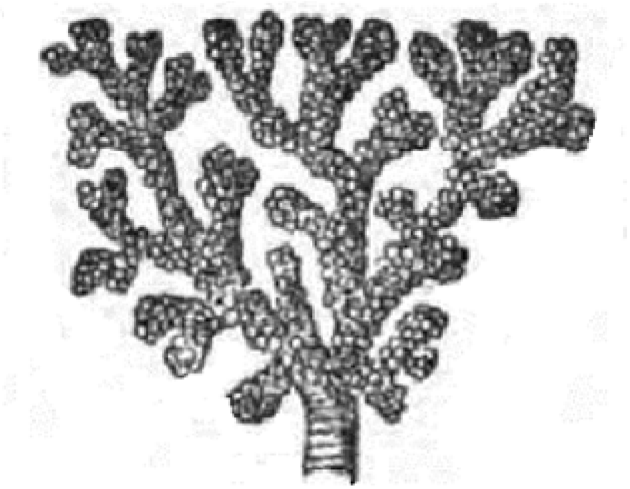

FIGURE 2. The fractal appearance of the bronchial ramifications.

The objective of this work is to determine a modeling technique in order to estimate the fractal dimension of the human lungs. So, after a brief overview on the pulmonary anatomy and respiratory physiology in the first section, we will attempt in Section 2 to show and highlight the fractal structure of the lungs. Thereafter and in Section 3 we will, by a new approach, calculate the fractal dimension of the lungs using a modeling technique based on one of the variants of the Von Koch curve. We will close this paper with the conclusions and discussions.

2. Fractal dimension and fractal structure of lungs

2.1. Fractal dimension. To evaluate the fractal dimension, several methods are proposed in the literature. We will use in this work the self-similarity dimension. This method applies to fractal curves and figures with the property of self-similarity, such that the various component parts are constructed by successive iterations with a same reduction factor q. According to Sapoval [18], the self-similarity dimension corresponds to the logarithm of the number of pieces needed to cover the object, relative to the logarithm of the report of enlargement by aligning the pieces with the initial object.

More generally, according to Gouyet [4] and Falconer [2], this fractal dimension D_F is given by the following relation:

$$D_F = -\frac{\log N}{\log q}, \tag{1}$$

where N is the decomposition factor and q is the single reduction factor from one stage to another (which can be also written as $D_A = \frac{\log N}{\log \frac{1}{q}}$, where $1/q$ is the report of enlargement).

Note that the position of the elements in the set is not involved. Only count their numbers and their relative sizes. Different shapes may have the same dimension.

2.2. Fractal structure of lungs. The role of the bronchial tree is driving the ambient air, rich in oxygen and low in carbon dioxide, to the exchange surface with the blood within the acini. Figure 3 shows the molding of a human lung (according to Weibel [22]). In yellow, there are the airways, in red, the pulmonary veins and in blue, pulmonary arteries. The complexity of the structure is striking. One can observe the tree geometry of the lung, and specifically, that this tree is almost dichotomous. This observation allows us to consider it as a succession of generations, see Figure 4 (according to Weibel [22]). The first generation is the largest branch, the trachea. It has a diameter of about two centimeters. The latter is located at the bottom of the acinus, at the twenty-third generation; it has a diameter on the order of half a millimeter. The number of branches of this tree is roughly 2^{24}, i.e. more than sixteen millions. The bronchi and bronchioles, until the $17th$ generation of the tree, are structures whose the only role is to ensure the air conduction to the latest generations. Rather asymmetrical at first, especially because of the presence of the heart on the left, dichotomous bifurcations become quickly practically homothetic from one generation to the next [21].

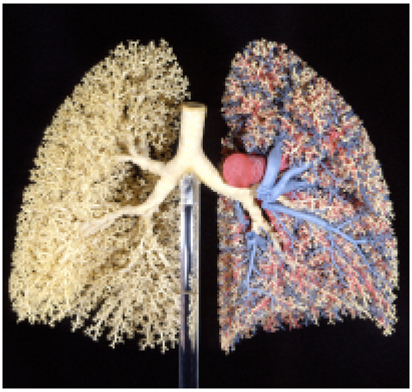

FIGURE 3. Molding of human lung.

The lower ducts, from the seventeenth generation, correspond to the breathing zone of lung, where gas exchange with blood held. Over there, we find neither cartilage nor smooth muscle. They are grouped into acini in number approximately 30000 by lung, dichotomous sub-trees, to six generations which the alveolar bags cling. These

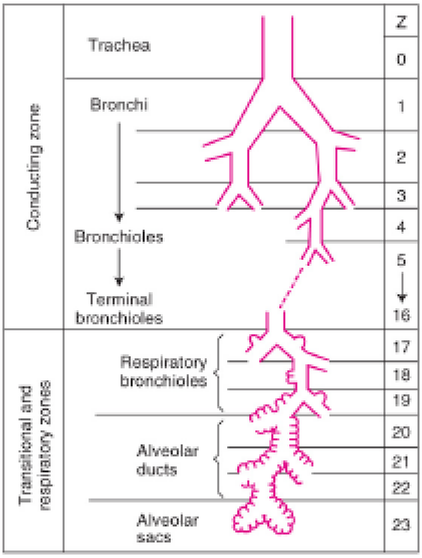

FIGURE 4. The different functional zones of lung airways.

bags are like clusters of alveoli and form the last generation of the lung. In fact, the channel diameter is constant in this region [21].

In the human adult, the gas exchange surface is of 140 square meters, which is the equivalent of half a tennis court. The area occupied by the lungs is small compared to the surface which allows gas exchange. The bronchial tree; which is a fractal structure, allows to increase the gas exchange area in a very significant way. Although the surface of the lungs involved in gas exchange is not infinite, it is considered the lungs and their self-similarity as a natural example of fractal. It enables major gas exchange on a yet reduced lung volume. This impressive gain of surface and space is proof of the interest of a fractal organization adopted by nature. The major advantage of these mathematical structures is that they allow a computer modeling of the lungs, and therefore a real quantification of the exchange zone [14].

3. Numerical simulation and results

3.1. Modeling technique. Lung bronchi are hollow tubes that branch like the branches of a tree, to distribute air evenly to both lungs. The trachea which leads air to the lungs falls within the thorax to be divided into two main bronchi, one for each lung. By phenomenon of self-similarity, bronchi then divide about 23 times (see Figure 4) to get the air to the alveoli [13]. The lung is a real fractal structure. We can mathematically model this fractal structure of the lung, using one of the variants of the Von Koch curve [2], [10]. The construction of this curve is based on the basic principle illustrated in Figure 5. Let $[AB]$ be a line segment of unit length. The transformation consists in removing the segment $[CD]$ and replacing it by the other tow sides of the isosceles triangle based on the removed segment (see Figure 5). Obtaining the points C, E, D from the points A and B is based on the following

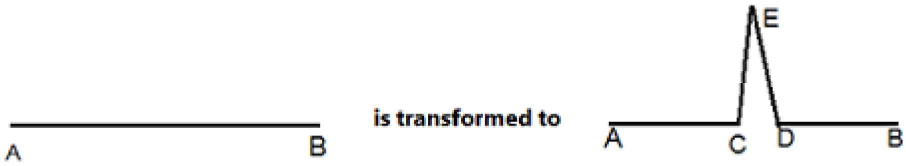

FIGURE 5. Principal and technique of modeling.

formulas:

$$AC = \frac{5}{11}AB \quad \text{and} \quad AD = \frac{6}{5}AC, \tag{2}$$

such that $AC = CE = ED = DB$ and that $(CD, CE) = \alpha$.
Express the two equalities of equation (2) in cartesian coordinates :

$$\begin{aligned} x_C &= \frac{5}{11}(x_B - x_A) + x_A \quad \text{and} \quad y_C = \frac{5}{11}(y_B - y_A) + y_A, \\ x_D &= \frac{6}{5}(x_C - x_A) + x_A \quad \text{and} \quad y_D = \frac{6}{5}(y_C - y_A) + y_A. \end{aligned}$$

Now, to be able to trace this fractal curve, we need to determine the coordinates of the point E. For this, we express CE in terms of CD. In fact, we have

$$CD = AD - AC = \frac{1}{5}AC.$$

However, in length $AC = CE$, which allows to assert that $CE = 5CD$. It follows that CE is the image of $5CD$ by rotation $\Re(C, \alpha)$. Therefore, the coordinates of point E are:

$$\begin{aligned} x_E &= 5\left[(x_D - x_C)\cos(\alpha) - (y_D - y_C)\sin(\alpha)\right] + x_C \\ y_E &= 5\left[(x_D - x_C)\sin(\alpha) + (y_D - y_C)\cos(\alpha)\right] + y_C. \end{aligned}$$

In the other hand, it is possible to give an approximate value of α.
Indeed $CD = \frac{1}{11}$ then $\frac{1}{2}CD = \frac{1}{22}$. So, we can apply trigonometric formula to get:

$$\cos(\alpha) = \frac{1}{2}\frac{CD}{CE} = \frac{1}{10}.$$

Then, an approximate value in radians of α is $\alpha \approx 1,47 rad$.
Thus, it is possible to carry out the iterations of this fractal representing the structure of the lung (see Figure 6).

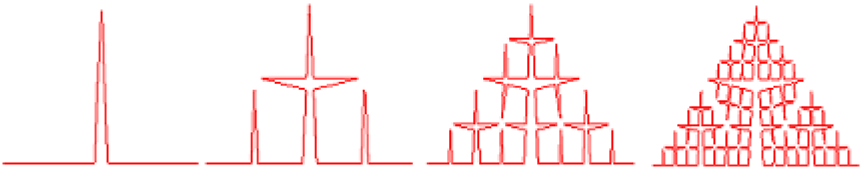

FIGURE 6. Modeling curve: construction steps.

The fourth iteration corresponds to the function of the air sacs composed of lung alveoli for gas exchange.

3.2. Results. The main artery divides into two and then the following arteries divide into two as well. This process is repeated 23 times. In the end, we get about 2^{23} separations of arteries, i.e. 8388608 arterioles, namely the equivalent of sixteen millions of branches.

Alveoli have a diameter $L_{al} = 0.2\ mm$. For the calculations, we use a cube of side $I = L_{al}$ to represent alveoli. On this cube, five faces will be allocated to the exchange surface while the last face cannot be used because it is necessary that the air enters the alveoli. Thus, we can estimate an exchange surface S_{al} by alveoli as

$$S_{al} = 5 \times (0.2)^2 = 2 \times 10^{-3}\ cm^2. \tag{3}$$

From an anatomical point of view, there are 2^{16} acini in lungs, i.e. 65536 acini. Knowing too that the total exchange surface of the lungs is about $140m^2$ [12]. Then, the exchange surface S_{ac} for each acini can be calculated as

$$S_{ac} = \frac{140 \times 10^4}{65536} = 21,36\ cm^2. \tag{4}$$

On the other hand, the pulmonary volume of a man is about 5 liters. It follows that the volume of each acini is

$$V_{ac} = \frac{5 \times 10^3}{65536} = 0,076\ cm^3. \tag{5}$$

Hence, from equation (5), the size L_{ac} of each acini is approximately equal to:

$$L_{ac} = \sqrt[3]{0,076} = 4,23\ mm. \tag{6}$$

To calculate the fractal dimension D_F of the lungs, we proceed as follows:
For each acini, stacking the alveoli on a fractal of dimension D_F, to get a total exchange surface S_{ac}. So, the number N of small cubes stacked on each acini is given by the relation

$$N = \frac{S_{ac}}{S_{al}}. \tag{7}$$

Replacing the values obtained from equations (3) and (4) in equation (7), we deduce that

$$N = 10680. \tag{8}$$

On the other side, from equation (6), the reduction factor q is expressed as

$$q = \frac{L_{al}}{L_{ac}} = \frac{0,2}{4,23}. \tag{9}$$

Then, the factor q is approximately equal to $0,04$. The fractal dimension D_F is expressed as

$$D_F = -\frac{\log N}{\log q}.$$

Consequently, the value of D_F can be calculated as follows:

$$D_F = -\frac{\log(10680)}{\log(0,04)}. \tag{10}$$

The fractal dimension of the lungs is therefore

$$D_F = 2,88. \tag{11}$$

4. Conclusions and discussions

According to the calculations and mathematical simulation used in this work, we arrive at a value of the fractal dimension of lungs of 2.88. This value is indeed a non-integer dimension (non-Euclidean) and therefore confirms the fractal structure of the lungs. The literature cited others values of fractal dimension of lungs, different from ours, but always non-integer.

In [16], Nelson and Manchester found that the fractal dimension D_F of the lungs varies between 2.64 and 2.76, this using the airway lengths as the measuring stick. As for Nelson, West and Goldenberg [17], from experimental data, leaded to values slightly lower. Indeed, they found $D_F = 2.4$, based on power scaling of the airways' length and $D_F = 2.26$ when the basis was the airways'diameter. Afterwards, Weibel [20] achieved a value $D_F = 2.35$, based on scaling of the average airways'diameter.

This heterogeneity of the results is directly related to the different experimental methods used and the choice of mathematical modeling type. But nevertheless, all these values share non-Euclidean property and confirm all "fractality" of the pulmonary system. Note that this fractal property of the lungs occurs very early, even when the formation of the lungs during the fetal stage. This fractal structure gives the lungs very advantageous properties, all working for a fundamental objective: an area of maximum gas exchange in a very small volume.

If have used the geometry of a sphere (Euclidean geometry), for example, to increase the surface, it would increase the radius. However, the lungs are in a closed and limited environment (rib cage). Counting the gas exchange surface of a human, there are approximately 140 square meters. Note that for the case of a sphere, the radius of the lung should measure a gigantic value of 3.3 meters !

The surface of lungs is tiny relative to the gas exchange surface as possible. Consequently, fractals allow living beings and for man in particular to have a very efficient respiratory system, while having a realistic volume.

It would be ideal and very wise to find a unified value of fractal dimension of lungs, which would do a "biological constant." Such an outcome would be very useful in the medical field and especially in the pulmonary diseases include among others asthma, emphysema, respiratory failure, amputation of lung lobes (case of cancers), ... Moreover, such a value would have a major impact in the sporting field, including monitoring and evaluation of performance of athletes high levels.

References

[1] G. Cherbit, *Fractals: Dimensions non entires et applications*, Masson, Paris, 1991.

[2] K.J. Falconer, *Fractal geometry: Mathematical foundations and applications*, John Wiley & Sons London, 1990.

[3] L. Goldberger, Non-linear dynamics for clinicians: chaos theory, fractals and complexity at the bed side, *Lancet.* **346** (1996), 1312–1314.

[4] J.F. Gouyet, *Physique et structures fractales*; Masson, Paris, 1992.

[5] C. Grandmont, Inspiration mathmatique : la modlisation du poumon, Brochure "Mathmatiques, l'explosion continue" (2013), 43–49.

[6] S. Kyriacos, Réseaux vasculaires : Analyse fractale et modélisation de la croissance, Thèse de Doctorat en Sciences Pharmaceutiques, Faculté de Pharmacie, Universit de Montréal, Canada, 1997.

[7] K. Lamrini Uahabi, *Fractal dimension of the family of trinomial arcs M(p,k,r,n)*, *International Mathematical Forum* **3** (2008), no. 6, 291–298.

[8] K. Lamrini Uahabi, M. Zaoui, *Rsolution des quations trinomiales*, *Ann. Sci. Math. Qubec* **28** (2004), no. 1-2, 189–197.

[9] P.M. Lannaccone, M. Khokha, *Fractal geometry in biological systems - An analytical approach*, CRC Press. Inc., 1996.

[10] B.B. Mandelbrot, *Les objets fractals*, Flammarion, 4ème dition, 1995.

[11] B.B. Mandelbrot, *The fractal geometry of nature*, W.H. Freeman and company, 1983.

[12] E.N. Marieb, K.N. Hoehn, *Human anatomy and physiology*, Pearson, 10th edition, 2014.

[13] B. Mauroy, Hydrodynamique dans le poumon, relations entre flux et géométries, Thèse de Doctorat, Ècole Normale Supérieure de Cachan, 2004.

[14] B. Mauroy, M. Filoche, E.R. Weibel, B. Sapoval,, An optimal bronchial tree may be dangerous, *Nature* **427** (2004), 633–636.

[15] B. Mauroy, M. Filoche, J.S. Andrade Jr., B. Sapoval, Interplay between geometry and flow distribution in an airway tree, *Phys. Rev. Lett.* **90** (2003), 148101, 1–4.

[16] T.R. Nelson, D.K. Manchester, Modeling of lung morphogenesis using fractal geometries, *IEEE Transactions on Medical Imaging* **7** (1988), 321–327.

[17] T.R. Nelson, B.J. West, A.L. Goldenberg,, The fractal lung: Universal and species-related scaling patterns, *Experimentia Basel* **46** (1990), 251–254.

[18] B. Sapoval, *Universalités et fractales, jeux d'enfant ou délits d'initié*, Flammarion, Paris, 1997.

[19] C. Tricot, *Courbes et dimension fractale, Springer-Verlag*, Editions Sciences et Culture, Paris, 1993.

[20] E.R. Weibel, Fractal geometry: A design principle for living organisms, *American Journal of Physiology* **261** (*Lung Cellular and Molecular Physiology* **5**) (1991), L361–L369.

[21] E.R. Weibel, *Morphometry of the human lung*, Springer-Verlag, 1963.

[22] E.R. Weibel, *The pathway for oxygen: structure and function in the mammalian respiratory system*, Harvard Univ. Press, 1984.

(Kaoutar Lamrini Uahabi, Mohamed Atounti) LABORATORY OF APPLIED MATHEMATICS AND INFORMATION SYSTEMS, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, MULTIDISCIPLINARY FACULTY OF NADOR, UNIVERSITY MOHAMMED FIRST OF MOROCCO, SELOUANE - NADOR, MOROCCO
E-mail address: `lamrinika@yahoo.fr, atounti@hotmail.fr`

Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 87–93
ISSN: 1223-6934

Multiple solutions for a Robin problem involving the $p(x)$-biharmonic operator

Abdesslem Ayoujil and Abdel Rachid El Amrouss

Abstract. This article is devoted to the solvability of Robin boundary problem involving the $p(x)$-biharmonic operator with two parameters. Using as main tool a result due to Ricceri, we obtain the existence of at least three nontrivial solutions.

2010 Mathematics Subject Classification. 35J65, 35J60, 47J30, 58E05.
Key words and phrases. $p(x)$-biharmonic equation; Navier boundary condition; Multiple solutions; Ricceri's variational principle.

---

1. Introduction

In recent years, various mathematical problems with variable exponent growth condition have been received considerable attention (see [5, 10, 16]). The interest in studying such problems arise from nonlinear elasticity theory, electrorheological fluids (cf. [19, 22]) and image processing (cf. [4]). We point out that, this kind of problems have been the subject of a large literature and many results have been obtained. We can cite, among others, the articles [1, 2, 3, 9, 13, 17, 21] and references therein for details.

Here, we are concerned with the following fourth-order quasilinear elliptic equation with Robin boundary conditions

$$\begin{aligned} \Delta^2_{p(x)}u = \lambda f(x,u) + \mu g(x,u), \quad \text{in } \Omega,\\ |\Delta u|^{p(x)-2}\frac{\partial u}{\partial \nu} + m(x)|u|^{p(x)-2}u = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, $p \in C(\overline{\Omega})$ with $p(x) > 1$ for all $x \in \overline{\Omega}$, $\Delta^2_{p(x)}u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic operator of fourth order, f, $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, $\lambda, \mu > 0$ are real numbers and $m \in L^\infty(\Omega)$ with $ess\, inf_{x\in\Omega} m(x) = m_0 > 0$.

Precise that elliptic equations involving the $p(x)$-biharmonic equations are not trivial generalizations of similar problems studied in the constant case since the $p(x)$-biharmonic operator is not homogeneous and, thus, some techniques which can be applied in the case of the p-biharmonic operators will fail in that new situation, such as the Lagrange Multiplier Theorem.

Received January 14, 2016.

To our best of knowledge, there seems few results about multiple solutions to $p(x)$-biharmonic equation. Although a natural extension of the theory, the problem addressed here is a natural continuation of recent papers. In [15], for the $p(x)$-laplacian Neumann problem, authors have obtained at least three weak solutions, which generalizes the corresponding result of [12]. In [6], the authors show the existence of at least three solutions to a Navier boundary problem involving the $p(x)$-biharmonic operator.

Motivated by the above papers and the ideas introduced in [15], the purpose of this work is to extend the results of [15] to the case of $p(x)$-biharmonic equation with Robin boundary condition. Our technical approach is an adaptation of variational method. More precisely, we assume $f(x,u)$ and $g(x,u)$ satisfies the following conditions:

($\mathbf{f_1}$)

$$|f(x,s)| \leq a_1 + a_2|s|^{\alpha(x)-1}, \quad \forall (x,s) \in \Omega \times \mathbb{R},$$

($\mathbf{g_1}$)

$$|g(x,s)| \leq b_1 + b_2|s|^{\beta(x)-1}, \quad \forall (x,s) \in \Omega \times \mathbb{R},$$

for some $\alpha, \beta \in C_+(\overline{\Omega})$ with $\alpha^+ < p^-$ and a_i, b_i $(i = 1,2)$ are positive constants, where

$$C_+(\overline{\Omega}) := \Big\{ p \in C(\overline{\Omega}) : \quad p(x) > 1, \forall x \in \overline{\Omega} \Big\},$$

and

$$h^- = \min_{x \in \overline{\Omega}} h(x), \quad h^+ = \max_{x \in \overline{\Omega}} p(x) \text{ for any } \in C_+(\overline{\Omega}).$$

($\mathbf{f_2}$)

$$\begin{aligned} &|f(x,s)| < 0, \quad \text{for } s \in (0, s_0), \\ &|f(x,s)| > M > 0, \quad \text{for } s \in (s_0, +\infty), \end{aligned}$$

where M and s_0 are positive constants.

Using the three critical points theorem of Ricceri [18] which is a powerful tool to study boundary problem of differential equation (see, for example, [3, 14]), we prove that problem 1 has at least three weak solutions for λ sufficiently large and requiring μ small enough.

The paper consists of three sections. In the the second section, we list some well known definitions, basic properties, recall some background facts concerning generalized Lebesgue-Sobolev spaces and introduce some notations used below. In third section, we recall Ricceri's three critical points theorem at first, then prove our main result.

2. Preliminaries and main result

For completeness, we introduce some theories of Lebesgue-Sobolev space with variable exponent. The detailed description can be found in, for example, [7, 8, 11, 20, 21].

For any $p \in C_+(\overline{\Omega})$, as in the constant exponent case, define the generalized Lebesgue space by

$$L^{p(x)}(\Omega) := \Big\{ u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty \Big\}.$$

Equipped with the so called Luxemburg norm

$$|u|_{p(x)} = \inf \big\{\nu > 0 : \int_\Omega |\frac{u(x)}{\nu}|^{p(x)} dx \leq 1\big\}$$

the space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a separable and reflexive Banach space.

For any positive integer k, the generalized Sobolev space $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) = \big\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \leq k\big\}.$$

Endowed with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x)},$$

$W^{k,p(x)}(\Omega)$ is also a separable and reflexive Banach space.

For any $x \in \overline{\Omega}$ and $k \geq 1$,

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ \infty & \text{if } kp(x) \geq N, \end{cases}$$

denote the critical exponent. Obviously, $p(x) < p_k^*(x)$ for all $x \in \overline{\Omega}$

Proposition 2.1. [7] *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$), there is a continuous and compact embedding $W^{k,p(x)}(\overline{\Omega}))$ into $L^{r(x)}(\overline{\Omega}))$.*

Define

$$\|u\|_m = \inf \big\{\nu > 0 : \rho\Big(\frac{u}{\nu}\Big) \leq 1\big\} \qquad \text{for } u \in X,$$

with

$$\rho_m(u) = \int_\Omega |\Delta u|^{p(x)} dx + \int_{\partial\Omega} m(x)|u|^{p(x)} d\sigma, \qquad \text{for } u \in X,$$

where $d\sigma$ is the measure on the boundary $\partial\Omega$. In view of $m_0 > 0$, it is easy to see that $\|.\|_m$ which will be used, is a norm equivalent to the norm $\|.\|_{2,p(x)}$. Moreover, similar to [7, Theorem 3.1], we have

Proposition 2.2. *The following statements hold true:*

(1) $\rho_m\Big(u\Big/|u|_{p(x)}\Big) = 1$.

(2) $\|u\|_m < 1 (= 1, > 1) \iff \rho_m(u) < 1 (= 1 > 1)$.

(3) $\|u\|_m < 1 \Longrightarrow \|u\|_m^{p^+} \leq \rho_m(u) \leq \|u\|_m^{p^-}$.

(4) $\|u\|_m > 1 \Longrightarrow \|u\|_m^{p^-} \leq \rho_m(u) \leq \|u\|_m^{p^+}$.

Here, problem (1) is stated in the framework of the generalized Sobolev space $X := W^{2,p(x)}(\Omega)$. A function $u \in X$ is said to be a weak solution of problem (1) if

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v dx + \int_{\partial\Omega} m(x)|u|^{p(x)-2}uv d\sigma = \lambda \int_\Omega f(x,u)v dx + \mu \int_\Omega g(x,u)v dx,$$

for all $v \in X$.

Now, we can state our main result as follows.

Theorem 2.3. *If* $(\mathbf{f_1})$, $(\mathbf{f_2})$ *hold and* $\frac{N}{2} < p^-$. *Then, there exist an open interval $\Lambda \subseteq (0, +\infty)$ and a positive real number $\rho > 0$ such that each $\lambda \in \Lambda$ and every function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ which satisfying* $(\mathbf{g_1})$, *there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$ problem* (1) *has at least three solutions whose norms are less than ρ.*

3. Proof of main result

Throughout the sequel, the letters $a_i, i = 1, 2, ...,$ denote positive constants which may vary from line to line but are independent of the terms which will take part in any limit process.

To prove the existence of at least three weak solutions for each of the given problem (1), we will use the revised form of Ricceri's three critical points theorem stated as follows.

Theorem 3.1. [18] *Let X be a reflexive real Banach space. $\Phi\colon X \to \mathbb{R}$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X' and Φ is bounded on each bounded subset of X; $\Psi\colon X \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact; $I \subseteq \mathbb{R}$ an interval. Assume that*

(i) $\lim_{\|x\|\to+\infty}(\Phi(x) + \lambda\Psi(x)) = +\infty,$ *for all* $\lambda > 0$,

(ii) *there exist* $r \in \mathbb{R}$ *and* $u_0, u_1 \in X$ *such that* $\Phi(u_0) < r < \Phi(u_1)$,

(iii) $\displaystyle\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) > \frac{\big(\Phi(u_1) - r\big)\Psi(u_0) + \big(r - \Phi(u_0)\big)\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}.$

Then, there exists an open interval $\Lambda \subseteq (0, \infty)$ and a positive real number ρ with the following property: for every $\lambda \in \Lambda$ and every C^1 functional $J\colon X \mapsto \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the equation

$$\Phi'(x) + \lambda\Psi'(x) + \mu J'(x) = 0$$

has at least three solutions in X whose norms are less than ρ.

Let $H : X \to \mathbb{R}$ be the energy functional corresponding to problem (1) defined by

$$H(u) = \Phi(u) + \lambda\Psi(u) + \mu J(u), \tag{2}$$

where

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx + \int_{\partial\Omega} \frac{m(x)}{p(x)}|u|^{p(x)}d\sigma, \tag{3}$$

$$\Psi(u) = -\int_\Omega F(x, u)dx, \tag{4}$$

$$J(u) = -\int_\Omega G(x, u)dx, \tag{5}$$

where $F(x, u) = \int_0^u f(x, s)ds$ and $G(x, u) = \int_0^u g(x, s)ds$.

It is well known that $\Phi, \Psi, J \in C^1(X, R)$ with the derivatives given by

$$\langle\Phi^{'}(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v dx + \int_{\partial\Omega} m(x)|u|^{p(x)-2}uvdx,$$

$$\langle\Psi^{'}(u), v\rangle = -\int_\Omega f(x, u)vdx,$$

$$\langle J^{'}(u), v\rangle = -\int_\Omega g(x, u)vdx,$$

for any $u, v \in X$.

Arguments similar to those used in the proof of [1, Proposition 4.2], we have the following

Proposition 3.2. $\Phi^{'} : X \to X^{'}$ *is a*

1. *continuous, bounded, of type* $(S)^{+}$ *and strictly monotone operator,*
2. *homeomorphism.*

Now, it is enough to verify that Φ, Ψ and J satisfy the hypotheses of Theorem 3.1. Obviously, by proposition 3.2, $(\Phi^{'})^{-1} : X^{'} \to X$ exists and continuous. Moreover, in view of $(f1)$ and [11], $\Psi^{'}, J^{'} : X \to X^{'}$ are completely continuous, which imply $\Psi^{'}$ and $J^{'}$ are compact. Thus, the precondition of Theorem 3.1 is satisfied. It remains to verify that the conditions (i), (ii) and (iii) are fulfilled.

First, we claim that condition (i) is satisfied. In fact, by Proposition 2.2, we have

$$\Phi(u) = \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \frac{m(x)}{p(x)} |u|^{p(x)} d\sigma \geq \frac{1}{p^{+}} \|u\|_{m}^{p^{-}}, \tag{6}$$

for every $u \in X$ with $\|u\|_{m} > 1$.

On the other hand, due to the assumption $(f1)$, we have

$$|F(x,s)| \leq a_1 |s| + \frac{a_2}{\alpha(x)} |s|^{\alpha(x)}, \quad \text{a.e.} x \in \Omega, \forall s \in \mathbb{R}.$$

Therefore

$$\begin{aligned} \Psi(u) &= -\int_{\Omega} F(x,u) dx \geq -a_1 \int_{\Omega} |u| dx - a_2 \int_{\Omega} \frac{1}{\alpha(x)} |u| dx^{\alpha(x)} \\ &\geq -a_3 \|u\|_{m} - \frac{a_2}{\alpha^{+}} \int_{\Omega} \big(|u|^{\alpha^{+}} + |u|^{\alpha^{-}} \big) dx = -a_3 \|u\|_{m} - a_4 \big(|u|_{\alpha^{+}}^{\alpha^{+}} + |u|_{\alpha^{-}}^{\alpha^{-}} \big). \end{aligned}$$

Since X is continuously embedded in $L^{\alpha^{+}}(\Omega)$ and $L^{\alpha^{-}}(\Omega)$, it follows

$$\Phi(u) \geq -a_3 \|u\|_{m} - a_5 \big(\|u\|_{m}^{\alpha^{+}} + \|u\|_{m}^{\alpha^{-}} \big). \tag{7}$$

So, combining the two inequalities (6) and (7), for any $\lambda > 0$ we obtain

$$\Phi(u) + \lambda \Psi(u) \geq \frac{1}{p^{+}} \|u\|_{m}^{p^{-}} - \lambda a_3 \frac{1}{p^{+}} \|u\|_{m} - \lambda a_5 \big(\|u\|_{m}^{\alpha^{+}} + \|u\|_{m}^{\alpha^{-}} \big),$$

for $u \in X$ with $\|u\|_{m} > 1$. As $1 < \alpha^{+} < p^{-}$, one has $\lim\limits_{\|u\|_{m} \to \infty} \Phi(u) + \lambda \Psi(u) = \infty$ and the condition (i) is verified.

Secondly, we will verify the conditions (ii). Precise that, from assumption $(\mathbf{f_2})$, $F(x,t)$ is increasing for $t \in (s_0, 1)$ and decreasing for $t \in (0, s_0)$, uniformly with respect to x. Moreover, $F(x,t) \to \infty$ as $t \to \infty$, so, there exists a real number $\delta > s_0$ such that

$$F(u,t) \geq 0 = F(u,0) \geq F(u,s), \quad \forall u \in X, t > \delta, s \in (0, s_0).$$

Furthermore, since $\frac{N}{2} < p^{-}$, there is a continuous embedding of X into $W^{2,p^{-}}(\Omega)$ which is continuously embedded in $C(\overline{\Omega})$. Then, there exists a constant $k > 0$ such that

$$\|u\|_{\infty} := \sup_{x \in \overline{\Omega}} |u(x)| \leq k \|u\|_{m}, \quad \forall u \in X. \tag{8}$$

Let choose A and B two real numbers such that $0 < A < \min\{t_0, k\}$ and $B > \delta$ satisfying

$$B^{p^{\pm}} \|m\|_{L^{1}(\partial\Omega)} > 1, \quad \text{where} \quad p^{\mp} = \begin{cases} p^{-}, & \text{if } B > 1, \\ p^{+}, & \text{if } B < 1. \end{cases}$$

Thus, for $t \in [0, A]$, we have $F(x,t) \leq F(x,0)$ which implies

$$\int_\Omega \sup_{t\in[0,A]} F(x,t)dx \leq \int_\Omega F(x,0)dx = 0. \tag{9}$$

Since $B > \delta$, we can get $\int_\Omega F(x,B)dx > 0$ and so,

$$\frac{A^{p^+}}{k^{p^+}B^{p^\mp}} \int_\Omega F(x,B)dx > 0. \tag{10}$$

Next, consider $u_0, u_1 \in X$ with $u_0(x) = 0$ and $u_1(x) = B$ for any $x \in \Omega$. Obviously, $\Phi(u_0) = \Psi(u_0) = 0$ and

$$\Phi(u_1) = \int_{\partial\Omega} \frac{1}{p(x)} m(x) B^{p(x)} dx \geq \frac{B^{p^\mp}}{p^+}\|m\|_{L^1(\partial\Omega)} > \frac{1}{p^+} > \frac{1}{p^+}\Big(\frac{A}{k}\Big)^{p^+}$$

Consequently, if we put $r = \frac{1}{p^+}\Big(\frac{A}{k}\Big)^{p^+}$, it follows

$$\Phi(u_0) < r < \Phi(u_1),$$

which ensures the condition (ii).

Finally, we will show the condition (iii). A simple calculation yields

$$\begin{aligned} -\frac{\big(\Phi(u_1) - r\big)\Psi(u_0) + \big(r - \Phi(u_0)\big)\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)} =& - r\frac{\Psi(u_1)}{\Phi(u_1)} \\ =& r\frac{\int_\Omega F(x,B)dx}{\int_{\partial\Omega} \frac{1}{p(x)} m(x) B^{p(x)} dx} > 0 \end{aligned} \tag{11}$$

Now, let $u \in \Phi^{-1}((-\infty, r])$. Then, $I_m(u) \leq rp^+ = \Big(\frac{A}{k}\Big)^{p^+} < 1$ which, by Proposition 3.2, implies $\|u\|_m < 1$. Consequently,

$$\frac{1}{p^+}\|u\|_m^{p^+} \leq \Phi(u) < r.$$

Therefore, by 8, we infer that

$$|u(x)| \leq \|u\|_\infty \leq k\|u\|_m \leq k\Big(rp^+\Big)^{1/p^+} = A, \quad \forall x \in \Omega,$$

for all $u \in X$ with $\Phi(u) \leq r$. The above inequality shows that

$$- \inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) = \sup_{u\in\Phi^{-1}((-\infty,r])} -\Psi(u) \leq \int_\Omega \sup_{t\in[0,A])} F(x,t)dx \leq 0.$$

From (11), we deduce that

$$- \inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) < r\frac{\int_\Omega F(x,B)dx}{\int_{\partial\Omega} \frac{1}{p(x)} m(x) B^{p(x)} dx},$$

that is,

$$\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) > \frac{\big(\Phi(u_1) - r\big)\Psi(u_0) + \big(r - \Phi(u_0)\big)\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}$$

which means that condition (iii) holds. At this point, conclusion follows from Theorem 3.1.

Acknowledgments. The authors would like to thank the anonymous referees for their clear valuable comments and constructive suggestions.

References

[1] A. Ayoujil, A.R. El Amrouss, On the spectrum of a fourth order elliptic equation with variable exponent, *Nonlinear Anal.* **71** (2009), 4916–4926.

[2] A. Ayoujil, A.R. El Amrouss, Continuous spectrum of a fourth order nonhomogeneous elliptic with variable exponent, *Electr. J. Diff. Eqns.* **24** (2011), 1–12.

[3] G. Bonanno, P. Candito, Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian, *Arch. Math.* (Basel) **80** (2003), 424–429.

[4] Y.M. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.* **66** (2006), 1383–1406.

[5] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, Lebesgue and Sobolev Spaces with Variable Exponents, *Lecture Notes in Mathematics* vol. 2017, Springer, Heidelberg, 2011.

[6] L. Ding, L. Li, W.W. Pan, Existence of multiple solutions for a $p(x)$- Biharmonic equation, *Elec.Jour.Diff.Equa.* **139** (2013), 1–10.

[7] X.L. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263** (2001), 424–446.

[8] X.L. Fan, J.S. Shen, D. Zhao, Sobolev embedding theorems for spaces $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **262** (2001), 749–760.

[9] B. Ge, Q.M. Zhou, Multiple solutions for a Robin-type differential inclusion problem involving the p(x)-Laplacian, **Math. Meth. Appl. Sci.** (2013). doi: 10.1002/mma.2760.

[10] P. Harjulehto, P. Hästö, Ú. Lê, M. Nuortio, Overview of differential equations with nonstandard growth, *Nonlinear Anal.* **72** (2010), 4551–4574.

[11] O. Kováčik, J.Răkosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.* **41** (1991), 592-618.

[12] R. Mashiyev, Three Solutions to a Neumann Problem for Elliptic Equations with Variable Exponent, *Arab. J. Sci. Eng.* **36** (2011), 1559–1567.

[13] M. Mihăilescu, V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. R. Soc. Lond. Ser. A* **462** (2006), 2625–2641.

[14] M. Mihăilescu, Existence and multiplicity of solutions for a Neumann problem involving the p(x)-Laplace operator, *Nonlinear Anal.* **67** (2007), 1419–1425.

[15] W.W. Pan, G.A. Afrouzi, L. Li, Three solutions to a $p(x)$-Laplacian problem in weighted-variable-exponent Sobolev space, *An. St. Univ. Ovidius Constanta* **21** (2013), no. 2, 195-205.

[16] V. R adulescu, Nonlinear elliptic equations with variable exponent: old and new, *Nonlinear Anal. TMA* **121** (2015), 336–369.

[17] V. Rădulescu, D. Repovš, *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*, CRC Press, Taylor & Francis Group, Boca Raton FL, 2015.

[18] B. Ricceri, A three critical points theorem revisited, *Nonlinear Anal.* **70** (2009), 3084–3089.

[19] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, 1748, Springer-Verlag, Berlin, 2000.

[20] S. Samko, Denseness of $C_0^\infty(\mathbb{R}^N)$ in the generalized Sobolev spaces $W^{m,p(x)}(\mathbb{R}^N)$, In *Direct and inverse problems of mathematical physics* (Newark, DE, 1997), vol. 5 of Int. Soc. Anal. Appl. Comput. 333–342, Kluwer Acad. Publ., Dordrecht, 2000.

[21] A.B. Zang, Y. Fu, Interpolation inequalities for derivatives in variable exponent Lebesgue Sobolev spaces, *Nonlinear Anal. TMA* **96** (2008), 3629–3636.

[22] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Mathematics of the USSR-Izvestiya* **9** (1987), 33–66.

(Abdesslem Ayoujil) REGIONAL CENTRE OF TRADES EDUCATION AND TRAINING, OUJDA, MOROCCO
E-mail address: `abayoujil@gmail.com`

(Abdel Rachid El Amrouss) UNIVERSITY MOHAMED I, FACULTY OF SCIENCES, DEPARTMENT OF MATHEMATICS, OUJDA, MOROCCO
E-mail address: `elamrouss@fso.ump.ma`

Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 94–99
ISSN: 1223-6934

On the existence of positive solutions for boundary value problems with sign- changing weight and Caffarelli-Kohn-Nirenberg exponents

A. Firouzjai, G.A. Afrouzi, and S. Talebi

Abstract. In this paper we consider the existence of positive solutions to the singular infinite semipositone problems with sign-changing weight. We use the method of sub-supersolution to establish our existence result.

2010 Mathematics Subject Classification. 35J25, 35J55, 35J60.
Key words and phrases. Caffarelli-Kohn-Nirenberg exponents; Infinite semipositone problem; Positive solution; Singular problem; Sub-supersolution.

---

1. Introduction

The study of positive solutions of singular partial differential equations or systems has been an extremely active research topic during the past few years. Such singular nonlinear problems arise naturally and they occupy a central role in the interdisciplinary research between analysis, geometry, biology, elasticity, mathematical physics, etc.

In this paper, we are concerned with the existence of positive solutions to the boundary value problem

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u) = \lambda |x|^{-(\alpha+1)p+\beta} g(x)(f(u) - \frac{1}{u^{\gamma}}), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ with $0 \in \Omega$ with smooth boundary, $1 < p < N$, $0 \leq \alpha < \frac{N-p}{p}$, $\gamma \in (0,1)$, λ, β are positive constants, $g(x)$ is a C^1 sign-changing function that maybe negative near the boundary and be positive in the interior and $f : (0,\infty) \to (0,\infty)$ is a C^1 nondecreasing function. Elliptic problems involving more general operator, such as the degenerate quasilinear elliptic operator given by $-div(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u)$, where motivated by the following Caffarelli, Kohn and Nirenberg's inequality (see [3, 15]). The study of this type of problems motivated by it's various applications, for example, in fluid mechanics, in newtonian fluids, in flow through porous media and in glaciology (see [1, 4]). So the study of singular elliptic problems has more practical meaning. We refer to ([11, 6, 2, 7]) for additional result on elliptic problem, we study problem (1) in the semipositone case. See [10], where the authors discussed the problem (1) when $g \sim 1$, $\alpha = 0$ and $\beta = p = 2$. In [9], the authors extended the study of [10] to the case when $p > 1$. In [12], the

---

Received February 23, 2016.

prpblem in [9] was studid with weight function $g(x)$. Here we focus on futher extending the study in [12] for quasilinear elliptic problem involving singularity. Due to this singularity in the weights, the extensions are challenging and nontrivial. Our approach is based on the method of sub-supper solution (see[5, 8, 13]). In this paper, we denote $W_0^{1,p}(\Omega, |x|^{-\alpha p})$, the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = (\int_\Omega |x|^{-\alpha p}|\nabla u|^p dx)^{\frac{1}{p}}$. To precisely state our existence result, we consider the eigenvalue problem

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla\phi|^{p-2}\nabla\phi) = \lambda|x|^{-(\alpha+1)p+\beta}|\phi|^{p-2}\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

Let ϕ be the eigenfunction corresponding to the first eigenvalue λ_1 of (2) such that $\phi(x) > 0$ in Ω, and $\|\phi\|_\infty = 1$.
Let $m, \sigma, \delta > 0$ be such that

$$\sigma \le \phi \le 1, \qquad x \in \Omega - \overline{\Omega}_\delta, \tag{3}$$

$$|x|^{-\alpha p}\Big(1 - \frac{\gamma p}{p-1+\gamma}\Big)|\nabla\phi|^p \ge m, \qquad x \in \overline{\Omega}_\delta, \tag{4}$$

where $\overline{\Omega}_\delta = \{x \in \Omega \mid d(x, \partial\Omega) \le \delta\}$. This is possible since $|\nabla\phi| \ne 0$ on $\partial\Omega$ while $\phi = 0$ on $\partial\Omega$. We will also consider the unique solution $e \in W_0^{1,p}(\Omega, |x|^{-\alpha p})$ of the boundary value problem

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla e|^{p-2}\nabla e) = |x|^{-(\alpha+1)p+\beta}, & x \in \Omega, \\ e = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result. It is known that $e > 0$ in Ω and $\dfrac{\partial e}{\partial n} < 0$ on $\partial\Omega$.

Here we assume that the weight function $g(x)$ takes negative values in $\overline{\Omega}_\delta$, but require $g(x)$ be strictly positive in $\Omega - \overline{\Omega}_\delta$.

To be precise we assume that there exist positive constants a, b such that $g(x) \ge -a$, on $\overline{\Omega}_\delta$ and $g(x) \ge b$ on $\Omega - \overline{\Omega}_\delta$.

2. Existence result

A non-negative function ψ is said to be a subsolution of problem (1), if it satisfy $\psi \ge 0$ on $\partial\Omega$ and

$$\int_\Omega |x|^{-\alpha p}|\nabla\psi|^{p-2}\nabla\psi\cdot\nabla w dx \le \int_\Omega \lambda|x|^{-(\alpha+1)p+\beta}g(x)\Big[f(\psi) - \frac{1}{\psi^\gamma}\Big]w dx \quad \forall w \in W,$$

where $W = \{w \in C_0^\infty(\Omega) : \ w \ge 0 \text{ for all } x \in \Omega\}$ (see [14]).
A function z is said supersolution of (1), if it satisfy $z \ge 0$ on $\partial\Omega$, and

$$\int_\Omega |x|^{-\alpha p}|\nabla z|^{p-2}\nabla z\cdot\nabla w dx \ge \int_\Omega \lambda|x|^{-(\alpha+1)p+\beta}g(x)\Big[f(z) - \frac{1}{z^\gamma}\Big]w dx, \quad \forall w \in W.$$

Then the following result holds:

Lemma 2.1. *(see* [8]*). If there exist a sub-solution ψ and supersolution z such that $\psi \le z$ in Ω then (1) has a weak-solution u such that $\psi \le u \le z$.*
We make the following assumptions:
(H_1) $f : (0,\infty) \to (0,\infty)$ *is* C^1 *nondecreasing function.*

(H_2) $\lim\limits_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0$.

(H_3) *suppose that there exist* $\epsilon > 0$ *such that*

i) $f\Big(\frac{\epsilon^{\frac{1}{p-1}}(p-1+\gamma)\sigma}{p}\Big) > \Big(\frac{p}{\epsilon^{\frac{1}{p-1}}\sigma(p-1+\gamma)}\Big)^{\gamma}$,

ii) $\frac{\epsilon^{\frac{\gamma+p-1}{p-1}}\lambda_1(p-1+\gamma)^{\gamma}}{ap^{\gamma}} < \frac{m\epsilon}{af(\epsilon^{\frac{1}{p-1}})}$,

iii) $\frac{\epsilon\lambda_1}{Nb} < \frac{m\epsilon}{af\Big(\epsilon^{\frac{1}{p-1}}\Big)}$, *where* $N = f\Big(\frac{\epsilon^{\frac{1}{p-1}}(p-1+\gamma)\sigma}{p}\Big) - \Big(\frac{p}{\epsilon^{\frac{1}{p-1}}\sigma(p-1+\gamma)}\Big)^{\gamma}$.

iv) Let $\eta > 0$ *be such that* $\eta \geq \max |x|^{-(\alpha+1)p+\beta}$, *in* $\overline{\Omega}_\delta$.

We are now ready to give our existence result.

Theorem 2.2. *Let* $(H_1) - (H_3)$ *hold, then there exists positive weak solution of (1) for every* $\lambda \in [\lambda_*(\epsilon), \lambda^*(\epsilon)]$, *where*

$\lambda^* = \frac{m\epsilon}{\eta af\Big(\epsilon^{\frac{1}{p-1}}\Big)}$ *and* $\lambda_* = \max\left\{\frac{\epsilon^{\frac{\gamma+p-1}{p-1}}\lambda_1(p-1+\gamma)^{\gamma}}{ap^{\gamma}}, \frac{\epsilon\lambda_1}{Nb}\right\}$.

Remark 2.1. Note that (H_3) implies $\lambda_* < \lambda^*$.

Proof. Now we construct a positive sub-solution of (1). For this, we let

$$\psi = \frac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}.$$

Let $w \in W$. Since $\nabla\psi = \epsilon^{\frac{1}{p-1}}\phi^{\frac{1-\gamma}{p-1+\gamma}}\nabla\phi$, then a calculation shows that

$$\begin{aligned}
&\int_\Omega |x|^{-\alpha p}|\nabla\psi|^{p-2}\nabla\psi\cdot\nabla w dx = \epsilon\int_\Omega |x|^{-\alpha p}\phi^{\frac{(p-1)(1-\gamma)}{p-1+\gamma}}|\nabla\phi|^{p-2}\nabla\phi\cdot\nabla w dx\\
&= \epsilon\int_\Omega |x|^{-\alpha p}|\nabla\phi|^{p-2}\nabla\phi\Big[\nabla(\phi^{1-\frac{\gamma p}{p-1+\gamma}}w) - \nabla(\phi^{1-\frac{\gamma p}{p-1+\gamma}})w\Big]dx\\
&= \epsilon\int_\Omega |x|^{-\alpha p}|\nabla\phi|^{p-2}\nabla\phi\cdot\nabla(\phi^{1-\frac{\gamma p}{p-1+\gamma}}w)dx - \epsilon\int_\Omega |x|^{-\alpha p}|\nabla\phi|^{p-2}\nabla\phi\cdot\nabla(\phi^{1-\frac{\gamma p}{p-1+\gamma}})w dx\\
&= \epsilon\int_\Omega |x|^{-(\alpha+1)p+\beta}\lambda_1\phi^{\frac{-\gamma p}{p-1+\gamma}}\phi^p w dx - \epsilon\int_\Omega |x|^{-\alpha p}\Big(1-\frac{\gamma p}{p-1+\gamma}\Big)\phi^{-\frac{\gamma p}{p-1+\gamma}}|\nabla\phi|^p w dx\\
&= \epsilon\int_\Omega \Big[|x|^{-(\alpha+1)p+\beta}\lambda_1\phi^{p-\frac{\gamma p}{p-1+\gamma}} - |x|^{-\alpha p}\Big(1-\frac{\gamma p}{p-1+\gamma}\Big)\phi^{-\frac{\gamma p}{p-1+\gamma}}|\nabla\phi|^p\Big]w dx.
\end{aligned}$$

First we consider the case when $x \in \overline{\Omega}_\delta$. We have $|x|^{-\alpha p}\Big(1-\frac{\gamma p}{p-1+\gamma}\Big)|\nabla\phi|^p \geq m$ and $g(x) \geq -a$. Hence since $\lambda \leq \lambda^* = \frac{m\epsilon}{\eta af(\epsilon^{\frac{1}{p-1}})}$, we have

$$\begin{aligned}
-|x|^{-\alpha p}\epsilon\Big(1-\tfrac{\gamma p}{p-1+\gamma}\Big)\phi^{\frac{-\gamma p}{p-1+\gamma}}|\nabla\phi|^p &\leq -m\epsilon\phi^{\frac{-\gamma p}{p-1+\gamma}} \leq -m\epsilon\\
&\leq -\lambda a\eta f(\epsilon^{\frac{1}{p-1}}) \leq -\lambda a|x|^{-(\alpha+1)p+\beta}f\Big(\tfrac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\Big),
\end{aligned} \tag{5}$$

and since $\lambda \geq \lambda_* = \dfrac{\epsilon^{\frac{\gamma+p-1}{p-1}}\lambda_1(p-1+\gamma)^\gamma}{ap^\gamma}$, we have

$$|x|^{-(\alpha+1)p+\beta}\epsilon\phi^{-\frac{\gamma p}{p-1+\gamma}}\lambda_1\phi^p \leq \frac{\lambda|x|^{-(\alpha+1)p+\beta}ap^\gamma}{\epsilon^{\frac{\gamma}{p-1}}(p-1+\gamma)^\gamma} \leq \frac{|x|^{-(\alpha+1)p+\beta}\lambda a}{\Big(\frac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\Big)^\gamma}. \tag{6}$$

By combining (5) and (6) we see that

$$\begin{aligned}&\epsilon\Big[|x|^{-(\alpha+1)p+\beta}\phi^{p-\frac{\gamma p}{p-1+\gamma}}\lambda_1 - |x|^{-\alpha p}\Big(1-\frac{\gamma p}{p-1+\gamma}\Big)\phi^{-\frac{\gamma p}{p-1+\gamma}}|\nabla\psi|^p\Big]\\ &\quad\leq \lambda|x|^{-(\alpha+1)p+\beta}g(x)\Big[f\Big(\frac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\Big) - \frac{1}{\Big(\frac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\Big)^\gamma}\Big].\end{aligned}$$

On the other hand, on $\Omega - \overline{\Omega}_\delta$, we have $g(x) \geq b$ and $\sigma \leq \phi^{\dfrac{p}{p-1+\gamma}} \leq 1$. Thus for $\lambda \geq \lambda_* \geq \dfrac{\epsilon\lambda_1}{Nb}$, we have

$$\begin{aligned}&\epsilon\Big(|x|^{-(\alpha+1)p+\beta}\phi^{-\frac{\gamma p}{p-1+\gamma}}\lambda_1\phi^p - |x|^{-\alpha p}\Big(1-\frac{\gamma p}{p-1+\gamma}\Big)\phi^{-\frac{\gamma p}{p-1+\gamma}}|\nabla\phi|^p\Big)\\ &\quad\leq |x|^{-(\alpha+1)p+\beta}\epsilon\lambda_1\phi^{p-\frac{\gamma p}{p-1+\gamma}} \leq |x|^{-(\alpha+1)p+\beta}\epsilon\lambda_1 \leq |x|^{-(\alpha+1)p+\beta}\lambda bN\\ &\quad\leq |x|^{-(\alpha+1)p+\beta}\lambda b\Big[f\Big(\frac{p-1+\gamma}{p}\Big)\sigma\epsilon^{\frac{1}{p-1}} - \frac{1}{\Big((\frac{p-1+\gamma}{p})\sigma\epsilon^{\frac{1}{p-1}}\Big)^\gamma}\Big]\\ &\quad\leq |x|^{-(\alpha+1)p+\beta}\lambda g(x)\Big[f\Big(\frac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\Big) - \frac{1}{\Big((\frac{p-1+\gamma}{p})\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\Big)^\gamma}\Big]\\ &\quad= \lambda|x|^{-(\alpha+1)p+\beta}g(x)\Big[f(\psi) - \frac{1}{\psi^\gamma}\Big].\end{aligned}$$

Hence

$$\begin{aligned}&\int_\Omega |x|^{-\alpha p}|\nabla\psi|^{p-2}\nabla\psi\cdot\nabla w dx\\ &\quad\leq \epsilon\int_\Omega\Big[|x|^{-(\alpha+1)p+\beta}\lambda_1(\phi^{p-\frac{\gamma p}{p-1+\gamma}}) - |x|^{-\alpha p}\Big(1-\frac{\gamma p}{p-1+\gamma}\Big)\phi^{-\frac{\gamma p}{p-1+\gamma}}|\nabla\phi|^p\Big]wdx\\ &\quad\leq \int_\Omega \lambda|x|^{-(\alpha+1)p+\beta}g(x)\Big[f\Big(\frac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\Big) - \frac{1}{\Big(\frac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\Big)^\gamma}\Big]wdx\\ &\quad= \int_\Omega \lambda|x|^{-(\alpha+1)p+\beta}g(x)\Big[f(\psi) - \frac{1}{\psi^\gamma}\Big]wdx.\end{aligned}$$

So ψ is a sub- solution of (1) for $\lambda \in [\lambda_*, \lambda^*]$.

Now we will construct a supersolution of (1). For this, we let $z := ce$ and $w \in W$. Since $\nabla z = c\nabla e$ then a calculation shows that

$$\begin{aligned}&\int_\Omega |x|^{-\alpha p}|\nabla z|^{p-2}\nabla z\cdot\nabla w dx = c^{p-1}\int_\Omega |x|^{-\alpha p}|\nabla e|^{p-2}\nabla e\cdot\nabla w dx\\ &\quad= -c^{p-1}\textstyle\int_\Omega div(|x|^{-\alpha p}|\nabla e|^{p-2}\nabla e)wdx \ = c^{p-1}\displaystyle\int_\Omega |x|^{-(\alpha+1)p+\beta}wdx.\end{aligned}$$

By (H_2) we can choose c large enough so that

$$(c\|e\|_\infty)^{p-1}(\lambda\|g(x)\|_\infty\|e\|_\infty)^{-1} \geq f(c\|e\|_\infty).$$

Hence

$$c^{p-1} \geq \lambda \|g(x)\|_\infty f(c\|e\|_\infty) \geq \lambda g(x) f(ce) \geq \lambda g(x)\Big[f(ce) - \frac{1}{(ce)^\gamma}\Big] = \lambda g(x)\Big[f(z) - \frac{1}{z^\gamma}\Big].$$

Thus we have

$$\int_\Omega |x|^{-\alpha p} |\nabla z|^{p-2} \nabla z \cdot \nabla w dx = c^{p-1} \int_\Omega |x|^{-(\alpha+1)p+\beta} w dx$$

$$\geq \int_\Omega |x|^{-(\alpha+1)p+\beta} \lambda g(x)\Big[f(ce) - \tfrac{1}{(ce)^\gamma}\Big] w dx = \int_\Omega |x|^{-(\alpha+1)p+\beta} \lambda g(x)\Big[f(z) - \frac{1}{z^\gamma}\Big] w dx.$$

So z is a supersolution of (1) with $z \geq \psi$ for c large. Thus, there exist a positive weak solution u of (1) such that $\psi \leq u \leq z$. This completes the proof of Theorem 2.2. □

References

[1] C. Atkinson, K. El Kalli, Some boundary value problems for the Bingham model, *J. Non-Newtonian fluid Mech.* **41** (1992), 339–363.

[2] H. Bueno, G. Ercole, W. Ferreira, A. Zunpano, Existence and multiplicity of positive solutions for the $p-$Laplacian with nonlocal coefficient, *J. Math. Anal. Appl.* **343** (2008), 151–158.

[3] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalites with weights, *Compos. Math.* **53** (1984), 259–275.

[4] F. Cirstea, D. Motreanu, V. Rădulescu, Weak solutions of quasilinear problems with nonlinear boundary condition, *Nonlinear Anal.* **43** (2001), 623–636.

[5] P. Drabek, J. Hernandez, Existence and uniqueness of positive solutions for some quasilinear elliptic problems, *Nonlinear Anal.* **44** (2001), 189–204.

[6] P. Drabek, S.H. Rasouli, A quasilinear eigenvalue problem with Robin conditions on the non-smooth domain of finite measure, *Zeitschrist fur Analisis und ihre Anwendungen* **29** (2010), no. 4, 469–485.

[7] F. Fang, S. Liu, Nontrivial sloutions of superlinear $p-$Laplacian equation, *J. Math. Anal. Appl.* **351** (2009), 138–146.

[8] S. Gui, Existence and nonexistence of positive solution for singular semilinear elliptic boundary value problems, *Nonlinear Anal. TMA* **41** 2000, 149–176.

[9] E.K. Lee, R. Shivaji, J. Ye, Classes of infinite semipositone $n \times n$ systems, *Diff. Int. Eqns.* **24** (2001), no. 3-4, 361–370.

[10] M. Ramaswamy, R. Shivaji, J. Ye, Positiv solution for a class of infinite semipositone problems, *Diff. Integ. Eqns.* **20** (2007), 1423–1433.

[11] S.H. Rasouli, G.A. Afrouzi, The Nehari manifold for a calss of concave convex elliptic systems involving the $p-$Laplacian and nonlinear boundary condition, *Nonlinear Anal.* **73** (2010), 3390–3401.

[12] S. H. Rasouli, Z. Firouzjahi, On a class of singular $p-$Laplacian semipositone problems with sign-changing weight, *J. Appl. Anal. Comput.* **4** (2014), 383–388.

[13] S.H. Rasouli, Z. Halimi, Z. Mashhadban, A remark on the existence of positive weak solution for a class of (p,q)-Laplacian nonlinear system with sign-changing weight, *Nonlinear Anal.* **73** (2010), 385–389.

[14] R. Shivaji, J.Ye , Positive solutions for a class of infinite semipositone problems, *Diff. Int. Eqns.* **12** (2007), 1423–1433.

[15] B. Xuan, The solvability of quasilinear Brezis-Nirenberg- type problems with singuler weights, *Nonlinear Anal.* **62** (2005), 703-725.

(A. Firouzjai) Department of Mathematics, Faculty of Basic Sciences, Pyame Noor University, Tehran, Iran
E-mail address: `Firouzjai@phd.pnu.ac.ir`

(G.A. Afrouzi) Department of Mathematics, Faculty of mathematical Sciences, University of Mazandaran, Babolsar, Iran
E-mail address: `afrouzi@umz.ac.ir`

(S. Talebi) Department of Mathematics, Faculty of Basic Sciences, Pyame Noor University, Mashhad, Iran
E-mail address: `talebi_s@yahoo.com`

Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 100–114
ISSN: 1223-6934

Three critical solutions for variational - hemivariational inequalities involving $p(x)$-Kirchhoff type equation

M. Alimohammady and F. Fattahi

Abstract. In this paper, we study the existence of three solutions to the p(x)-Kirchhoff type equations in $\mathbb{R}^N$. By means of nonsmooth three critical points theorem and the theory of the variable exponent Sobolev spaces, we establish the existence of three critical points for the problem. Moreover, we study the existence of three radially symmetric solutions for a class of quasilinear elliptic inclusion problem with discontinuous nonlinearities in $\mathbb{R}^N$. Our approach is based on critical point theory for locally Lipschitz functionals due to Iannizzotto.

2010 Mathematics Subject Classification. Primary 49J40; Secondary 47J20, 47J30, 35J35.
Key words and phrases. Nonsmooth critical point theory, principle of symmetric criticality, variational-hemivariational inequality, $p(x)-$Kirchhoff type operator.

---

1. Introduction

In this paper, we are concerned with the following nonlinear elliptic differential inclusion with $p(x)-$Kirchhoff-type problem

$$\begin{cases} M\Big(\int_{\mathbb{R}^N}\frac{1}{p(x)}(|\nabla u|^{p(x)}-|u|^{p(x)})dx\Big)[\Delta_{p(x)}u-|u|^{p(x)-2}u] & \\ \qquad\qquad\qquad \in -\lambda\partial F(x,u)-\mu\partial G(x,u) & in\,\mathbb{R}^N \\ u=0 & on\,\mathbb{R}^N, \end{cases} \tag{1}$$

where $p(x)\in C(\mathbb{R}^N)$ is continuous function satisfying

$$1<p^-=\inf_{x\in\mathbb{R}^N}p(x)\le p(x)\le p^+=\sup_{x\in\mathbb{R}^N}p(x)<+\infty,$$

and $\lambda,\mu>0$. $F,G:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a function in which $F(\cdot,u)$ is measurable for every $u\in\mathbb{R}$ and $F(x,\cdot)$ is locally Lipschitz for a.e. $x\in\mathbb{R}^N$. $\partial F(x,u)$ and $\partial G(x,u)$ denotes the generalized Clarke gradient of $F(x,u)$ and $G(x,u)$ at $u\in\mathbb{R}$.

Let X be real Banach space. We assume that it is also given a functional $\chi: X\to\mathbb{R}\cup\{+\infty\}$ which is convex, lower semicontinuous, proper whose effective domain $dom(\chi)=\{x\in X:\chi(x)<+\infty\}$ is a (nonempty, closed, convex) cone in X. Our aim is to study the following variational-hemivariational inequality problem: Find $u\in\mathcal{B}$ (it is called a weak solution of problem (1)) if for all $v\in\mathcal{B}$,

$$M\Big(\int_{\mathbb{R}^N}\frac{1}{p(x)}(|\nabla u|^{p(x)}-|u(x)|^{p(x)})dx\Big)\int_{\mathbb{R}^N}(|\nabla u|^{p(x)-2}\nabla u\nabla v-|u|^{p(x)-2}uv)dx$$

Received March 25, 2016.

$$-\lambda \int_{\mathbb{R}^N} F^0(x,u;v)dx - \mu \int_{\mathbb{R}^N} G^0(x,u;v)dx \geq 0, \tag{2}$$

where $\mathcal{B}$ is a closed convex subset of $X = W_0^{1,p(\cdot)}(\mathbb{R}^N)$, and F^0, G^0 are the generalized directional derivatives of the locally Lipschitz functions F, G.

The operator $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$ is the so-called $p(x)-$Laplacian, which becomes $p-$Laplacian when $p(x) \equiv p$ is a constant. More recently, the study of $p(x)-$Laplacian problems has attracted more and more attention (cf. [2]).

The problem (1) is a generalization of an equation introduced by Kirchhoff (cf. [20]). The study of Kirchhoff model has already been extended to the case involving the p-Laplacian (cf. [8], [10]) and $p(x)-$Laplacian (cf. [6], [15]).

Applications of problems involving the p(x)-Laplace operator is applied to the modeling of various phenomena such as elastic mechanics, thermorheological and electrorheological fluids, mathematical mathematical biology and plasma physics (cf. [10], [30], [31]). In recent years, differential equations and variational problems have been studied in many papers, we refer to some interesting works (cf. [27], [28]).

Many authors investigated variational methods to a class of non-differentiable functionals to prove some existence theorems for PDE with discontinuous nonlinearities. In [33] author studied a priori bounds for a class of variational inequalities involving general elliptic operators of second-order and terms of generalized directional derivatives; in [4], authors studied variational-hemivariational inequalities involving the p-Laplace operator and a nonlinear Neumann boundary condition; in [1], authors studied variational-hemivariational inequality by using the mountain pass theorem.

However, authors appeared some technical difficulties for studying problem on unbounded domains (cf. [3]). Therefore, to resolve this issue the space of radially symmetric functions was introduced. For instance, the existence of radially symmetric solutions for a class of differential inclusion problems was considered by many authors. In [32] author studied infinitely many radially symmetric solutions for a class of hemivariational inequalities with the Cerami compactness condition and the principle of symmetric criticality for locally Lipschitz functions; in [24] author studied the existence of infinitely many radial respective non-radial solutions for a class of hemivariational inequalities; in [18] authors studied the existence of infinitely many radially symmetric solutions for a class of perturbed elliptic equations with discontinuous nonlinearities under some hypotheses on the behavior of the potential.

More recently, the study of the three-critical-points for nonsmooth functionals was investigated. In [23] authors studied the existence of three critical points which extends the variational principle of Ricceri [29] to nonsmooth functionals. In [19] author studied three-critical-points theorem based on a minimax inequality and on a truncation argument which extended to Motreanu-Panagiotopoulos functionals. In [34], authors studied the existence of at least three critical points for a p(x)-Laplacian differential inclusion based on the nonsmooth analysis.

The purpose of this paper is to prove the existence of at least three solutions for a variational-hemivariational inequality depending on two parameters in $W_0^{1,p(x)}(\mathbb{R}^N)$. In fact, the existence result for $p(x)-$Kirchhoff-type problem with locally Lipschitz functions under special hypotheses on F and G is investigated. Also, for the second part under further additional assumptions, the quasilinear elliptic inclusion problem is considered. A major problem is that the compact embedding for $W_0^{1,p(x)}(\mathbb{R}^N)$ into

$L^\infty(\mathbb{R}^N)$ is required. Hence, we overcome this gap by using the subspace of radially symmetric functions of $W_0^{1,p(x)}(\mathbb{R}^N)$, denoted by $W_{0,r}^{1,p(x)}(\mathbb{R}^N)$, can be embedded compactly into $L^\infty(\mathbb{R}^N)$.

The paper is organized as follows. We prepare the basic definitions and properties in the framework of the generalized Lebesgue and Sobolev spaces. Besides, some basic notions about generalized directional derivative and hypotheses on F, G are given. Next, we give the main results about the existence of three solutions in theorem 3.7. The final part of this paper is concerned with the existence of three radially symmetric solutions in theorem 4.5.

2. Preliminaries

We recall some basic facts about the variable exponent Lebesgue-Sobolev (cf. [11],[13],[16]).
The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\mathbb{R}^N) = \{u : \mathbb{R}^N \longrightarrow \mathbb{R} : \int_{\mathbb{R}^N} |u(x)|^{p(x)}dx < \infty\}$$

and is endowed with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\{\ \lambda > 0 : \int_{\mathbb{R}^N} |\frac{u(x)}{\lambda}|^{p(x)}dx\} \leq 1\}.$$

Note that, when $p \equiv Const.$, the Luxemburg norm $\|\cdot\|_{p(\cdot)}$ coincides with the standard norm $\|\cdot\|_p$ of the Lebesgue space $L^p(\mathbb{R}^N)$.

The generalized Lebesgue-Sobolev space $W^{L,p(\cdot)}(\mathbb{R}^N)$ for $L = 1, 2, ...$ is defined as

$$W^{L,p(\cdot)}(\mathbb{R}^N) = \{u \in L^{p(\cdot)}(\mathbb{R}^N) : D^\alpha u \in L^{p(\cdot)}(\mathbb{R}^N), |\alpha| \leq L\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n}$ with $\alpha = (\alpha_1, \alpha_2, \cdot\cdot\cdot, \alpha_N)$ is a multi-index and $|\alpha| = \Sigma_{i=1}^N \alpha_i$.
The space $W^{L,p(\cdot)}(\mathbb{R}^N)$ with the norm

$$\|u\|_{W^{L,p(\cdot)}}(\mathbb{R}^N) = \sum_{|\alpha|\leq L} \|D^\alpha u\|_{p(\cdot)},$$

is a separable reflexive Banach space(cf. [12]).
The space $W_0^{L,p(\cdot)}(\mathbb{R}^N)$ denotes the closure in $W^{L,p(\cdot)}(\mathbb{R}^N)$ of the set of all $W^{L,p(\cdot)}(\mathbb{R}^N)$ -functions with compact support. Hence, an equivalent norm for the space $W_0^{L,p(\cdot)}(\mathbb{R}^N)$ is given by

$$\|u\|_{W_0^{L,p(\cdot)}(\Omega)} = \sum_{|\alpha|=L} \|D^\alpha u\|_{p(\cdot)}.$$

If $\Omega \subset \mathbb{R}^N$ is open bounded domain, let p_L^* denote the critical variable exponent related to p, defined for all $x \in \bar{\Omega}$ by the pointwise relation

$$p_L^*(x) = \begin{cases} \frac{Np(x)}{N-Lp(x)} & Lp(x) < N, \\ +\infty & Lp(x) \geq N. \end{cases} \tag{3}$$

For every $u \in W_0^{L,p(\cdot)}(\Omega)$ the Poincaré inequality holds, where $C_p > 0$ is a constant

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C_p \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

(see (cf. [17])).

Proposition 2.1. *(cf.* [16]*) Let p' be the function obtained by conjugating the exponent p pointwise, that is $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$ for all $x\in\bar{\Omega}$, then p' belongs to $C_+(\overline{\Omega})$. For any $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$, the following Hölder type inequality valid,*

$$\int_\Omega |u(x)v(x)|dx \le (\frac{1}{p^-}+\frac{1}{p'^-})\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)},$$

where $L^{\acute{p}(\cdot)}(\Omega)$ is the conjugate space of $L^{p(\cdot)}(\Omega)$.

Proposition 2.2. *For $\phi(u)=\int_{\mathbb{R}^N}[|\nabla u|^{p(x)}-|u(x)|^{p(x)}]dx$, and $u,u_n\in X$, we have*

(i) $\|u\|<(=;>)1\Leftrightarrow \phi(u)<(=;>)1$,

(ii) $\|u\|\le 1\Rightarrow \|u\|^{p^+}\le\phi(u)\le\|u\|^{p^-}$,

(iii) $\|u\|\ge 1\Rightarrow \|u\|^{p^-}\le\phi(u)\le\|u\|^{p^+}$,

(iv) $\|u_n\|\to 0\Leftrightarrow \phi(u_n)\to 0$,

(v) $\|u_n\|\to\infty\Leftrightarrow \phi(u_n)\to\infty$.

Proof is similar to that in (cf. [16]).

Proposition 2.3. *(cf.* [16],[21]*) For $p,q\in C_+(\overline{\Omega})$ in which $q(x)\le p^*_L(x)$ for all $x\in\overline{\Omega}$, there is a continuous embedding*

$$W^{L,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega).$$

If we replace $\le$ with $<$, the embedding is compact.

Remark 2.1. (i) By the proposition (2.3) there is a continuous and compact embedding of $W_0^{1,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$, where $q(x)<p^*(x)$ for all $x\in\overline{\Omega}$.
(ii) Denote by

$$\|u\|=\inf\{\lambda>0\ :\ \int_{\mathbb{R}^N}[|\frac{\nabla u}{\lambda}|^{p(x)}-|\frac{u}{\lambda}|^{p(x)}]dx\le 1\},$$

which is a norm on $W_0^{1,p(\cdot)}(\mathbb{R}^N)$.

Here, we recall some definitions and basic notions of the theory of generalized differentiation for locally Lipschitz functions. We refer the reader to (cf. [5], [7], [25], [26]).
Let X be a Banach space and $X^\star$ its topological dual. By $\|\cdot\|$ we will denote the norm in X and by $<\cdot,\cdot>_X$ the duality brackets for the pair $(X,X^\star)$.
A function $h:X\to\mathbb{R}$ is said to be locally Lipschitz continuous, when to every $x\in X$ there correspond a neighborhood V_x of x and a constant $L_x\ge 0$ such that

$$|h(z)-h(w)|\le L_x\|z-w\|,\ \forall z,w\in V_x.$$

For a locally *Lipschitz* function $h:X\to\mathbb{R}$, the generalized directional derivative of h at $u\in X$ in the direction $\gamma\in X$ is defined by

$$h^0(u;\gamma)=\limsup_{w\to u,t\to 0^+}\frac{h(w+t\gamma)-h(w)}{t}.$$

The generalized gradient of h at $u \in X$ is

$$\partial h(u) = \{x^\star \in X^\star \ : < x^\star, \gamma >_X \leq h^0(u;\gamma), \ \forall \gamma \in X\},$$

which is non-empty, convex and $w^\star$−compact subset of $X^\star$, where $< \cdot, \cdot >_X$ is the duality pairing between $X^\star$ and X.

Proposition 2.4. *(cf. [7]) Let $h, g : X \to \mathbb{R}$ be locally Lipschitz functionals. Then, for any $u, v \in X$ the following hold:*
(1) *$h^0(u;\cdot)$ is subadditive, positively homogeneous;*
(2) *∂h is convex and weak* compact;*
(3) $(-h)^0(u;v) = h^0(u;-v)$;
(4) *the set-valued mapping $h : X \to 2^{X^*}$ is weak* u.s.c.;*
(5) $h^0(u;v) = \max_{u^* \in \partial h(u)} < u^*, v >$;
(6) *$\partial(\lambda h)(u) = \lambda \partial h(u)$ for every $\lambda \in \mathbb{R}$;*
(7) $(h+g)^0(u;v) \leq h^0(u;v) + g^0(u;v)$;
(8) *the function $m(u) = \min_{\nu \in \partial h(u)} \nu_{X^*}$ exists and is lower semicontinuous; i.e., $\liminf_{u \to u_0} m(u) \geq m(u_0)$;*
(9) $h^0(u;v) = \max_{u^* \in \partial h(u)} \langle u^*, v \rangle \leq L\|v\|$.

Proposition 2.5. *(cf. [7])(Lebourg's mean value theorem) Let $h : X \to \mathbb{R}$ be a locally Lipschitz functional. Then, for every $u, v \in X$ there exists $w \in [u,v]$, $w^* \in \partial h(u)$ such that $h(u) - h(v) = \langle w^*, u - v \rangle$.*

Definition 2.1. (cf. [26]) Let X be a Banach space, $\mathcal{I} : X \to (-\infty, +\infty]$ is called a Motreanu-Panagiotopoulos-type functional, if $\mathcal{I} = h + \chi$, where $h : X \to \mathbb{R}$ is locally Lipschitz and $\chi : X \to (-\infty, +\infty]$ is convex, proper and lower semicontinuous.

Definition 2.2. (cf. [19]) An element $u \in X$ is called a critical point for $\mathcal{I} = h + \chi$ if

$$h^0(u; v-u) + \chi(v) - \chi(u) \geq 0, \quad \forall v \in X.$$

The Euler-Lagrange functional associated to problem (1) is given by

$$\mathcal{I}(u) = \widehat{M}\Big(\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} - |u|^{p(x)})dx\Big) - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} G(x,u)dx,$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$ and $M(t)$ is supposed to verify the following assumptions:
(M_1) There exist m_1 and m_0 in which $m_1 \geq m_0 > 0$ and for all $t \in \mathbb{R}^+$, $m_0 \leq M(t) \leq m_1$;
(M_2) For all $t \in \mathbb{R}^+$, $\widehat{M}(t) \geq M(t)t$.
Denote $\Phi : W_0^{1,p(\cdot)}(\mathbb{R}^N) \to \mathbb{R}$, as follows

$$\Phi(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}[|\nabla u|^{p(x)} - |u|^{p(x)}]dx.$$

The next lemma characterizes some properties of Φ (cf. [14]).

Proposition 2.6. *Let $\Phi(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}[|\nabla u|^{p(x)} - |u|^{p(x)}]dx$. Then*
(i) $\Phi : X \to \mathbb{R}$ is sequentially weakly lower semicontinuous.
(ii) Φ' is of (S_+) type.
(iii) Φ' is a homeomorphism.

Proposition 2.7. *(cf. [7]) Let* $F, G : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *be locally Lipschitz function and set* $\mathcal{F}(u) = \int_{\mathbb{R}^N} F(x, u(x))dx$, $\mathcal{G}(u) = \int_{\mathbb{R}^N} G(x, u(x))dx$. *Then* $\mathcal{F}$, $\mathcal{G}$ *are well-defined and*

$$\mathcal{F}^0(u; v) \leq \int_{\mathbb{R}^N} F^0(u(x); v(x))dx, \quad \mathcal{G}^0(u; v) \leq \int_{\mathbb{R}^N} G^0(u(x); v(x))dx, \forall u, v \in X.$$

3. Three solutions for a differential inclusion problem

For the reader's convenience, we recall the nonsmooth three critical points theorem.

Theorem 3.1. [19] *Let* X *be a separable and reflexive Banach space,* Λ *a real interval and* $\mathcal{B}$ *a nonempty, closed, convex subset of* X. $\Phi \in C^1(X, \mathbb{R})$ *a sequentially weakly l.s.c. functional and bounded on any bounded subset of* X *such that* Φ' *is of type* $(S)_+$, *suppose that* $\mathcal{F} : X \to \mathbb{R}$ *is a locally Lipschitz functional with compact gradient. Assume that:*
(i) $lim_{\|u\| \to +\infty}[\Phi - \lambda\mathcal{F}] = +\infty, \quad \forall \lambda \in \Lambda,$
(ii) *There exists* $\rho_0 \in \mathbb{R}$ *such that*

$$\sup_{\lambda \in \Lambda} \inf_{u \in X} [\Phi + \lambda(\rho_0 - \mathcal{F}(u))] < \inf_{u \in X} \sup_{\lambda \in \Lambda} [\Phi + \lambda(\rho_0 - \mathcal{F}(u))].$$

Then, there exist $\lambda_1, \lambda_2 \in \Lambda$ $(\lambda_1 < \lambda_2)$ *and* $\sigma > 0$ *such that for every* $\lambda \in [\lambda_1, \lambda_2]$ *and every locally Lipschitz functional* $\mathcal{G} : X \to \mathbb{R}$ *with compact derivative, there exists* $\mu_1 > 0$ *such that for every* $\mu \in]0, \mu_1[$ *the functional* $\Phi - \lambda\mathcal{F} + \mu\mathcal{G}$ *has at least three critical points whose norms are less than* σ.

Let us introduce the following conditions of our problem.
We assume that $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which is locally Lipschitz in the second variable and satisfies the following properties:
(F_1) $|\xi| \leq K(|s|^{t(x)-1} + |s|^{z(x)-1})$ for all $\xi \in \partial F(x, s)$ with $(x, s) \in \mathbb{R}^N \times \mathbb{R}$ $(1 \leq p^- \leq p(x) \leq p^+ < z^- \leq z(x) \leq z^+ < t^- \leq t(x) \leq t^+ < p^*(x))$;
(F_2) $|F(x, s)| \leq H(|s|^{\alpha(x)} + |s|^{\beta(x)})$ for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$ $(H > 0,\ 1 \leq \alpha^- \leq \alpha(x) \leq \alpha^+ < \beta^- \leq \beta(x) \leq \beta^+ < p^- \leq p(x) \leq p^+ < p^*(x))$;
(F_3) $F(x, 0) = 0$ for a.e. $x \in \mathbb{R}^N$ and there exists $\hat{u} \in W_0^{1,p(\cdot)}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} F(x, \hat{u})dx > 0$ for a.e. $x \in \mathbb{R}^N$;
(G) $|\xi| \leq K'(1 + |s|^{r(x)-1})$ for all $\xi \in \partial G(x, s)$ with $(x, s) \in \mathbb{R}^N \times \mathbb{R}$ $(1 \leq p^- \leq p(x) \leq p^+ < r^- \leq r(x) \leq r^+ < p^*(x))$.

We need the following lemmas in the proof of our main result.

Lemma 3.2. *If* (F_1) *holds, then* $\mathcal{F} : X \to \mathbb{R}$ *is locally Lipschitz functional with compact gradient.*

Proof. First we prove that $\mathcal{F}$ is Lipschitz continuous on each bounded subset of X. Let $u, v \in B(0, M)$ $(M > 0)$ and $\|u\|, \|v\| \leq 1$. From proposition 2.5, the Hölder inequality and the embedding of X in $L^{t(x)}(\mathbb{R}^N)$ and $L^{z(x)}(\mathbb{R}^N)$

$$\begin{aligned} |\mathcal{F}(u) - \mathcal{F}(v)| &\leq \int_{\mathbb{R}^N} |F(x, u(x)) - F(x, v(x))|dx \\ &\leq \int_{\mathbb{R}^N} K(|u(x)|^{t(x)-1} + |v(x)|^{t(x)-1} + |u(x)|^{z(x)-1} + |v(x)|^{z(x)-1})|u(x) - v(x)|dx \end{aligned}$$

$$\begin{aligned}&\leq K(\||u(x)|^{t(x)-1}+|v(x)|^{t(x)-1}\|)_{L^{t'(x)}(\mathbb{R}^N)}\|u-v\|_{L^{t(x)}(\mathbb{R}^N)}\\&\qquad +K(\||u(x)|^{z(x)-1}+|v(x)|^{z(x)-1}\|)_{L^{z'(x)}(\mathbb{R}^N)}\|u-v\|_{L^{z(x)}(\mathbb{R}^N)}\\&\leq 2K(c_1M^{z^--1}+c_2M^{t^--1})\|u-v\|,\end{aligned}$$

where $c_1,\ c_2$ are positive constants.
We prove that $\partial\mathcal{F}$ is compact. Let $\{u_n\}$ be a sequence in X such that $\|u_n\|\leq M$ and choose $u_n^*\in\partial\mathcal{F}(u_n)$ for any $n\in\mathbb{N}$. From (F_1) it follows that for any $n\in\mathbb{N},\ v\in X$,

$$\begin{aligned}<u_n^*,v> &\leq \int_{\mathbb{R}^N}|u_n^*(x)||v(x)|dx\leq\int_{\mathbb{R}^N}K(|u(x)|^{t(x)-1}+|u(x)|^{z(x)-1})|v(x)|dx\\&\leq (c_3M^{t^--1}+c_4M^{z^--1})\|v\|,\end{aligned}$$

where $c_3,\ c_4$ are positive constants.
Consequently,

$$\|u_n^*\|_{X^*}\leq(c_3M^{t^--1}+c_4M^{z^--1}).$$

The sequence $\{u_n^*\}$ is bounded and hence, up to a subsequence, $u_n^*\rightharpoonup u^*$.
Suppose on the contrary; we assume that there exists $\epsilon>0$ for which $\|u_n^*-u^*\|_{X^*}>\epsilon$ (choose a subsequence if necessary). For every $n\in\mathbb{N}$, we can find $\{v_n\}\in X$ with $\|v_n\|<1$ and

$$\langle u_n^*-u^*,v_n\rangle>\epsilon. \tag{4}$$

Then, $\{v_n\}$ is a bounded sequence and up to a subsequence, $v_n\rightharpoonup v,\ \|v_n-v\|_{L^{t(x)}(\Omega)}\to 0$ and $\|v_n-v\|_{L^{z(x)}(\Omega)}\to 0$. Hence,

$$|\langle u_n^*-u^*,v\rangle|<\frac{\epsilon}{4},\quad|\langle u^*,v_n-v\rangle|<\frac{\epsilon}{4},$$
$$\|v_n-v\|_{L^{t(x)}}<\frac{\epsilon}{4Kc_3M^{t^--1}},\quad\|v_n-v\|_{L^{z(x)}}<\frac{\epsilon}{4Kc_4M^{z^--1}}.$$

It follows that,

$$\begin{aligned}\langle u_n^*-u^*,v_n\rangle &\leq \langle u_n^*,v_n-v\rangle+\langle u_n^*-u^*,v\rangle+\langle u^*,v-v_n\rangle\\&\leq \int_{\mathbb{R}^N}|u_n^*(x)||v_n(x)-v(x)|dx+\langle u_n^*-u^*,v\rangle+\langle u^*,v-v_n\rangle\\&\leq K(c_3M^{t^--1}\|v_n-v\|_{L^{t(x)}}+c_4M^{z^--1}\|v_n-v\|_{L^{z(x)}})\\&\qquad+\langle u_n^*-u^*,v\rangle+\langle u^*,v-v_n\rangle\to 0,\end{aligned}$$

which contradicts (15). ☐

Lemma 3.3. *Let G be satisfied. Then $\mathcal{G}$ is a locally Lipschitz functional with compact gradient.*

The proof is similar to lemma (3.2).

The next lemma points out the relationship between the critical points of $\mathcal{I}(u)$ and solutions of Problem (2).

Lemma 3.4. *Every critical point of the functional $\mathcal{I}$ is a solution of Problem* (1).

Proof. Let $u\in X$ be a critical point of $\mathcal{I}(u)=\Phi(u)-\lambda\mathcal{F}(u)-\mu\mathcal{G}(u)+\chi(u)$. Then $u\in\mathcal{B}$ and by definition 2.2

$$\langle\Phi'u,v-u\rangle+\lambda(-\mathcal{F})^0(u;v-u)+\mu(-\mathcal{G})^0\langle u;v-u\rangle\geq 0,\quad\forall v\in X.$$

Using proposition 2.7 and proposition 2.4, we obtain the desired inequality. ☐

Lemma 3.5. *(cf. [19]) Let (F_1) and (F_3) be satisfied. Then, there exists $\hat{u} \in \mathcal{B}$ such that $\mathcal{F}(\hat{u}) > 0$.*

Lemma 3.6. *If (F_2) holds, then for any $\lambda \in (0, +\infty)$, the function $\Phi - \lambda\mathcal{F}$ is coercive.*

Proof. For $u \in X$ such that $\|u\| \geq 1$

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} F(x,u)dx \leq \int_{\mathbb{R}^N} H(|u|^{\alpha(x)} + |u|^{\beta(x)})dx \leq H(\|u\|^{\alpha^+}_{L^{\alpha(x)}(\mathbb{R}^N)} + \|u\|^{\beta^+}_{L^{\beta(x)}(\mathbb{R}^N)}).$$

By the embedding theorem for suitable positive constant c_5, c_6 it implies that

$$\mathcal{F}(u) \leq H(c_5\|u\|^{\alpha^+}_X + c_6\|u\|^{\beta^+}_X).$$

Consequently, by using proposition 2.2, for any $\lambda > 0$,

$$\Phi(u) - \lambda\mathcal{F}(u) \geq \frac{1}{p^+}\|u\|^{p^-}_X - H(c_5\|u\|^{\alpha^+}_X + c_6\|u\|^{\beta^+}_X).$$

Since $p^- > \min\{\alpha^+, \beta^+\}$, it follows that

$$\lim_{\|u\| \to +\infty} [\Phi - \lambda\mathcal{F}] = +\infty, \quad \forall u \in X, \ \lambda \in (0, +\infty).$$

□

Theorem 3.7. *Let F_1, F_2, F_3 are satisfied. Then there exist $\lambda_1, \lambda_2 > 0 (\lambda_1 < \lambda_2)$ and $\sigma > 0$ such that for every $\lambda \in [\lambda_1, \lambda_2]$ and every $\mathcal{G}$ satisfying G, there exists $\mu_1 > 0$ such that for every $\mu \in]0, \mu_1[$ problem (1) admits at least three solutions whose norms are less than σ.*

Proof. Due to Lemma 3.4, we are going to prove the existence of a critical point of functional $\mathcal{I}$. First, we check if $\mathcal{I}$ satisfies the conditions of the nonsmooth three critical points theorem 3.1. It is clear that Lemma 2.6 shows that Φ satisfies the weakly sequentially lower semicontinuous property and Φ' is of type (S_+). Moreover, according to Lemma 3.2, the functional $\mathcal{F}$ is weakly sequentially semicontinuous.
Since Lemma 3.6, implies that $\Phi - \lambda\mathcal{F}$ is coercive on X for all $\lambda \in \Lambda =]0, +\infty[$, so, the assumption (i) of theorem 3.1, satisfies.

Case 1. Let us assume that $\|u\| \leq 1$.
Set for every $r > 0$,

$$\theta_1(r) = \sup\{\mathcal{F}(u); u \in X, \frac{m_1}{p^-}\|u\|^{p^-} \leq r\},$$

we indicate that

$$\lim_{r \to 0^+} \frac{\theta_1(r)}{r} = 0. \tag{5}$$

From (F_1), it is follows that for every $\epsilon > 0$, there exists $c(\epsilon) > 0$ such that for every $x \in \Omega, u \in \mathbb{R}$ and $\xi \in \partial F(x,u)$

$$|\xi| \leq \epsilon|u|^{t(x)-1} + c(\epsilon)|u|^{z(x)-1}. \tag{6}$$

Applying Lebourgs mean value theorem and using the Sobolev embedding theorem for every $u \in X$, there exist suitable positive constants c_7 and c_8

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} F(x,u)dx \leq \int_{\mathbb{R}^N} K(|u|^{t(x)} + |u|^{z(x)})dx \leq K(\|u\|^{t^+}_{L^{t(x)}(\mathbb{R}^N)} + \|u\|^{z^+}_{L^{z(x)}(\mathbb{R}^N)})$$

$$\leq Kc_7(\|u\|^{t^+}_X + \|u\|^{z^+}_X) \leq Kc_8(r^{\frac{t^+}{p^-}} + r^{\frac{z^+}{p^-}}).$$

It follows from $\min\{t^+, z^+\} > p^-$ that

$$\lim_{r\to 0^+} \frac{\theta_1(r)}{r} = 0.$$

From Lemma (3.5), $\hat{u} \neq 0$. Hence, in view of (5), there is $r \in \mathbb{R}$ in which $0 < r < \frac{m_1}{p^-}\|\hat{u}\|^{p^-}$, $\qquad 0 < \frac{\theta_1(r)}{r} < \frac{\mathcal{F}(\hat{u})}{\frac{m_1}{p^-}\|\hat{u}\|^{p^-}}$.

Choose $\rho_0 > 0$ such that

$$\theta_1(r) < \rho_0 < \frac{r\mathcal{F}(\hat{u})}{\frac{m_1}{p^-}\|\hat{u}\|^{p^-}}, \tag{7}$$

especially, $\rho_0 < \mathcal{F}(\hat{u})$.

We claim that

$$\sup_{\lambda\in\Lambda} \inf_{u\in\mathcal{B}} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] < r. \tag{8}$$

It is obvious that the mapping

$$\lambda \mapsto \sup_{\lambda\in\Lambda} \inf_{u\in\mathcal{B}} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))]$$

is upper semicontinuous on Λ and

$$\lim_{\lambda\to+\infty} \inf_{u\in\mathcal{B}} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] \leq \lim_{\lambda\to+\infty} [\frac{m_1}{p^-}\|\hat{u}\|^{p^-} + \lambda(\rho_0 - \mathcal{F}(\hat{u}))] = -\infty.$$

Therefore, there exists $\bar{\lambda} \in \Lambda$ in which

$$\sup_{\lambda\in\Lambda} \inf_{u\in\mathcal{B}} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] = \inf_{u\in\mathcal{B}} [\frac{m_1}{p^-}\|u\|^{p^-} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))].$$

We consider two cases:

(I) If $\bar{\lambda}\rho_0 < r$, we obtain

$$\inf_{u\in\mathcal{B}} [\frac{m_1}{p^-}\|u\|^{p^-} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))] \leq \bar{\lambda}\rho_0 < r.$$

(II) If $\bar{\lambda}\rho_0 \geq r$, from (7) we obtain

$$\inf_{u\in\mathcal{B}} [\frac{m_1}{p^-}\|u\|^{p^-} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))] \leq \frac{m_1}{p^-}\|\hat{u}\|^{p^-} + \bar{\lambda}(\rho_0 - \mathcal{F}(\hat{u})) \leq$$

$$\leq \frac{m_1}{p^-}\|\hat{u}\|^{p^-} + \frac{r}{\rho_0}(\rho_0 - \mathcal{F}(\hat{u})) \leq r.$$

We claim that

$$\inf_{u\in\mathcal{B}} \sup_{\lambda\in\Lambda} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] \geq r. \tag{9}$$

Infact, for every $u \in \mathcal{B}$ there are two cases:

(I) If $\mathcal{F}(u) < \rho_0$,

$$\sup_{\lambda\in\Lambda} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] = +\infty.$$

(II) If $\mathcal{F}(u) \geq \rho_0$, by (7)

$$\sup_{\lambda\in\Lambda} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] = \Phi(u) \geq \frac{m_0}{p^+}\|u\|^{p^+} \geq r.$$

From (8), (9) and the assumption (ii) of Theorem 3.1, this case verified.

Case 2. Assume that $\|u\| \geq 1$.
Similar to case 1:
Set for every $r > 0$

$$\theta_2(r) = \sup\{\mathcal{F}(u); u \in X, \frac{m_1}{p^-}\|u\|^{p^+} \leq r\}.$$

We claim that

$$\lim_{r\to 0^+} \frac{\theta_2(r)}{r} = 0. \tag{10}$$

In order to Proposition 2.3, for every $u \in X$ by continuous and compact embedding, it implies the existence of c_9 and c_{10} such that

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} F(x,u)dx \leq \int_{\mathbb{R}^N} K(|u|^{t(x)} + |u|^{z(x)})dx \leq K(\|u\|^{t^+}_{L^{t(x)}(\mathbb{R}^N)} + \|u\|^{z^+}_{L^{z(x)}(\mathbb{R}^N)})$$

$$\leq Kc_9(\|u\|^{t^+}_X + \|u\|^{z^+}_X) \leq Kc_{10}(r^{\frac{t^+}{p^+}} + r^{\frac{z^+}{p^+}}).$$

It follows from $\min\{t^+, z^+\} > p^+$ that

$$\lim_{r\to 0^+} \frac{\theta_2(r)}{r} = 0.$$

Using Lemma 3.5 $\hat{u} \neq 0$, therefore, due to (10), there is some $r \in \mathbb{R}$ such that

$$0 < r < \frac{m_1}{p^-}\|\hat{u}\|^{p^+}, \qquad 0 < \frac{\theta_2(r)}{r} < \frac{\mathcal{F}(\hat{u})}{\frac{m_1}{p^-}\|\hat{u}\|^{p^+}}.$$

Let $\rho_0 > 0$ such that

$$\theta_2(r) < \rho_0 < \frac{r\mathcal{F}(\hat{u})}{\frac{m_1}{p^-}\|\hat{u}\|^{p^+}}. \tag{11}$$

We claim that

$$\sup_{\lambda\in\Lambda} \inf_{u\in\mathcal{B}} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] < r. \tag{12}$$

Because of the mapping

$$\lambda \mapsto \sup_{\lambda\in\Lambda} \inf_{u\in\mathcal{B}} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))]$$

is upper semicontinuous on Λ, so

$$\lim_{\lambda\to+\infty} \inf_{u\in\mathcal{B}} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] \leq \lim_{\lambda\to+\infty} [\frac{m_1}{p^-}\|\hat{u}\|^{p^+} + \lambda(\rho_0 - \mathcal{F}(\hat{u}))] = -\infty.$$

Therefore, there exists $\bar{\lambda} \in \Lambda$

$$\sup_{\lambda\in\Lambda} \inf_{u\in\mathcal{B}} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] = \inf_{u\in\mathcal{B}} [\frac{m_1}{p^-}\|u\|^{p^+} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))].$$

We consider two cases:
(I) If $\bar{\lambda}\rho_0 < r$, we obtain

$$\inf_{u\in\mathcal{B}} [\frac{m_1}{p^-}\|u\|^{p^+} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))] \leq \bar{\lambda}\rho_0 < r.$$

(II) If $\bar{\lambda}\rho_0 \geq r$, from (11) we obtain

$$\inf_{u\in\mathcal{B}} [\frac{m_1}{p^-}\|u\|^{p^+} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))] \leq \frac{m_1}{p^-}\|\hat{u}\|^{p^+} + \bar{\lambda}(\rho_0 - \mathcal{F}(\hat{u})) \leq$$

$$\leq \frac{1}{p^-}\|\hat{u}\|^{p^+} + \frac{r}{\rho_0}(\rho_0 - \mathcal{F}(\hat{u})) \leq r.$$

Next, we claim that

$$\inf_{u\in\mathcal{B}} \sup_{\lambda\in\Lambda}[\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] \geq r. \tag{13}$$

For every $u \in \mathcal{B}$ two cases can occur:
(I) If $\mathcal{F}(u) < \rho_0$ we have

$$\sup_{\lambda\in\Lambda}[\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] = +\infty.$$

(II) If $\mathcal{F}(u) \geq \rho_0$ we have by (11)

$$\sup_{\lambda\in\Lambda}[\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] = \Phi(u) \geq \frac{m_0}{p^+}\|u\|^{p^-} \geq r.$$

For function $\mathcal{G}$ which satisfies (G), it follows from Lemma 3.3, that the functional $\mathcal{G} : X \to \mathbb{R}$ is locally Lipschitz with weakly sequentially semicontinuous. From Theorem 3.1 there exist $\lambda_1, \lambda_2 \in \Lambda$ (without loss of generality we may assume $0 < \lambda_1 < \lambda_2$) and $\sigma > 0$ with the following property that, for $\lambda \in [\lambda_1, \lambda_2]$ there exists $\mu_1 > 0$ in which: for every $\mu_1 \in]0, \mu[$, the functional $\Phi - \lambda\mathcal{F} - \mu\mathcal{G}$ admits at least three critical points $u_0, u_1, u_2 \in \mathcal{B}$ with $\|u_i\| < \sigma$. So by Lemma 3.4 u_0, u_1, u_2 are three solutions of the problem (1). □

4. Three radially symmetric solutions for a differential inclusion problem

In this part we apply Theorem 3.1 to show the existence of at least three radially symmetric solutions for a variational-hemivariational inequality. The main difficulty in studying our problem is that there is no compact embedding of $W_0^{1,p(x)}(\Omega)$ to $L^\infty(\mathbb{R}^N)$. However, the subspace of radially symmetric functions of $W_0^{1,p(x)}(\mathbb{R}^N)$, denoted by $W_{0,r}^{1,p(x)}(\mathbb{R}^N)$ can be embedded compactly into $L^\infty(\mathbb{R}^N)$ whenever $N < p^- \leq p^+ < +\infty$.

Choosing $X = W_{0,r}^{1,p(\cdot)}(\mathbb{R}^N)$ and applying the nonsmooth version of the principle of symmetric criticality we consider the differential inclusion problem

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u \in \lambda\partial a(x)F(x,u) + \mu\partial b(x)G(x,u) & \text{on } \mathbb{R}^N \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases} \tag{14}$$

where λ, μ are positive parameters and $F, G : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz functions. $a, b \in L^\infty(\mathbb{R}^N)$, are radially symmetric and $a, b \geq 0$.

Let $O(N)$ be the group of orthogonal linear transformations in $\mathbb{R}^N$. We say that a function $l : \mathbb{R}^N \to \mathbb{R}$ is radially symmetric if $l(gx) = l(x)$ for every $g \in O(N)$ and $x \in \mathbb{R}^N$. The action of the group $O(N)$ on $W_0^{1,p(\cdot)}(\mathbb{R}^N)$ can be defined by $(gu)(x) := u(g^{-1}x)$, for every $g \in O(N)$ and $u \in W_0^{1,p(\cdot)}(\mathbb{R}^N)$. We can define the subspace of radially symmetric functions of $W_0^{1,p(\cdot)}(\mathbb{R}^N)$ by

$$W_{0,r}^{1,p(\cdot)}(\mathbb{R}^N) = \{u \in W_0^{1,p(\cdot)}(\mathbb{R}^N) : gu = u, \forall g \in O(N)\}.$$

Proposition 4.1. [9] *The embedding* $W_{0,r}^{1,p(\cdot)}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, *is compact whenever* $N < p^- \leq p^+ < +\infty$.

The energy functional $\tilde{\mathcal{I}} : W_{0,r}^{1,p(\cdot)}(\mathbb{R}^N) \to \mathbb{R}$ associated to problem (14) is given by

$$\tilde{\mathcal{I}} = \Phi(u) - \lambda\tilde{\mathcal{F}}(u) - \mu\tilde{\mathcal{G}}(u) + \chi(u)$$

such that

$$\tilde{\mathcal{F}}(u) = \int_{\mathbb{R}^N} a(x)F(x,u)dx, \qquad \tilde{\mathcal{G}}(u) = \int_{\mathbb{R}^N} b(x)G(x,u)dx, \qquad \forall u \in W_{0,r}^{1,p(\cdot)}(\mathbb{R}^N),$$

where $\chi(u)$ is the indicator function of the set $\mathcal{B}$.
By the principle of symmetric criticality of Krawcewicz and Marzantowicz (cf. [22]), u is a critical point of $\mathcal{I}$ if and only if u is a critical point of $\tilde{\mathcal{I}}^r = \mathcal{I}|_{W_{0,r}^{1,p(\cdot)}(\mathbb{R}^N)}$.

Lemma 4.2. *Assuming (F_1) satisfies, $\mathcal{F} : X \to \mathbb{R}$ will be locally Lipschitz functional and sequentially weakly semicontinuous.*

Proof. By similar argument of Lemma 3.2 we show that $\mathcal{F}$ is Lipschitz continuous on each bounded subset of X. Let $u, v \in B(0, M)$ $(M > 0)$, and $\|u\|, \|u\| \leq 1$. From proposition 2.5 and thanks to proposition 2.3

$$\begin{aligned}
|\mathcal{F}(u) - \mathcal{F}(v)| &\leq \int_{\mathbb{R}^N} |a(x)(F(x,u(x)) - F(x,v(x)))|dx \\
&\leq \int_{\mathbb{R}^N} Ka(x)(|u(x)|^{t(x)-1} + |v(x)|^{t(x)-1} + |u(x)|^{z(x)-1} + |v(x)|^{z(x)-1}) \\
&\qquad\qquad \times|u(x) - v(x)|dx \\
&\leq K\|a\|_\infty\|u - v\|_\infty \int_{\mathbb{R}^N} |u(x)|^{t(x)-1}dx + \int_{\mathbb{R}^N} |v(x)|^{t(x)-1}dx \\
&\quad + \int_{\mathbb{R}^N} |u(x)|^{z(x)-1}dx + \int_{\mathbb{R}^N} |v(x)|^{z(x)-1}dx \\
&\leq K\|a\|_\infty\|u - v\|_\infty\Big(\|u\|_{L^{t(x)}}^{t^- -1} + \|v\|_{L^{t(x)}}^{t^- -1} + \|u\|_{L^{z(x)}}^{z^- -1} + \|v\|_{L^{z(x)}}^{z^- -1}\Big) \\
&\leq K\|a\|_\infty\|u - v\|_X\Big(\|u\|_X^{t^- -1} + \|v\|_X^{t^- -1} + \|u\|_X^{z^- -1} + \|v\|_X^{z^- -1}\Big) \\
&\leq 2K\|u - v\|_X(c_{11}M^{t^- -1} + c_{12}M^{z^- -1})
\end{aligned}$$

where $c_{11},\ c_{12}$ are positive constants.
We show $\partial\mathcal{F}$ is compact. Let $\{u_n\}$ be a sequence in X such that $\|u_n\| \leq M$ and choose $u_n^* \in \partial\mathcal{F}(u_n) \subseteq \int_{\mathbb{R}^N} a(x)\partial F(x,u_n(x))dx$ for any $n \in \mathbb{N}$. From (F_1) it follows that for any $n \in \mathbb{N},\ v \in X$,

$$\begin{aligned}
< u_n^*, v > \quad &\leq \quad \int_{\mathbb{R}^N} |u_n^*(x)||v(x)|dx \leq \int_{\mathbb{R}^N} K|a(x)|(|u(x)|^{t(x)-1} + |u(x)|^{z(x)-1})|v(x)|dx \\
&\leq \quad K\|a\|_{L^\infty}(c_{13}M^{t^- -1} + c_{14}M^{z^- -1})\|v\|,
\end{aligned}$$

where $c_{13},\ c_{14}$ are positive constants.
Therefore,

$$\|u_n^*\|_{X^*} \leq K\|a\|_{L^\infty}(c_{13}M^{t^- -1} + c_{14}M^{z^- -1}).$$

The sequence $\{u_n^*\}$ is bounded and hence, up to a subsequence, $u_n^* \rightharpoonup u^*$.
Suppose on the contrary; there exists $\epsilon > 0$ for which $\|u_n^* - u^*\|_{X^*} > \epsilon$ (choose a subsequence if necessary). For every $n \in \mathbb{N}$, we can find $v_n \in X$ with $\|v_n\| < 1$ and

$$\langle u_n^* - u^*, v_n\rangle > \epsilon. \tag{15}$$

Then, $\{v_n\}$ is a bounded sequence and up to a subsequence, $\{v_n\}$ be a sequence in $W^{1,p(\cdot)}_{r,0}(\Omega)$ which converges weakly to $v \in W^{1,p(\cdot)}_{r,0}(\Omega)$. By proposition 4.1, $v_n \to v$ strongly in $L^\infty(\Omega)$. Therefore,

$$|\langle u_n^* - u^*, v\rangle| < \frac{\epsilon}{4}, \ \ |\langle u^*, v_n - v\rangle| < \frac{\epsilon}{4}, \ \ \|v_n - v\|_{L^\infty} < \frac{\epsilon}{2K\|a\|_{L^\infty}(c_3M^{t^- -1} + c_4M^{z^- -1})}.$$

It follows that,

$$\begin{aligned}\langle u_n^* - u^*, v_n\rangle &\le \langle u_n^*, v_n - v\rangle + \langle u_n^* - u^*, v\rangle + \langle u^*, v - v_n\rangle \\ &\le \int_{\mathbb{R}^N} |u_n^*(x)||v_n(x) - v(x)|dx + \langle u_n^* - u^*, v\rangle + \langle u^*, v - v_n\rangle \\ &\le K\|a\|_{L^\infty}(c_{13}M^{t^- -1} + c_{14}M^{z^- -1})\|v_n - v\|_{L^\infty} \\ &\qquad + \langle u_n^* - u^*, v\rangle + \langle u^*, v - v_n\rangle \to 0,\end{aligned}$$

which contradicts (15). □

Lemma 4.3. *If G satisfies, then $\mathcal{G}$ is a locally Lipschitz functional with compact gradient.*

The proof is similar to Lemma (4.2).

Lemma 4.4. *If (F_2) holds, then for any $\lambda \in (0, +\infty)$, the function $\Phi - \lambda\mathcal{F}$ is coercive.*

Proof. For $u \in X$ such that $\|u\| \ge 1$

$$\begin{aligned}\mathcal{F}(u) &= \int_{\mathbb{R}^N} a(x)F(x,u)dx \le \int_{\mathbb{R}^N} H|a(x)|(|u|^{\alpha(x)} + |u|^{\beta(x)})dx \\ &\le H\|a\|_{L^\infty}(\|u\|^{\alpha^+}_{L^{\alpha(x)}(\mathbb{R}^N)} + \|u\|^{\beta^+}_{L^{\beta(x)}(\mathbb{R}^N)}).\end{aligned}$$

By the embedding theorem for suitable positive constant c_{15}, c_{16}

$$\mathcal{F}(u) \le H\|a\|_{L^\infty}(c_{15}\|u\|_X^{\alpha^+} + c_{16}\|u\|_X^{\beta^+}).$$

Hence, from Proposition 2.2, for any $\lambda > 0$,

$$\Phi(u) - \lambda\mathcal{F}(u) \ge \frac{1}{p^+}\|u\|_X^{p^-} - H\|a\|_{L^\infty}(c_{15}\|u\|_X^{\alpha^+} + c_{16}\|u\|_X^{\beta^+}).$$

Since $p^- > \min\{\alpha^+, \beta^+\}$, it implies that

$$\lim_{\|u\| \to +\infty} [\Phi - \lambda\mathcal{F}] = +\infty, \quad \forall u \in X, \ \lambda \in (0, +\infty).$$

□

Theorem 4.5. *Let $a, b \in L^\infty(\Omega)$ be two radial functions and F_1, F_2, F_3 are satisfied. Then there exist $\lambda_1, \lambda_2 > 0 (\lambda_1 < \lambda_2)$ and $\tilde{\sigma} > 0$ such that for every $\lambda \in [\lambda_1, \lambda_2]$ and every $\mathcal{G}$ satisfying G, there exists $\mu_1 > 0$ such that for every $\mu \in]0, \mu_1[$ problem* (14) *admits at least three distinct, radially symmetric solutions whose norms are less than $\tilde{\sigma}$.*

Proof. **Case 1.** Let us assume that $\|u\| < 1$.
Put for every $r > 0$,

$$\theta_1(r) = \sup\{\mathcal{F}(u); u \in X, \frac{m_1}{p^-}\|u\|^{p^-} \leq r\},$$

we prove that

$$\lim_{r \to 0^+} \frac{\theta_1(r)}{r} = 0. \tag{16}$$

In view of (F_1), it is follows that for every $\epsilon > 0$, there exists $c(\epsilon) > 0$ such that for every $x \in \mathbb{R}^N, u \in \mathbb{R}$ and $\xi \in \partial F(x,u)$

$$|\xi| \leq \epsilon |u|^{t(x)-1} + c(\epsilon)|u|^{z(x)-1}. \tag{17}$$

Applying Lebourgs mean value theorem and using the Sobolev embedding theorem for every $u \in X$, there exist suitable positive constants c_{17} and c_{18}

$$\begin{aligned}
\mathcal{F}(u) &= \int_{\mathbb{R}^N} F(x,u)dx \leq \int_{\mathbb{R}^N} Ka(x)(|u|^{t(x)} + |u|^{z(x)})dx \\
&\leq K\|a\|_{L^\infty}(\|u\|^{t^+}_{L^{t(x)}(\mathbb{R}^N)} + \|u\|^{z^+}_{L^{z(x)}(\mathbb{R}^N)}) \\
&\leq K\|a\|_{L^\infty} c_{17}(\|u\|^{t^+}_X + \|u\|^{z^+}_X) \leq K\|a\|_{L^\infty} c_{18}(r^{\frac{t^+}{p^-}} + r^{\frac{z^+}{p^-}}).
\end{aligned}$$

By using $\min\{t^+, z^+\} > p^-$ we conclude that

$$\lim_{r \to 0^+} \frac{\theta_1(r)}{r} = 0.$$

The remainder proof for the existence of three radially symmetric solutions of problem (14) is similarly to Theorem 3.7. □

References

- [1] M. Alimohammady, F. Fattahi, Existence of solutions to hemivaritional inequalities involving the p(x)-biharmonic operator, *Electron. J. Diff. Equ.* **2015** (2015), 1–12.
- [2] M. Allaoui, Existence of Solutions For A Robin Problem Involving The $p(x)-$Laplacian, *Applied Mathematics E-Notes* **14** (2014), 107–115.
- [3] T. Bartsch, M. Willem, Infinitely many nonradial solutions of a Euclidean scalar field equation, *J. Funct. Anal.* **117** (1993), 447–460.
- [4] G. Bonannoa, P. Winkert, Multiplicity results to a class of variational-hemivariational inequalities, *Topological methods in nonlinear analysis* **43**(2) (2014), 493–516.
- [5] S. Carl, V.K. Le, D. Motreanu, *Nonsmooth variational problems and their inequalities*, Springer Monographs in Mathematics, Springer, New York, 2007.
- [6] N.T. Chung, On some $p(x)-$Kirchhoff type equations with weights, *Rocky Mountain J. Appl. Math. Informatics* **32** (2014), 113–128.
- [7] F.H. Clarke, *Optimization and nonsmooth analysis*, Wiley, 1983.
- [8] F.J.S.A. Corrêa, G.M. Figueiredo, On an elliptic equation of p-Kirchhoff type via variational methods, *Bull. Aust. Math. Soc.* **74** (2006), 263–277
- [9] G.W. Dai, Infinitely many solutions for a $p(x)-$Laplacian equation in $\mathbb{R}^N$, *Nonlinear Anal.* **71** (2009), 1133–1139.
- [10] S. Dhanalakshmi, R. Murugesu, Existence of fractional order mixed type functional integro-differential equations with nonlocal conditions, *Int. J. Adv. Appl. Maths. and Mech.* **1**(3) (2014), 11–21.
- [11] L. Diening, Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Math. Nachr.* **268** (2004), 31–43.

[12] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents,* Lecture Notes, Vol. 2017, Springer-Verlag, Berlin, 2011.

[13] D.E. Edmunds, J. Rákosník, Density of smooth functions in $W^{k,p(x)}(\Omega)$, *Proc. R. Soc. A* **437** (1992), 229–236.

[14] A.R. El Amrouss, A. Ourraoui, Existence of solutions for a boundary problem involving $p(x)-$biharmonic operator, *Bol. Soc. Paran. Mat.* **31**(1) (2013), 179–192.

[15] L. Eugenio Cabanillas, LL. Adrian, G. Aliaga, M. Willy Barahona, V. Gabriel Rodriguez, Existence of solutions for a class of p(x)-Kirchhoff type equation via topological methods, *Rocky Mountain Int. J. Adv. Appl. Math. and Mech.* **2**(4) (2015), 64–72.

[16] X.L. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263** (2001), 424–446.

[17] X.L. Fan, Q.H. Zhang, Existence of solutions for $p(x)-$Laplacian Dirichlet problems, *Nonlinear Anal.* **52** (2003), 1843–1852.

[18] SH. Heidarkhani, F. Gharehgazlouei, A. Solimaninia, Existence of infinitely many symmetric solutions to perturbed elliptic equations with discontinuous nonlinearities in $\mathbb{R}^N$, *Electronic Journal of Differential Equations* 2015 (2015), 1–17.

[19] A. Iannizzotto, Three critical points for perturbed nonsmooth functionals and applications, *Nonlinear Analysis TMA* **72** (2010), 1319–1338.

[20] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

[21] O. Kováčik , J. Rákosnínk , On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.* **41** (1991), 592–618.

[22] W. Krawcewicz, W. Marzantowicz, Some remarks on the Ljusternik-Schnirelman method for non-differentiable functionals invariant with respect to a finite group action, *Rocky Mountain J. Math.* **20** (1990), 1041–1049.

[23] A. Kristály, W. Marzantowicz, C. Varga, A nonsmooth three critical points theorem with applications in differential inclusions, *J. Glob. Optim.* (2009).

[24] A. Kristály, Infinitely many radial and non-radial solutions for a class of hemivariational inequalities, *Rocky Mountain Journal of Mathematics* **35**(4) (2005), 1173–1190.

[25] D. Motreanu, P. D. Panagiotopoulos, *Minimax theorems and qualitative properties of the solutions of hemivariational inequalities*, Kluwer Academic Publishers, Dordrecht, 1999.

[26] D. Motreanu, V. Rădulescu, *Variational and non-variational methods in nonlinear analysis and boundary value problems*, Kluwer Academic Publishers, Boston-Dordrecht-London, 2003.

[27] V. Rădulescu, Nonlinear elliptic equations with variable exponent: old and new, *Nonlinear Analysis TMA* **121** (2015), 336–369.

[28] V. Rădulescu, D. Repovš, *Partial differential equations with variable exponents*, Variational Methods and Qualitative Analysis, CRC Press, Taylor & Francis Group, Boca Raton, 2015.

[29] B. Ricceri, Minimax theorems for limits of parametrized functions having at most one local minimum lying in a certain set, *Topol. Appl.* **153** (2006), 3308–3312.

[30] M. Ruzicka, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2002.

[31] R. Temam, A nonlinear eigenvalue problem: the shape at equilibrium of a confined plasma, *Arch. Ration. Mech. Anal.* **60**(1) (1975) 51–73.

[32] C. Varga, Existence and infinitely many solutions for an abstract class of hemivariational inequalities, *Journal of Inequalities and Applications* **2005** (2005), 89–105.

[33] P. Winkert, On the Boundedness of Solutions to Elliptic Variational Inequalities, *Set-Valued and Var. Anal.* **22**(4) (2014), 763–781.

[34] Z. Yuan, L. Huang, Nonsmooth extention of a three critical points theorem by Ricceri with an application to $p(x)-$laplacian differential inclusions, *Electron. J. Diff. Equ.* **2015** (2015), 1–16.

(M. Alimohammady, F. Fattahi) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MAZANDARAN
BABOLSAR, IRAN
E-mail address: `Amohsen@umz.ac.ir, F.Fattahi@stu.umz.ac.ir`

Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 115–125
ISSN: 1223-6934

Łukasiewicz Implication Prealgebras

Aldo V. Figallo and Gustavo Pelaitay

Abstract. In this paper we revise the Łukasiewicz implication prealgebras which we will call Łukasiewicz $I-$prealgebras to sum up. They were used by Antonio Jesús Rodríguez Salas on his doctoral thesis under the name of Sales prealgebras. These structures are a natural generalization of the notion of $I-$prealgebras, introduced by A. Monteiro in 1968 aiming to study using algebraic techniques the $\{\to\}$-fragment of the three-valued Łukasiewicz propositional calculus. The importance of Łukasiewicz $I-$prealgebras focuses on the fact that from these structures we can directly prove that Lindembaun-Tarski algebra in the $\{\to\}$-fragment of the infinite-valued Łukasiewicz implication propositional calculus is a Łukasiewicz residuation BCK-algebra in the sense of Berman and Blok [1]. This last result is indicated without a proof on Komori's paper ([8]) and it is suggested on his general lines on the Rodriguez Salas thesis.

2010 Mathematics Subject Classification. Primary 03G25, Secondary 06F35.
Key words and phrases. Łukasiewicz implication prealgebras, $I-$prealgebras, Łukasiewicz residuation $BCK-$algebras.

---

1. Introduction and preliminaries

In 1982, A. Iorgulescu said that she came up with the idea of the $I-$prealgebras after reading about preboolean sets in [11, 12] and about Nelson algebras and Łukasiewicz algebras in [10], on one side, and about $I-$algebras [13]. For details please go to [7].

On the other hand, in 1980, A. Monteiro introduced a particular class of I-prealgebras. In this paper, we will use Monteiro therminology.

In 1930 Łukasiewicz considered the matrix $Ł_{n+1} = \langle C_{n+1}, \to, \sim, D\rangle$ and $Ł = \langle [0,1], \to, \sim, D\rangle$, where:

(i) $C_{n+1} = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$, n is a positive integer and $[0,1]$ is the real interval;

(ii) If $p, q \in C_{n+1}$ or $p, q \in [0,1]$, then the implication, $\to$, is defined by the formula $p \to q = min\,\{1, 1-p+q\}$, the negation, $\sim$, by $\sim p = 1-p$; and

(iii) $D = \{1\}$ is the set of designated elements.

For the ones interested in focusing on the many algebraization of the Łukasiewicz propositional calculus, we recommend reading the important book [2] indicated in the references section.

In the following, we will denote with $(n+1)-$IŁ, $n \geq 1$, and with $\omega-$IŁ to the propositional calculus determined by the implicative parts of $Ł_{n+1}$ and Ł respectively.

In 1956, Rose [16] indicated an axiomatization of the $\omega-$IŁ, where he proved the substitution rules, the modus ponens and the axioms:

(C1) $p \to (q \to p)$,

Received April 3, 2016.
The support of CONICET is gratefully acknowledged by Gustavo Pelaitay.

(C2) $(p \to q) \to ((q \to r) \to (p \to r))$,
(C3) $((p \to q) \to q) \to ((q \to p) \to p)$,
(C4) $((p \to q) \to (q \to p)) \to (q \to p)$,
are sufficient. On the same article adding the axiom scheme:
(C5) $((x \to_n y) \to x) \to x$,
determined an axiomatization of the $(n+1)-$IŁ, where
(Ab1) $p \to_0 q = q$ and $p \to_{n+1} q = p \to (p \to_n q)$, for $n = 0, 1, 2 \ldots$.

In [9] Monteiro, with the purpose of studying the $3-$IŁ with algebraic techniques, introduced the concepts of I_3-prealgebras and 3-valued Łukasiewicz implication algebra. The results obtained by this author were exposed in 1968 in a course given at Universidad Nacional del Sur but they have not been published yet.

On this work, we take our research based on the algebraization method proposed by Monteiro, who has shown his excellent studies on many propositional calculus. To begin with, we consider the $I-$prealgebras and then the $I_{n+1}-$prealgebras, as generalizations of the I_3-prealgebras of Monteiro and we redo some proofs of the properties needed for the rest of the work exposed here, indicated by Monteiro in [9]. In particular, we concentrate on those properties in which the axiom referring to the n-valence, of the Definition 4.1, does not take place here.

2. Łukasiewicz $I-$prealgebras

Definition 2.1. The system $\langle A, \to, D\rangle$ is a Łukasiewicz implication prealgebra (or Łukasiewicz $I-$prealgebra) if we verify:

(i) $\langle A, \to\rangle$ is an algebra of type 2,
(ii) D is a non-empty subset of A such that for every $p, q, r \in D$ the conditions are verified:
  (R1) $p \to (q \to p) \in D$,
  (R2) $(p \to q) \to ((q \to r) \to (p \to r)) \in D$,
  (R3) $(p \vee q) \to (q \vee p) \in D$,
  (R4) $(p \to q) \vee (q \to p) \in D$, where (Ab2) $p \vee q = (p \to q) \to q$.
  And the modus ponens rule:
  (MP) $\dfrac{p \in D, p \to q \in D}{q}$,

Example 2.1.

(i) If $\langle A, \to\rangle$ is an algebra of type 2, then $\langle A, \to, A\rangle$ is an Łukasiewicz $I-$prealgebra.
(ii) The matrix $\langle C_{n+1}, \to, \{1\}\rangle$, $n \geq 1$, and $\langle [0,1], \to, \{1\}\rangle$ are Łukasiewicz $I-$ prealgebras.
(iii) If $\langle For(G), \to\rangle$ is an algebra of type 2 absolutely free and $\mathcal{T}$ is the set of the thesis of the $\omega-$IŁ, then $\langle For(G), \to, \mathcal{T}\rangle$ is an Łukasiewicz $I-$prealgebra.

Throughout this section A is the underlying set of the $I-$prealgebra $\langle A, \to, D\rangle$.

Definition 2.2. Let $\langle A, \to, D\rangle$ be a Łukasiewicz $I-$prealgebra. Let $p, q \in A$, we say that $p \preceq q$ if $p \to q \in D$.

Lemma 2.1. *Let $\langle A, \to, D\rangle$ be a Łukasiewicz $I-$prealgebra. Then the following properties are verified:*

(I1) *If* $p \preceq q$ *and* $q \preceq r$ *then* $p \preceq r$.
(I2) *If* $d \in D$ *and* $p \in A$, *then* $p \preceq d$.
(I3) $(q \to p) \to r \preceq p \to r$.
(I4) $p \preceq q \vee p$.
(I5) $q \preceq q \vee p$.
(I6) $(q \vee r) \to s \preceq q \to s$.
(I7) $(q \vee r) \to (p \to r) \preceq q \to (p \to r)$.
(I8) $p \to (q \to r) \preceq (q \vee r) \to (p \to r)$.
(I9) $p \to (q \to r) \preceq q \to (p \to r)$.
(I10) $p \to (q \to p) \preceq q \to (p \to p)$.
(I11) $q \preceq (p \to p)$.
(I12) $p \preceq p$.
(I13) $q \to r \preceq (p \to q) \to (p \to r)$.
(I14) *If* $q \preceq r$, *then* $p \to q \preceq p \to r$.

Proof. (I1):
(1) $p \to q \in D$, [hip.]
(2) $q \to r \in D$, [hip.]
(3) $p \to r \in D$, [(1), (2), R2, MP]
(4) $p \preceq r$. [(3), Definition 2.2]

(I2):
(1) $d \in D$ and $p \in A$, [hip.]
(2) $d \to (p \to d) \in D$, [R1]
(3) $p \to d \in D$. [(1), (2), MP]
(4) $p \preceq d$. [(3), Definition 2.2]

(I3):
(1) $p \to (q \to p) \in D$, [R1]
(2) $(p \to (q \to p)) \to (((q \to p) \to r) \to (p \to r)) \in D$, [R2]
(3) $((q \to p) \to r) \to (p \to r) \in D$. [(1), (2), MP]
(4) $(q \to p) \to r \preceq p \to r$. [(3), Definition 2.2]

(I4):
(1) $p \to ((q \to p) \to p) \in D$, [R1]
(2) $p \to (q \vee p) \in D$, [(1), Ab2]
(3) $p \preceq q \vee p$. [(2), Definition 2.2]

(I5):
(1) $p \preceq q \vee p$, [(I4)]
(2) $q \vee p \preceq p \vee q$, [R3, Definition 2.2]
(3) $p \preceq p \vee q$. [(1), (2), (I1)]

(I6):
(1) $(q \to (q \vee r)) \to (((q \vee r) \to s) \to (q \to s)) \in D$, [R2, Definition 2.2]
(2) $q \to (q \vee r) \in D$, [(I5), Definition 2.2]
(3) $((q \vee r) \to s) \to (q \to s) \in D$, [(2), (1), Definition 2.2, R2, MP]
(4) $(q \vee r) \to s \preceq q \to s$. [(3), Definition 2.2]

(I7):

We obtained it replacing s by $p \to r$ in (I6).

(I8):
(1) $(p \to (q \to r)) \to (((q \to r) \to r) \to (p \to r)) \in D$, [R2]
(2) $p \to (q \to r) \preceq (q \vee r) \to (p \to r)$. [(1), Ab2, Definition 2.2]

(I9):

(1) $p \to (q \to r) \preceq ((q \vee r)) \to (p \to r)$, [(I8)]

(2) $(q \vee r) \to (p \to r) \preceq q \to (p \to r)$, [(I7)]

(3) $p \to (q \to r) \preceq q \to (p \to r)$. [(1), (2), (I1)]

(I10):

We get this result substituting r for p in (I9).

(I11):

(1) $(p \to (q \to p)) \to (q \to (p \to p)) \in D$, [Definition 2.2, (I10)]

(2) $q \to (p \to p) \in D$. [(1), R1, MP]

(I12):

(1) $(p \to (q \to p)) \to (p \to p) \in D$, [(I11)]

(2) $p \to p \in D$. [(1), R1, MP]

(I13):

(1) $((p \to q) \to ((q \to r) \to (p \to r))) \to ((q \to r) \to ((p \to q) \to (p \to r))) \in D$, (I9)]

(2) $(p \to q) \to ((q \to r) \to (p \to r)) \in D$, [R2]

(3) $(q \to r) \to ((p \to q) \to (p \to r)) \in D$. [(2), (1), MP]

(I14):

(1) $q \to r \in D$, [hip.]

(2) $(p \to q) \to (p \to r) \in D$, [(1), (I13), MP]

(3) $p \to q \preceq p \to r$. [(2), Definition 2.2]

□

Theorem 2.2. *$\langle A, \preceq \rangle$ is a quasiorder set.*

Proof. The proof is followed by (I1) and (I12). □

Definition 2.3. Let $\langle A, \to, D \rangle$ be a Łukasiewicz I–prealgebra. Let $p, q \in A$. We will say that $p \equiv q$ if $p \preceq q$ and $q \preceq p$.

Theorem 2.3. *The relation $\equiv$ has the following properties:*

(i) *$p \preceq q$ and $q \equiv r$ imply $p \preceq r$,*

(ii) *$p \preceq q$ and $p \equiv s$ imply $s \preceq q$,*

(iii) *$p \preceq q$, $p \equiv s$ and $q \equiv r$ imply $s \preceq r$.*

Proof. The proof is followed by (I1). □

Theorem 2.4. *The relation $\equiv$ is compatible with the operation $\to$.*

(i) If $p \equiv q$ then $p \to r \equiv q \to r$:

(1) $p \to q \in D$, [hip.]

(2) $(p \to q) \to ((q \to r) \to (p \to r)) \in D$, [R2]

(3) $(q \to r) \to (p \to r) \in D$, [(1), (2), MP]

(4) $q \to r \preceq p \to r$, [(3), Definition 2.2]

In an analogous way, we can prove that:

(5) $p \to r \preceq q \to r$.

(ii) If $p \equiv q$ then $r \to p \equiv r \to q$:

(1) $p \to q \in D$, [hip.]

(2) $(p \to q) \to ((r \to p) \to (r \to q)) \in D$, [(I13)]

(3) $(r \to p) \to (r \to q) \in D$, [(1), (2), MP]

(4) $r \to p \preceq r \to q$, [(3), Definition 2.2]

Similarly, we show that:
(5) $r \to q \preceq r \to p$.

Theorem 2.5. *If* $t \in D$ *then* $[t] = D$*, where* $[t] = \{p \in A : p \equiv t\}$.

(i) $D \subseteq [t]$:
Indeed, let d be an element of D, then
(1) $t \in D$, [hip.]
(2) $d \in D$, [hip.]
(3) $d \to t \in D$, [(1), (I2)]
(4) $t \to d \in D$, [(2), (I2)]
(5) $d \preceq t$, [(3), Definition 2.2]
(6) $t \preceq d$, [(4), Definition 2.2]
(7) $d \equiv t$, [(5), (6), Definition 2.3]
(8) $d \in [t]$, [(7)]
(9) $D \subseteq [t]$, [(2), (8)]

(ii) $[t] \subseteq D$:
Indeed:
(1) $p \in [t]$, [hip.]
(2) $p \equiv t$, [(1)]
(3) $t \to p \equiv t \to t$, [(2), Theorem 2.4 (ii)]
(4) $t \to p \in D$, [(3), (I12)]
(5) $t \in D$, [hip.]
(6) $p \in D$. [(5), (4), MP]
(7) $[t] \subseteq D$. [(1), (6)]

3. Łukasiewicz $I-$prealgebras of the Lindenbaum-Tarski algebras

As a consequence of Theorem 2.4 we can explain the quotient set. If $[p] \to [q] = [p \to q]$ and $D = \mathbf{1}$, then $\langle A/\equiv, \to, \mathbf{1}\rangle$ is an algebra of type $(2, 0)$.

Definition 3.1. The algebra $\langle A/\equiv, \to, \mathbf{1}\rangle$ is called the Lindenbaum-Tarski algebra of the Łukasiewicz $I-$prealgebra $\langle A, \to, D\rangle$.

With the intention of indicating important properties of the Lindenbaum-Tarski algebra we previously noted a list of additional properties valid in every Łukasiewicz $I-$prealgebra:

Lemma 3.1. *Let* $\langle A, \to, D\rangle$ *be a Łukasiewicz* $I-$*prealgebra. Then the following properties can be verified:*
(I15) $q \vee q \equiv q$
(I16) $(q \to r) \to (p \to r)) \preceq (p \vee r) \to (q \vee r)$.
(I17) $p \to q \preceq (p \vee r) \to (q \vee r)$.
(I18) $p \to r \preceq (p \vee q) \to (r \vee q)$.
(I19) $p \to q \preceq (r \vee p) \to (r \vee q)$.
(I20) $q \to r \preceq (r \vee q) \to (r \vee r)$.

Proof. (I15):
(1) $q \to ((q \to q) \to q) \in D$, [R1]
(2) $q \to (q \vee q) \in D$, [(1), Ab2]

(3) $(q \to q) \to ((q \to q) \vee q) \in D$, [(I5), Definition 2.3]
(4) $(q \to q) \to ((q \to q) \to ((q \to q) \vee q)) \in D$, [(2), (I2)]
(5) $(q \to q) \vee q \in D$, [(4), (I12), MP]
(6) $((q \to q) \to q) \to q \in D$, [(5), Ab2]
(7) $(q \vee q) \to q \in D$, [(6), Ab2]
(8) $q \vee q \equiv q$. [(2), (7), Definition 2.3]

(I16):
(1) $((q \to r) \to (p \to r)) \to (((p \to r) \to r) \to ((q \to r) \to r)) \in D$, [R2]
(2) $((q \to r) \to (p \to r)) \to ((p \vee r) \to (q \vee r)) \in D$. [(1), Ab2]
(3) $(q \to r) \to (p \to r)) \preceq (p \vee r) \to (q \vee r)$. [(2), Definition 2.2]

(I17):
(1) $p \to q \preceq (q \to r) \to (p \to r)$, [R2, Definition 2.2]
(2) $(p \vee r) \to (q \vee r) \in D$, [R3]
(3) $(q \to r) \to (p \to r) \preceq (p \vee r) \to (q \vee r)$, [(2), (I2), Definition 2.2]
(4) $p \to q \preceq (p \vee r) \to (q \vee r)$. [(1), (3), (I1)]

(I18):
Comes from (I3) replacing q by r and r by q.

(I19):
(1) $p \vee r \equiv r \vee p$, [R3, Definition 2.2, Definition 2.3]
(2) $q \vee r \equiv r \vee q$, [R3, Definition 2.2, Definition 2.3]
(3) $p \to q \preceq (q \to r) \to (p \to r)$, [R2, Definition 2.2]
(4) $(q \to r) \to (p \to r) \preceq ((p \vee r) \to r) \to ((q \to r) \to r)$, [R2, Definition 2.2]
(5) $p \to q \preceq (p \vee r) \to (q \vee r)$, [(3), (4), (I1), Ab2]
(6) $(p \vee r) \to (q \vee r) \equiv (r \vee p) \to (q \vee r)$ [(1), Theorem 2.4]
$\equiv (r \vee p) \to (r \vee q)$, [(2), Theorem 2.4]
(7) $p \to q \preceq (r \vee p) \to (r \vee q)$. [(5), (6), Theorem 2.3]

(I20):
Results from (I19) substituting p for q and q for r. □

Now we can analyze the order given by the algebra $A/\equiv$. For that we give some results.

Definition 3.2. Let $p, q \in A$; $[p] \leq [q]$ if and only if $p \preceq q$.

The pair $\langle A/\equiv, \leq \rangle$ is an ordered set which has the properties mentioned in Theorem 3.2.

Theorem 3.2. *$\langle A/\equiv, \leq \rangle$ is an ordered set with a last element $\mathbf{1}$. Besides, it is a join-lattice where the greatest of the elements $[p]$ and $[q]$ is $[p] \vee [q] = [p \vee q]$.*

Proof. (i) $\leq$ is an order: It is a consequence of Theorem 2.2 and the Definition 2.3.
(ii) $[p] \leq \mathbf{1}$, for every $p \in A$: Let $p \in A$, then:
(1) It exists $t \in D$, [Definition 2.1]
(2) $p \to t \in D$, [(I2), Definition 2.2]
(3) $p \to (p \to t) \in D$, [(2), (I2), Definition 2.2]
(4) $[p \to t] = \mathbf{1}$ [(2), Theorem 2.5]
(5) $[p] \leq \mathbf{1}$. [(3), (4), Definition 2.2, Definition 3.2]
(iii) $[p \vee q]$ is the supremum of $[p]$ and $[q]$: Indeed, we can verify:
(s1) $p \preceq p \vee q$, [(I5)]
(s2) $q \preceq p \vee q$, [(I6)]

(s3) $p \preceq r$ and $q \preceq r$ imply $p \vee q \preceq r$:
(1) $p \preceq r$, [hip.]
(2) $q \preceq r$, [hip.]
(3) $p \vee q \preceq r \vee q$, [(1), (I18), Definition 2.2, MP]
(4) $r \vee q \preceq r \vee r$, [(2), (I20), Definition 2.2, MP]
(5) $r \vee r \equiv r$, [(I1)]
(6) $p \vee q \preceq r$. [(3), (4), (5), (I1), Definition 2.2, Theorem 2.3]
□

On the other hand, we verify:

Theorem 3.3. *The Lindenbaum-Tarski algebra of the Łukasiewicz I−prealgebra* $\langle A, \rightarrow, D\rangle$ *satisfy the properties:*
(W1) $1 \rightarrow x = x$,
(W2) $x \rightarrow (y \rightarrow x) = 1$,
(W3) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,
(W4) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(W5) $((x \rightarrow y) \rightarrow (y \rightarrow x)) \rightarrow (y \rightarrow x) = 1$.

This is, the Łukasiewicz $I-$prealgebras of the Lindenbaum-Tarski algebras are the algebras that satisfy the identities W1, ..., W5.

From the third of Example 2.1 and the Theorem 3.3 we get a proof that the Lindenbaum-Tarski algebra of the $\omega-$IŁ is a Łukasiewicz residuation algebra [1].

4. Łukasiewicz $I_{n+1}-$prealgebras

In this section, we will analize a particular class of Łukasiewicz I–prealgebras.

Definition 4.1. A Łukasiewicz $I-$prealgebra $\langle A, \rightarrow, D\rangle$ is a Łukasiewicz $I_{n+1}-$ prealgebra if for every p, $q \in A$ the following property is verified:
(R5) $(p \rightarrow_n q) \vee p \in D$.

Onwards, to sum up, we write
(Ab3) $p \mapsto q = p \rightarrow_n q$,

The operation $\mapsto$, which we will call weak implication, defined in (Ab3), has the following properties.

Theorem 4.1. *In very Łukasiewicz* $I_{n+1}-$*prealgebra* $\langle A, \rightarrow, D\rangle$ *we verify:*
(DR1) $p \mapsto (q \mapsto p) \in D$,
(DR2) $(p \mapsto (q \mapsto r)) \mapsto ((p \mapsto q) \mapsto (p \mapsto r)) \in D$,
(DR3) $((p \mapsto q) \mapsto p) \mapsto p \in D$.

The proof of the Theorem 4.1 is a consequence of the following properties:

Lemma 4.2. *For every Łukasiewicz* $I_{n+1}-$*prealgebra* $\langle A, \rightarrow, D\rangle$ *the following properties are verified:*
(a) $p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \rightarrow r)$
(b) $p \rightarrow (q \mapsto p) \in D$.
(c) $p \mapsto (q \mapsto p) \in D$.
(d) $(p \mapsto q) \rightarrow p \equiv p$.
(e) $(p \mapsto q) \mapsto p \equiv p$.

(f) $((p \mapsto q) \mapsto p) \mapsto p \in D$.
(g) $p \to (p \mapsto q) \equiv p \mapsto q$.
(h) $p \mapsto (p \mapsto q) \equiv p \mapsto q$.
(i) $p \mapsto (q \to r) \equiv q \to (p \mapsto r)$.
(j) $p \mapsto (q \mapsto r) \equiv q \mapsto (p \mapsto r)$.
(k) $p \to q \preceq p \mapsto q$.
(l) $p \mapsto (q \to r) \preceq (p \mapsto q) \to (p \mapsto r)$.
(ll) $p \mapsto (q \to r) \preceq (p \mapsto q) \mapsto (p \mapsto r)$.
(m) $p \mapsto (q \mapsto r) \preceq (p \mapsto q) \mapsto (p \mapsto r)$.

Proof. (a) The proof is adjacent to (I9) and the Definition 2.3.

(b)
(1) $p \to (q \to p) \in D$, [R1]
(2) $q \to (p \to (q \to p)) \in D$, [(1), (I2)]
(3) $p \to (q \to (q \to p)) \in D$, [(2), (I9), MP]
(4) $p \to (q \to_2 p) \in D$. [(3), Ab1]

If $n = 2$, the proof is done. On the contrary:
(5) $q \to (p \to (q \to_2 p)) \in D$, [(4), (I2)]
(6) $p \to (q \to (q \to_2 p)) \in D$, [(5), (I9), MP]
(7) $p \to (q \to_3 p) \in D$. [(6), Ab1]

Repeating the process, we obtain:
$$p \to (q \mapsto p) \in D.$$

(c) It is a consequence of (b) and of Ab3.

(d)
(1) $((p \mapsto q) \to p) \to p \in D$, [R5, Ab3, Ab2]
(2) $p \to ((p \mapsto q) \to p) \in D$, [R1]
(3) $(p \mapsto q) \to p \equiv p$. [(1), (2), Definition 2.2]

(e)
(1) $(p \mapsto q) \to p \equiv p$, [(d)]
(2) $(p \mapsto q) \to ((p \mapsto q) \to p) \equiv (p \mapsto q) \to p$, [(1), Theorem 2.4]
$\equiv p$ [(1)]
(3) $(p \mapsto q) \to_2 p \equiv p$. [(2), Ab3]

If $n = 2$ the proof is finished, if $n \geq 3$ repeating the process we get to:
(j) $(p \mapsto q) \to_n p \equiv p$,
(j+1) $(p \mapsto q) \mapsto p \equiv p$. [(j), Ab3]

(f)
(1) $(p \mapsto q) \mapsto p \equiv p$, [(e)]
(2) $((p \mapsto q) \mapsto p) \to p \equiv p \to p$, [(1), Theorem 2.4]
(3) $((p \mapsto q) \mapsto p) \to p \in D$, [(2), (I12), Definition 2.2]
(4) $((p \mapsto q) \mapsto p) \to (((p \mapsto q) \mapsto p) \to p) \in D$, [(3), (I2)]
(5) $((p \mapsto q) \mapsto p) \to_2 p \in D$. [(4), Ab3]

If $n = 2$ the proof is finished. If not, repeating the process we get to:
(l) $((p \mapsto q) \mapsto p) \to_n p \in D$,
(l+1) $((p \mapsto q) \mapsto p) \mapsto p \in D$. [(l), Ab3]

(g)
(1) $((p \mapsto q) \to p) \to p \in D$, [R5, Ab2]
(2) $(p \to (p \mapsto q)) \to (p \mapsto q) \in D$, [(1), R3, MP]
(3) $p \to (p \mapsto q) \preceq p \mapsto q$, [(2), Definition 2.2]

(4) $p \mapsto q \preceq p \to (p \mapsto q)$, [R1, Definition 2.2]

(5) $p \to (p \mapsto q) \equiv p \mapsto q$. [(3), (4), Definition 2.3]

(h)

(1) $p \to (p \mapsto q) \equiv p \mapsto q$, [(g)]

(2) $p \to (p \to (p \mapsto q)) \equiv p \to (p \mapsto q)$ [(1), Theorem 2.4]

$\equiv p \mapsto q$, [(g)]

(3) $p \to_2 (p \mapsto q) \equiv p \mapsto q$. [(2), Ab1]

If $n = 2$ the proof is finished. If not, repeating the process, we get to:

(j) $p \to_n (p \mapsto q) \equiv p \mapsto q$,

(j+1) $p \mapsto (p \mapsto q) \equiv p \mapsto q$. [(j), Ab1]

(i)

(1) $p \to (q \to r) \equiv q \to (p \to r)$, [(a)]

(2) $p \to (p \to (q \to r)) \equiv p \to (q \to (p \to r))$ [(1), Theorem 2.4]

$\equiv q \to (p \to (p \to r))$, [(a)]

(3) $p \to_2 (q \to r) \equiv q \to (p \to_2 r)$. [(2), Ab1]

If $n = 2$, from (3) and Ab3 we obtain

(4) $p \mapsto (q \to r) \equiv q \to (p \mapsto r)$.

If $n > 2$, repeating the process we obtain:

(l) $p \to_n (q \to r) \equiv q \to (p \to_n r)$,

(l+1) $p \mapsto (q \to r) \equiv q \to (p \mapsto r)$. [(k), Ab3]

(j)

(1) $q \to (p \mapsto r) \equiv p \mapsto (q \to r)$, [(j)]

(2) $q \to (q \to (p \mapsto r)) \equiv q \to (p \mapsto (q \to r))$ [(1), Theorem 2.4]

$\equiv p \mapsto (q \to (q \mapsto r))$, [(j)]

(3) $q \to (q \mapsto r) \equiv q \mapsto r$, [(g)]

(4) $q \to_2 (p \mapsto r) \equiv p \mapsto (q \mapsto r)$. [(2), (3), Theorem 2.4, Ab1]

If $n = 2$, we obtain:

(5) $q \mapsto (p \mapsto r) \equiv p \mapsto (q \mapsto r)$. [(4), Ab3]

If $n > 2$, repeating the process we obtain:

(l) $q \to_n (p \mapsto r) \equiv p \mapsto (q \mapsto r)$,

(l+1) $q \mapsto (p \mapsto r) \equiv p \mapsto (q \mapsto r)$. [(l), Ab3]

(k)

(1) $p \to q \preceq p \to (p \to q)$, [R1, Definition 2.2]

(2) $p \to q \preceq p \to_2 q$. [(1), Ab1]

If $n = 2$, the proof is over. On the contrary, repeating the process we get:

(i) $p \to q \preceq p \to_n q$,

(i+1) $p \to q \preceq p \mapsto q$. [(i), Ab3]

(l)

(1) $q \preceq (q \to r) \to r$, [(I5), Ab2]

(2) $p \mapsto q \preceq p \mapsto ((q \to r) \to r)$, [(1), (I14), Ab3]

(3) $p \mapsto q \preceq (q \to r) \to (p \mapsto r)$, [(2), (i)]

(4) $(p \mapsto q) \to ((q \to r) \to (p \mapsto r)) \in D$, [(3), Definition 2.2]

(5) $(q \to r) \to ((p \mapsto q) \to (p \mapsto r)) \in D$, [(4), (I9), Definition 2.2]

(6) $q \to r \preceq (p \mapsto q) \to (p \mapsto r)$, [(5), Definition 2.2]

(7) $p \mapsto (q \to r) \preceq p \mapsto ((p \mapsto q) \to (p \mapsto r))$, [(6), (I14), Ab3]

(8) $p \mapsto (q \to r) \preceq (p \mapsto q) \to (p \to (p \mapsto r))$, [(7), (i)]

(9) $p \mapsto (q \to r) \preceq (p \mapsto q) \to (p \mapsto r)$, [(8), (g)]

(ll)
(1) $p \mapsto (q \to r) \preceq (p \mapsto q) \to (p \mapsto r)$, [(i)]
(2) $p \to (p \mapsto (q \to r)) \preceq p \to ((p \mapsto q) \to (p \mapsto r))$, [(1), (I14)]
(3) $p \mapsto (q \to r) \preceq (p \mapsto q) \to (p \to (p \mapsto r))$, [(2), (I9), (g)]
(4) $p \mapsto (q \to r) \preceq (p \mapsto q) \to (p \mapsto r))$. [(3), (g)]
(m)
(1) $p \mapsto (q \mapsto r) = p \mapsto (q \to (q \to_{n-1} r))$. [Ab1, Ab3]
For $n = 2$, we verify:
(2) $p \mapsto (q \mapsto r) \preceq (p \mapsto q) \to (p \mapsto (q \to_{n-1} r))$, [(1), (ll)]
(3) $p \mapsto (q \to_{n-1} r) = p \mapsto (q \to r)$ [$n = 2$]
$\preceq (p \mapsto q) \to (p \mapsto r)$, [(g)]
(4) $(p \mapsto q) \to (p \mapsto (q \to_{n-1} r)) \preceq (p \mapsto q) \to ((p \mapsto q) \to (p \mapsto r))$, [(3), Theorem 2.4 (ii)]
(5) $(p \mapsto q) \to (p \mapsto (q \to_{n-1} r)) \preceq (p \mapsto q) \mapsto (p \mapsto r)$, [(4), $n = 2$, Ab1, Ab3, (g)]
(6) $p \mapsto (q \mapsto r) \preceq (p \mapsto q) \mapsto (p \mapsto r)$. [(2), (5), (I1)]
For $n = 3$, we have:
(7) $p \mapsto (q \to (q \to_{n-1} r)) \preceq (p \mapsto q) \to (p \mapsto (q \to_{n-1} r))$, [(j)]
(8) $p \mapsto (q \to_{n-1} r) = p \mapsto (q \to (q \to_{n-2} r))$ [Ab1]
$\preceq (p \mapsto q) \to (p \mapsto (q \to_{n-2} r))$, [(j)]
(9) $p \mapsto (q \to_{n-2} r) = p \mapsto (q \to r)$, [$n = 3$]
(10) $p \mapsto (q \to_{n-2} r) \preceq (p \mapsto q) \to (p \mapsto r)$, [(4), (j)]
(11) $(p \mapsto q) \to (p \mapsto (q \to_{n-2} r)) \preceq (p \mapsto q) \to ((p \mapsto q) \to (p \mapsto r))$, [(5), Theorem 2.4(ii)]
(12) $p \mapsto (q \to_{n-1} r) \preceq (p \mapsto q) \to_2 (p \mapsto r)$, [(4), (11), (I1), Ab1]
(13) $(p \mapsto q) \to (p \mapsto (q \to_{n-1} r)) \preceq (p \mapsto q) \to_3 (p \mapsto r)$, [(12), Theorem 2.4(ii), Ab1]
(14) $p \mapsto (q \mapsto r) \preceq (p \mapsto q) \mapsto (p \mapsto r)$. [(1), (7), (13), (I1), Ab3, $n = 3$]
For $n \geq 4$ we proceed in an analogous way. ☐

An interesting result to remark is the following.

Lemma 4.3. *If $\mathcal{A} = \langle A, \to, D \rangle$ is a prealgebra which verifies* R1 *to* R5*, then the following conditions are equivalent:*
(i) *$\mathcal{A}$ verifies the modus ponens rule,*
(ii) *$\mathcal{A}$ verifies the weak modus ponens rule* (MPD) $\dfrac{p \in D, p \mapsto q \in D}{q}$,

(MP) $\Rightarrow$ (MPD):
(1) $p \in D$, [hip.]
(2) $p \mapsto q \in D$, [hip.]
(3) $p \to (p \to_{n-1} q) \in D$, [(2), Ab1]
(4) $p \to_{n-1} q \in D$, [(3), (1), (MP)]
if $n = 2$ the proof is done. On the contrary, repeating the process, we get to:
(j) $p \to q \in D$,
(j+1) $q \in D$. [(1), (j), (MP)]
(MPD) $\Rightarrow$ (MP):
(1) $p \in D$, [hip.]
(2) $p \to q \in D$, [hip.]

(3) $p \mapsto q \in D$, [(2), (I1)]
(4) $q \in D$. [(1), (3), (MPD)]

5. Lindenbaum-Tarski algebras of the Łukasiewicz I_{n+1}−prealgebras

Theorem 5.1. *If* $\mathcal{A} = \langle A, \to, D\rangle$ *is a Łukasiewicz* I_{n+1}*−prealgebra, then the algebra of Lindenbaum-Tarski* $\langle A/\equiv, \to, \mathbf{1}\rangle$ *of* $\mathcal{A}$ *is a Łukasiewicz residuation algebra that verifies the additional identity:*
(I6) $(x \to_n y) \vee x = 1$.

Proof. It is consequence of Theorem 3.3, (R5) and Theorem 2.5. □

That is to say, the Lindenbaum-Tarski algebra of the Łukasiewicz I_{n+1}−prealgebras are $(n+1)$−valued Łukasiewicz residuation algebras.

References

[1] J. Berman, W. Blok, Free Łukasiewicz and hoop residuation algebras, *Studia Logica* **77** (2004), 153–180.
[2] R. Cignoli, I. D'ottaviano, D. Mundice, *Ālgebras das Lōgicas de Łukasiewicz.* Centro de Lógica, Epistemologia e Historia da Ciécia. UNICAMP, COLEÇÃO CLE. 1995
[3] A.V. Figallo, A. Figallo Orellano, M. Figallo, A. Ziliani, Łukasiewicz residuation algebras with infimum, *Demonstratio Mathematica* **40**(2007), no. 4, 751–758.
[4] A.V. Figallo, I_{n+1}–álgebras con operaciones adicionales, Doctoral Thesis, Universidad Nacional del Sur, Bahía Blanca, Argentina, 1989.
[5] J.M. Font, A. Rodríguez, A. Torrens, Wajsberg Algebras, *Stochastica* **8** (1984), no. 1, 5–31.
[6] L. Iturrioz, O. Rueda, Algèbres Implicatives Trivalentes de Lukasiewicz Libres, *Discrete Mathematics* **18** (1977), 35–44.
[7] A. Iorgulescu, $\mathcal{I}$−prealgebras, Discrete Mathematics 126 (1994) 415–419.
[8] Y. Komori, The separation theorem of the ω–valued Lukasiewicz propositional logic, *Rep. Fas. of Sc. Shizuoka University* **12** (1978), 1–5.
[9] A. Monteiro, *Algebras implicativas trivalentes de Lukasiewicz*, Lectures given at the Univ. Nac. del Sur, Bahía Blanca, Argentina, 1968.
[10] A. Monteiro, Construction des algeèbres de Łukasiewicz trivalentes dans les algeèbres de Boole Monadiques-I, *Notas de Logica Mat.* **11** (1974).
[11] D. Ponasse, *Logique mathématique. Éléments de base: Calcul propositionnel, calcul des prédicats*, Éditions de l'Office Central de Librairie (O.C.D.L.), Paris, 1967.
[12] D. Ponasse, J.C. Carrega, *Algèbre et topologie booléennes*, Masson, Paris, 1979.
[13] H. Rasiowa, *An algebraic approach to non-classical logics*, North-Holland Publishing Co., Amsterdam-London, 1974.
[14] A.J. Rodriguez Salas, Un estudio algebraico de los cálculos proposicionales de Lukasiewicz, Doctoral Thesis, Univ. de Barcelona, 1980.
[15] A. Rose, Formalisation du calcul propositionnel implicatif a $\aleph_0$–valeurs de Lukasiewicz, *C. R. Acad. Sci. Paris* **243** (1956), 1183–1185.
[16] A. Rose, Formalisation du calcul propositionnel implicatif a m–valeurs de Lukasiewicz, *C. R. Acad. Sci. Paris* **243** (1956), 1263–1264.

(Aldo V. Figallo) Instituto de Ciencias Básicas, Universidad Nacional de San Juan, 5400 San Juan, Argentina
E-mail address: `avfigallo@gmail.com`

(Gustavo Pelaitay) Insituto de Ciencias Básicas, Universidad Nacional de San Juan, 5400 San Juan, Argentina and Departamento de Matemática, Universidad Nacional de San Juan, 5400 San Juan, Argentina
E-mail address: `gpelaitay@gmail.com`

Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 126–136
ISSN: 1223-6934

Generalized ring-groupoids

Mustafa Habil GÜRSOY

Abstract. In this work, we are going to present the concept of generalized ring-groupoid. Also, we are going to investigate some characterizations about the generalized ring-groupoids. We are going to introduce the concept of generalized subring-groupoid. So we construct the category of generalized ring-groupoids. Furthermore, we are going to discuss a new class of the generalized ring-groupoids, which we will say it "M-ring-groupoid". In the end of the paper, we are going to give the product of generalized ring-groupoids.

2010 Mathematics Subject Classification. Primary 22A22; Secondary 57M10.
Key words and phrases. Groupoid, generalized ring, generalized ring-groupid.

---

1. Introduction

The concept of generalized ring was first defined by Molaei [13] in 2003. Later, some algebraic properties of the generalized ring which is a new concept in literature have been studied in [7]. There is the concept of generalized group in the structure of generalized ring. The concept was again defined by Molaei [12] is an interesting generalization of groups. While there is only one identity element in a group, each element in a generalized group has a unique identity element. With this property, every group is a generalized group.

Another algebraic notion covered in the present study is groupoid which was defined by Brandt [1] in 1926. But, in the category theoretical approach, a groupoid is a small category whose every morphism is an isomorphism. After introducing of topological and differentiable groupoids by Ehresmann [4] in 1950s, it has been studied by many mathematicians with different approaches [3, 9]. One of these different approaches is structured groupoid which is obtained with adding another algebraic structure such that the composition of groupoid is compatible with the operation of the added algebraic structure [2, 5, 10, 14]. The best knowns of the structured groupoids are the concepts of group-groupoid and ring-groupoid. The group-groupoid which is a group object in the category of groupoids was defined by Brown and Spencer [2]. The concept of ring-groupoid defined by [15] has been studied by many mathematicians [10, 11].

In this study, we extend the concept of ring-groupoid to the concept of generalized ring-groupoid by adding the structure of generalized ring to a groupoid such that the composition of the groupoid and the operations of the generalized ring are compatible. In other words, a generalized ring-groupoid is a generalized ring object in the category of groupoids. Thus, we construct the category of the generalized ring-groupoids. Also,

---

Received April 12, 2016.

we present two concept related to the generalized ring-groupoids: generalized subring-groupoid and M-ring-groupoid.

2. Preliminaries

This section of the paper is devoted to give basic definitions and concepts related to the generalized rings and groupoids. We will consider these concepts under two headings: generalized rings and groupoids.

2.1. Generalized Rings. In this subsection, it is given some basic recalls of the concept of generalized ring which was first defined by Molaei. Let us start with the definition of a generalized group that the existing in the structure of a generalized ring.

Definition 2.1. [12] A generalized group G is a non-empty set admitting an operation called multiplication subject to the set of rules given below:
i) $(ab)c = a(bc)$, for all $a, b, c \in G$
ii) For each $a \in G$, there exists a unique $e(a) \in G$ such that $ae(a) = e(a)a = a$
iii) For each $a \in G$, there exists $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e(a)$.

Let us list some properties of generalized groups via following lemma.

Lemma 2.1. [12] *Let G be a generalized group. Then,*
i) For each $a \in G$, there is a unique element $a^{-1} \in G$.
ii) For each $a \in G$, we have $e(a) = e(a^{-1})$ and $e(e(a)) = e(a)$.
iii) For each $a \in G$, we have $(a^{-1})^{-1} = a$.

It is easily from Definition 2.1 that every group is a generalized group. But it is not true in general that every generalized group is a group.

Let us state the relation between group and generalized group by the following lemma.

Lemma 2.2. [12] *Let G be a generalized group and $ab = ba$ for all $a, b \in G$. Then, G is a group.*

In other words, every abelian generalized group is a group.

Example 2.1. [12] Let $G = IR \times (IR \setminus \{0\})$. Then G with the multiplication $(a, b) \cdot (c, d) = (bc, bd)$ is a generalized group in which for all $(a, b) \in G$, $e(a, b) = (a/b, 1)$ and $(a, b)^{-1} = (a/b^2, 1/b)$.

Example 2.2. [5] Let G with the multiplication m be a generalized group. Then, $G \times G$ with the multiplication

$$m_1((a, b), (c, d)) = (m(a, c), m(b, d))$$

is a generalized group. For any element $(a, b) \in G \times G$, the identity element is $e_1(a, b) = (e(a), e(b))$ and the inverse element is $(a, b)^{-1} = (a^{-1}, b^{-1})$.

Definition 2.2. [12] If $e(ab) = e(a)e(b)$ for all $a, b \in G$, then G is called normal generalized group.

Definition 2.3. [12] A non-empty subset H of a generalized group G is a generalized subgroup of G if and only if for all $a, b \in H$, $ab^{-1} \in H$.

Definition 2.4. [12] A generalized subgroup N of the generalized group G is said to be normal if there exist a generalized group H and a homomorphism $f : G \to H$ such that for each $a \in G$, $N_a = kerf_a$ provided that $N_a \neq \emptyset$, where $N_a = N \cap G_a$.

Example 2.3. [12] Let G be a generalized group of Example 2.1. Then $N = \{(a, b) : a = b \text{ or } a = 3b\}$ is a generalized normal subgroup of G.

Definition 2.5. [12] Let G and H be two generalized groups. A generalized group homomorphism from G to H is a map $f : G \to H$ such that $f(ab) = f(a)f(b)$ for all $a, b \in G$.

Theorem 2.3. [12] *Let $f : G \to H$ be a homomorphism of the distinct generalized groups G and H. Then,*
i) $f(e(a)) = e(f(a))$ is an identity element in H for all $a \in G$.
ii) $f(a^{-1}) = (f(a))^{-1}$
iii) If K is a generalized subgroup of G, then $f(K)$ is a generalized subgroup of H.

Now we can give definition of a generalized ring.

Definition 2.6. [13] A generalized ring R is a non-empty set R with two different operations $(x, y) \mapsto x + y$ and $(x, y) \mapsto xy$ with the following axioms:
i) $(x + y) + z = x + (y + z)$, where $x, y, z \in R$
ii) For all $x \in R$, there exists a unique $e(x) \in R$ such that $x + e(x) = e(x) + x = x$
iii) For all $x \in R$, there exists $-x \in R$ such that $x + (-x) = (-x) + x = e(x)$.
iv) $(xy)z = x(yz)$, where $x, y, z \in R$
v) For all $x, y, z \in R$, $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$.

The properties (i), (ii) and (iii) mean that $(R, +)$ is a generalized group.

Remark 2.1. Using (iii) and the associavity of $+$, one easily verifies $e(x)+e(x) = e(x)$ for every $x \in R$. Hence $e(e(x)) = e(x)$ follows by definitions and so $e^2 = e$ for the corresponding function $e : R \to R$.

A generalized ring with its operations is a ring iff e is a constant function.

Example 2.4. [7] The two dimensional Euclidean space IR^2 with the operations $(a_1, b_1) + (a_2, b_2) = (a_1, b_2)$ and $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$ is a generalized ring.

A generalized ring R is called an M-ring if $e(xy) = e(x)e(y)$ and $e(x + y) = e(x) + e(y)$, for all $x, y \in R$.

R is an M-ring if $e(x + y) = e(x) + e(y)$, for all $x, y \in R$. In other words, the identity function e is a generalized ring homomorphism if $e(x + y) = e(x) + e(y)$, for all $x, y \in R$.

If there is $1 \in R$ such that $x.1 = 1.x = x$, for all $x \in R$, then R is called a generalized ring with an identity.

One can easily prove that the identity of a generalized ring is unique.

Theorem 2.4. [7] *If R is a generalized ring, then $e(ab) = e(a)e(b)$, for all $a, b \in R$.*

Proof. Let $a, b \in R$ be given $ab+ae(b) = a(b+e(b)) = ab$, $ae(b)+ab = a(e(b)+b) = ab$. So $e(ab) = ae(b)$, $e(a)e(b) + ae(b) = (e(a) + a)e(b) = ae(b)$, $ae(b) + e(a)e(b) = (a + e(a))e(b) = ae(b)$. So $e(ae(b)) = e(a)e(b)$. Hence $e(e(ab)) = e(a)e(b)$. Thus $e(ab) = e(a)e(b)$, because $e^2 = e$. □

Corollary 2.5. *If R is a generalized ring, then $e(a)e(b) = ae(b) = e(a)b = e(ab)$, for all $a, b \in R$.*

Previous theorem implies that a generalized ring R is an M-ring if and only if $(R, +)$ is a normal generalized group.

Theorem 2.6. [7] *If R is a generalized ring, and if there is $x \in R$ such that $Rx = \{e(y) \mid y \in R\}$, then R is an M-ring.*

Proof. If $a, b \in R$, then there are $a_x \in R$ and $b_x \in R$ such that $e(a) = a_x x$ and $e(b) = b_x x$. So $e(a) + e(b) = (a_x + b_x)x$. Thus $e(a) + e(b) = e(z)$ for some $z \in R$. Hence $e(e(a)+e(b)) = e(e(z)) = e(z) = e(a)+e(b)$. In Remark 2.3 of [6] it was proved that $e(e(a) + e(b)) = e(a + b)$. So $e(a + b) = e(a) + e(b)$. Thus R is an M-ring. □

A subset I of an M-ring R is called a g-ideal (see [13]) if there exist a generalized ring D and a generalized ring homomorphism $f : R \to D$ such that $kerf = I$, where $kerf = \{r \in R \mid f(r) = f(e(a))$ for some $a \in R\}$. The set $R/I = \{x + kerf_r \mid x \in R_r$ and $f_r = f|_{R_r}\}$ with the operations $(x + kerf_r) + (y + kerf_k) = (x + y) + kerf_{r+k}$ and $(x + kerf_r)(y + kerf_k) = (xy) + kerf_{rk}$ is an M-ring (for the proof see Theorem 2.3 of [13]).

Definition 2.7. [7] If R and K are generalized rings, then a mapping $f : R \to K$ is called an embedding if f is a monomorphism.

2.2. Groupoids.

In this section, we introduce the elementary concepts of the groupoid theory. Then, it is given some recalls about the concept of ring-groupoid which is a ring object in the category of groupoids.

Definition 2.8. [3, 9] A groupoid consists of two sets G and G_0, called respectively the groupoid and the base, together with two maps α and β from G to G_0, called respectively the source and the target maps, a map $\epsilon : G_0 \to G$, $x \mapsto \epsilon(x) = \widetilde{x} = 1_x$, called the object inclusion map, a map $i : G \to G$, $x \mapsto i(x) = x^{-1}$, called the inversion, and a partial multiplication $(x, y) \mapsto m(x, y) = xy$ in G defined on the set $G_2 = G * G = \{(x, y) \mid \beta(x) = \alpha(y)\}$. These maps verify the following conditions:

G1) (associativity): $x(yz) = (xy)z$ for all $x, y, z \in G$ such that $\alpha(x) = \beta(y)$ and $\alpha(y) = \beta(x)$.
G2) (units): For each $x \in G$, we have $(\epsilon(\alpha(x)), x) \in G_2$, $(x, \epsilon(\beta(x))) \in G_2$and $\epsilon(\alpha(x))x = x\epsilon(\beta(x)) = x$.
G3) (inverses): For each $x \in G$, we have $(x, i(x)) \in G_2$, $(i(x), x) \in G_2$and $xi(x) = \epsilon(\alpha(x))$, $i(x)x = \epsilon(\beta(x))$.

The maps $\alpha, \beta, m, \epsilon, i$ are called structure maps of groupoid. For a groupoid G on G_0 and $x, y \in G_0$, we will write $St_G x$ for $\alpha^{-1}(x)$, $CoSt_G y$ for $\beta^{-1}(y)$ and $G(x, y)$ for $St_G x \cap CoSt_G y$. The set $St_G x$ is the star of G at x and $CoSt_G y$ is the co-star of G at y. The set $G(x, x)$, obviously a group under the restriction of the partial multiplication in G, is called the vertex group at x.

The following examples of groupoids are well-known.

Example 2.5. [3, 9] A group can be regarded as a groupoid with only one object.

Example 2.6. [3, 9] Any set G can be regarded as a groupoid on itself with $\alpha = \beta = id_G$ and every element a unity.

Example 2.7. [3] For a set X, the cartesian product $X \times X$ is a groupoid over X, called the Banal groupoid. The maps α and β are the natural projections onto the second and first factors, respectively. The object inclusion map is $x \mapsto (x, x)$ and the partial multiplication is given by $(x, y)(y, z) = (x, z)$. The inverse of (x, y) is simply (y, x).

Definition 2.9. [3, 9] Let G and $G^{'}$ be groupoids on B and $B^{'}$, respectively. A homomorphism $G \to G^{'}$ is a pair of (f, f_0) of maps $f : G \to G^{'}$, $f_0 : B \to B^{'}$ such that $\alpha^{'} \circ f = f_0 \circ \alpha$, $\beta^{'} \circ f = f_0 \circ \beta$ and $f(ab) = f(a)f(b)$ $\forall (a, b) \in G_2$.

We denote the groupoid homomorphism (f, f_0) by f for brevity.

Thus, we can construct the category *Gpd* of the groupoids and their homomorphisms.

Now let us recall the concept of ring-groupoid which is a ring object in the category of groupoids.

Definition 2.10. [15] A ring-groupoid R is a groupoid endowed with a structure of ring such that following ring structure maps are groupoid homomorphisms.

i) $m : R \times R \to R$, $(a, b) \mapsto a + b$, group operation
ii) $n : R \times R \to R$, $(a, b) \mapsto ab$, ring operation
iii) $u : R \to R$, $a \mapsto -a$, inverse in group
iv) $e : * \to R$.

Also, there exist following interchange laws in a ring-groupoid R.

(1) $(c \circ a) + (d \circ b) = (c + d) \circ (a + b)$,
(2) $(c \circ a)(d \circ b) = (cd) \circ (ab)$.

A ring groupoid homomorphism is a groupoid homomorphism preserving ring structure.

Example 2.8. Given a ring R, we can construct a ring-groupoid $R\times R$ over R. In this ring-groupoid we define the ring operation by $(a, b)(c, d) = (ac, bd)$ for all $a, b, c, d \in R$ (for more details, see [15]).

Definition 2.11. [15] Let R and S be two ring-groupoids. A homomorphism $f : R \to S$ of ring-groupoids is a homomorphism of underlying groupoids preserving ring structure.

Thus, the ring-groupoids and their homomorphisms form a category which is denoted by *RGd*.

3. Generalized Ring-Groupoids

In this section we present the concept of generalized ring-groupoid which is a generalized ring object in the category of groupoids. In addition, we construct the category of generalized ring-groupoids. From [8] with this aim, let us recall the concept of generalized group-groupoid which is lie in the structure of a generalized ring-groupoid.

Definition 3.1. A generalized group-groupoid is a groupoid (G, G_0) such that the following conditions are hold:
i) (G, w, v, σ) and $(G_0, w_0, v_0, \sigma_0)$ are generalized groups.

ii) The maps $(w, w_0) : (G \times G, G_0 \times G_0) \to (G, G_0)$, $v : \{\lambda\} \to G$ and $(\sigma, \sigma_0) : (G, G_0) \to (G, G_0)$ are groupoid homomorphisms.

Also, there exists an interchange law between the groupoid composition and the generalized group operation:

$$w\left(m\left(b,a\right), m\left(d,c\right)\right) = m\left(w\left(b,d\right), w\left(a,c\right)\right).$$

We shall denote a generalized group-groupoid by $(G, G_0, \circ, +)$.

We use the following equality for interchange law:

$$\left(b \circ a\right) + \left(d \circ c\right) = \left(b + d\right) \circ \left(a + c\right).$$

In other words, a generalized group-groupoid is a groupoid endowed with a structure of generalized group such that the structure maps of groupoid are generalized group homomorphisms.

Example 3.1. [8] Let G be a generalized group. Then we constitute a generalized group-groupoid $G \times G$ with object set G. For each object $(x, y) \in G \times G$, the identity arrow is $(e(x), e(y))$, and the inverse is $(-x, -y)$.

A generalized group homomorphism $f : G \to H$ between the generalized group-groupoids G and H is a groupoid homomorphism preserving the structure of generalized group [8].

Therefore, the generalized group-groupoids and their homomorphisms form a category denoted by $GG - Gd$.

Now let us give definition of a generalized ring-groupoid.

Definition 3.2. A generalized ring-groupoid R is a groupoid R endowed with a structure of generalized ring such that the following maps are groupoid homomorphisms:
1) $m : R \times R \to R$, $(a, b) \mapsto a + b$, generalized group operation,
2) $u : R \to R$, $a \mapsto -a$,
3) $e : * \to R$, where $*$ is a singleton,
4) $n : R \times R \to R$, $(a, b) \mapsto ab$, generalized ring operation.

Also, there exist two interchange laws between the groupoid composition and the operations of the generalized ring:

$$(c \circ a) + (d \circ b) = (c + d) \circ (a + b)$$
$$(c \circ a) \cdot (d \circ b) = (c \cdot d) \circ (a \cdot b).$$

We shall denote a generalized ring-groupoid by $(R, R_0, \circ, +, \cdot)$.

In a generalized ring-groupoid, if e is the identity of R_0, then 1_e is that of R.

We can rewrite the definition of a generalized ring-groupoid in terms of the generalized group-groupoid as follows:

Definition 3.3. A generalized ring-groupoid R is a generalized group-groupoid R endowed with a structure of generalized ring such that the map $n : R \times R \to R$, defined by $(a, b) \mapsto ab$, is a homomorphism of groupoids. Also, in a generalized ring-groupoid, we have the following interchange law:

$$(c \circ a)(d \circ b) = (cd) \circ (ab).$$

Proposition 3.1. *Let R be a generalized ring-groupoid. Then, the maps of source, target and object are generalized ring homomorphisms.*

Proof. Since $R = ((R, R_0, \circ, +)$ is a generalized group-groupoid, the maps of source, target and object are generalized group homomorphisms. Let $a, b \in R$ and $x, y \in R_0$. Since n is a groupoid homomorphism, the equalities $\alpha n(a,b) = f_0(\alpha \times \alpha)(a,b)$, $\beta n(a,b) = f_0(\beta\times\beta)(a,b)$ and $\epsilon f_0(x,y) = n(\epsilon\times\epsilon)(x,y)$ imply to be $\alpha(a,b) = \alpha(a)\alpha(b)$, $\beta(a,b) = \beta(a)\beta(b)$ and $\epsilon(x,y) = \epsilon(x)\epsilon(y)$, respectively.

Thus, the maps of source, target and object are generalized ring homomorphisms. □

Example 3.2. Let R be a generalized ring. Then $R\times R$ is a generalized ring-groupoid with the object set R. We know from [8] that $R\times R$ with the operation $(x,y)+(z,t) = (x+z, y+t)$ is a generalized group-groupoid over R. So it is enough to show that $R\times R$ is a generalized ring, and then the generalized ring map $n : (R\times R)\times(R\times R) \to R\times R$ is a groupoid homomorphism. We also must verify the second interchange law.

If we show that the conditions (iv) and (v) in Definition 3.2 are hold, we conclude that $R \times R$ is a generalized ring. Now let us control these conditions.

We define the generalized ring operation of $R \times R$ as follows:

$$(x,y)(z,t) = (xz, yt).$$

iv) We have

$$\begin{aligned} ((x,y)(z,t))(p,s) &= (xz,yt)(p,s)\\ &= ((xz)p,(yt)s)\\ &= (x(zp),y(ts))\\ &= (x,y)(zp,ts)\\ &= (x,y)((z,t)(p,s)). \end{aligned}$$

So, fourth condition is hold.

v)

$$\begin{aligned} (x,y)[(z,t)+(p,s)] &= (x,y)(z+p,t+s)\\ &= (x(z+p),y(t+s))\\ &= (xz+xp,yt+ys)\\ &= (xz,yt)+(xp,ys)\\ &= (x,y)(z,t)+(x,y)(p,s)) \end{aligned}$$

and

$$\begin{aligned} [(x,y)+(z,t)](p,s) &= (x+z,y+t)(p,s)\\ &= ((x+z)p),(y+t)s)\\ &= (xp+zp,ys+ts)\\ &= (xp,ys)+(zp,ts)\\ &= (x,y)(p,s)+(z,t)(p,s)). \end{aligned}$$

Hence, the condition (v) also is hold. Therefore, $R \times R$ is a generalized ring.

Now let us show that the second interchange law is satisfied.

$$\begin{aligned} [(z,y)\circ(y,x)][(z^{'},y^{'})\circ(y^{'},x^{'})] &= (z,x)(z^{'},x^{'})\\ &= (zz^{'},xx^{'}) \end{aligned}$$

and

$$\begin{aligned}[(z,y)(z',y')]\circ[(y,x)(y',x')] &= (zz',yy')\circ(yy',xx') \\ &= (zz',xx').\end{aligned}$$

Hence, we have the equality

$$[(z,y)\circ(y,x)][(z',y')\circ(y',x')] = [(z,y)(z',y')]\circ[(y,x)(y',x')].$$

Consequently, $R\times R$ is a generalized ring-groupoid.

Definition 3.4. Let R and S be two generalized ring-groupoids. A generalized ring-groupoid homomorphism $f : R \to S$ is a groupoid homomorphism satisfying the generalized ring structure.

Therefore, the generalized ring-groupoids and their homomorphisms form a category denoted by $GR-Gd$.

Proposition 3.2. *There is a functor from the category GR of the generalized rings to the category $GR-Gd$ of the generalized ring-groupoids.*

Proof. Let R be a generalized ring. Then, from Example 3.2, the cartesian product $R\times R$ is a generalized ring-groupoid. If $f : R \to S$ is a homomorphism of the generalized rings, then

$$\begin{aligned}\Gamma(f) : R\times R &\longrightarrow S\times S \\ (a,b) &\longmapsto (f(a),f(b))\end{aligned}$$

is a homomorphism of the generalized ring-groupoids. Thus, Γ is a functor from the category GR to the category $GR-Gd$. ☐

Now let us define the concept of generalized subring-groupoid.

Definition 3.5. Let R be a generalized ring-groupoid and be $S\subset R$. S is called a generalized subring-groupoid if $(S,S_0,\circ,+,.)$ has a structure of generalized ring-groupoid.

Furthermore, S is wide, if $S_0 = R_0$, and S is full, if $S(x,y) = R(x,y)$ for all $x,y\in S_0$.

Proposition 3.3. *Let R be a generalized ring-groupoid. Then, the set of identities $\epsilon(R_0)$ is a wide generalized subring-groupoid.*

Proof. Denote by A the set of identities $\epsilon(R_0)$ for brevity. If $1_x, 1_y \in A$, then $1_x+\overline{1_y}\in A$. Hence (A,A_0) is a wide subgroupoid of R. It remains to prove that A is closed under the generalized ring operation.

Since the object map ϵ preserves the generalized ring structure, we have

$$1_x1_y = (1_x\circ 1_x)(1_y\circ 1_y) = (1_x1_y)\circ(1_x1_y) = 1_{xy}\circ 1_{xy} = 1_{xy}.$$

This implies that $1_x1_y\in A$.

On the other hand, for $1_z\in A$

$$1_x(1_y+1_z) = 1_x1_{y+z} = 1_{x(y+z)} = 1_{xy+xz} = 1_{xy}+1_{xz} = 1_x1_y+1_x1_z.$$

Therefore, $A=\epsilon(R_0)$ is a wide generalized subring-groupoid. ☐

We define a special class of the generalized ring-groupoids.

Definition 3.6. A generalized ring-groupoid R is called an M-ring-groupoid if R has a structure of M-ring.

It is obvious that the category of M-ring-groupoids is a subcategory of the category of generalized ring-groupoids. Also, every M-ring-groupoid is a generalized ring-groupoid.

Since the set of arrows and the set of objects in an M-ring-groupoid are M-rings, then we can define the concept of a g-ideal ring-groupoid as follows:

Definition 3.7. A generalized subgroup-groupoid S of an M-ring-groupoid R is a left g-ideal ring-groupoid if

$$\begin{aligned} l \quad : \quad & R\times S\to S \\ & (r,s)\mapsto rs, \forall r\in R, \forall s\in S \end{aligned}$$

is a groupoid homomorphism. Similarly, S is a right g-ideal ring-groupoid if

$$\begin{aligned} k \quad : \quad & S\times R\to S \\ & (s,r)\mapsto sr, \forall r\in R, \forall s\in S \end{aligned}$$

is a groupoid homomorphism. Furthermore, S is a g-ideal ring-groupoid if it is both left and right g-ideal ring-groupoid.

From Definition 3.7, the sets of arrows and objects of S are left g-ideal rings, because l is a groupoid homomorphism. Also, every left (right) g-ideal ring-groupoid is a generalized subring-groupoid.

Proposition 3.4. *Let S be a generalized subgroup-groupoid of an M-ring groupoid R. If the set of arrows of S is a left g-ideal ring, then S_0 is also a left g-ideal of R_0.*

Proof. Let $x\in S_0$ and $y\in R_0$. Then, we have $1_x\in S$ and $1_y\in R$. Since the set of arrows of S is a left g-ideal ring, then we have $1_y1_x=1_{yx}\in S$. Since S is a generalized subgroup-groupoid, then we have $yx\in S_0$. Thus, S_0 is a left g-ideal of R_0. □

The interchange law in a g-ideal ring-groupoid is hold as follows: Let R be an M-ring-groupoid and I be a left g-ideal ring-groupoid such that $a,c\in I$. For $b,d\in R$, if $a\circ c$ and $b\circ d$ are defined, then we have $(b\circ d)(a\circ c)=(ba)\circ(dc)$. Since the set of arrows of I is a left g-ideal, then $ba,dc\in I$. Also, since I is a generalized subgroup-groupoid, which means that ba and dc are defined in I, then we have $(ba)\circ(dc)\in I$.

A similar result to Proposition 3.4 can also be given for a right g-ideal ring-groupoid.

Finally, let us present the product of generalized ring-groupoids.

Proposition 3.5. *Let $\{R_i : i\in I\}$ be a family of generalized ring-groupoids. Then, $(R=\prod R_i, R_0=\prod (R_i)_0, \circ, +, .)$ is a generalized ring-groupoid.*

Proof. The arrows of R are all tuples $(r_i)_{i\in I}$ for each $r_i\in R_i$ and its objects are all tuples $(x_i)_{i\in I}$ for each $x_i\in (R_i)_0$. It is easily proved that $(R,R_0,\circ,+)$ is a generalized group-groupoid. We define the generalized ring operation on R as follows:

$$\begin{aligned} (r_i)_{i\in I}(s_i)_{i\in I} &= (r_is_i)_{i\in I}, for\ each\ (r_i,s_i)\in R_i\times R_i \\ (x_i)_{i\in I}(y_i)_{i\in I} &= (x_iy_i)_{i\in I}, for\ each\ (x_i,y_i)\in (R_i)_0\times (R_i)_0 \end{aligned}$$

For the source map α, since

$$\begin{aligned}\alpha((r_i)_{i\in I}+(s_i)_{i\in I}) &= \alpha((r_i+s_i)_{i\in I})\\ &= (\alpha_i(r_i+s_i))_{i\in I}\\ &= (\alpha_i(r_i))_{i\in I}+(\alpha_i(s_i))_{i\in I}\\ &= \alpha((r_i)_{i\in I})+\alpha((s_i)_{i\in I})\end{aligned}$$

and

$$\begin{aligned}\alpha((r_i)_{i\in I}(s_i)_{i\in I}) &= \alpha((r_is_i)_{i\in I})\\ &= (\alpha_i(r_is_i)_{i\in I}\\ &= (\alpha_i(r_i))_{i\in I}(\alpha_i(s_i))_{i\in I}\\ &= \alpha((r_i)_{i\in I})\alpha((s_i)_{i\in I}),\end{aligned}$$

then α is a generalized ring homomorphism. Similarly, it can be easily shown that β and ϵ are also generalized ring homomorphisms.

Let us show that the interchange law is hold. Let us take any elements $(r)_{i\in I}$, $(s)_{i\in I}$, $(t)_{i\in I}$ and $(v)_{i\in I}\in R$ such that $\alpha((r)_{i\in I})=\beta((s)_{i\in I})$ and $\alpha((t)_{i\in I})=\beta((v)_{i\in I})$. Then,

$$\begin{aligned}[(r_i)_{i\in I}\circ(s_i)_{i\in I}][(t_i)_{i\in I}\circ(v_i)_{i\in I}] &= (r_i\circ s_i)_{i\in I}(t_i\circ v_i)_{i\in I}\\ &= ((r_i\circ s_i)(t_i\circ v_i))_{i\in I}\\ &= ((r_it_i)\circ(s_iv_i))_{i\in I}\\ &= (r_it_i)_{i\in I}\circ(s_iv_i)_{i\in I}\\ &= (r_i)_{i\in I}(t_i)_{i\in I}\circ(s_i)_{i\in I}(v_i)_{i\in I}.\end{aligned}$$

Thus, the interchange law between the groupoid composition and the generalized ring operation is satisfied. Moreover, we have

$$\begin{aligned}(r_i)_{i\in I}[(s_i)_{i\in I}+(t_i)_{i\in I}] &= (r_i)_{i\in I}[(s_i+t_i)_{i\in I}\\ &= (r_i(s_i+t_i)_{i\in I}\\ &= (r_is_i+r_it_i)_{i\in I}\\ &= (r_is_i)_{i\in I}+(r_it_i)_{i\in I}.\end{aligned}$$

Consequently, R is a generalized ring-groupoid.

□

References

[1] H. Brandt, Uber eine Verallgemeinerung des Gruppenbegriffes, *Math. Ann.* **96** (1926), 360–366.

[2] R. Brown and C.B. Spencer, G-groupoids, Crossed Modules and the Fundamental Groupoid of a Topological Group, *P. K. Ned. Akad. A. Math.* **79** (1976), 196–302.

[3] R. Brown, *Topology and Groupoids*, BookSurge LLC, Deganwy, United Kingdom, 2006.

[4] C. Ehresmann, Catégories topologiques et catégories differentiables, *Colloque de Géometrie Différentielle Globale, Cenrtre Belge de Recherches Mathematiques, Bruxelles*, (1958), 137–150.

[5] M.R. Farhangdoost, T. Nasirzade, Geometrical Categories of Generalized Lie Groups and Lie Group-Groupoids, *Iran. J. Sci. Technol. A.* (2013), 69–73.

[6] F. Fatehi, M.R. Molaei, On Completely Simple Semigroups, *Acta Math. Acad. Paedagog. Nyhazi. (N.S.)* **28** (2012), 95–102.

[7] F. Fatehi, M.R. Molaei, Some Algebraic Properties of Generalized Rings, *An. Stiint. U. Al I-Mat*, DOI: 10.2478/aicu-2014-0045.

[8] M.H. Gürsoy, H. Aslan, I. Icen, Generalized crossed modules and group-groupoids, DOI: 10.3906/mat-1602-63, *Turkish J. Math.*, 2017.

[9] M.H. Gürsoy, I. Icen, The Homomorphisms of Topological Groupoids, *Novi Sad J. Math.* **44** (2014), no. 1, 129–141.

[10] M.H. Gürsoy, I. Icen, Coverings of Structured Lie Groupoids, accepted in *Hacet. J. Math. Stat.*, 2017.

[11] M.Q. Mann'a, Some Properties of Topological Ring-Groupoids, *Int. J. Contemp. Math. Sci.* **7** (2012), no. 11, 517–529.

[12] M. R. Molaei, Generalized groups, *Bul. Inst. Politeh. Iasi. Sect. I. Mat. Mec. Teor. Fiz.* **49** (1999), 21–24.

[13] M. R. Molaei, Generalized rings, *Ital. J. Pure Appl. Math.* **12** (2003), 105–111.

[14] O. Mucuk, T. Sahan, N. Alemdar, Normality and Quotients in Crossed Modules and Group-groupoids, *Appl. Categor. Struct.* **23** (2015), no. 3, 415–428.

[15] O. Mucuk, Coverings and Ring-Groupoids, *Georgian Math. J.* **5** (1998), no. 5, 475–482.

(Mustafa Habil GÜRSOY) Inonu University, Faculty of Art and Science, Department of Mathematics, 44280, Malatya, Turkey
E-mail address: `mhgursoy@gmail.com`

Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 137–148
ISSN: 1223-6934

Weak solutions of one-dimensional pollutant transport model

Brahima Roamba, Jean De Dieu Zabsonré, and Yacouba Zongo

Abstract. We consider a one-dimensional bilayer model coupling shallow water and Reynolds lubrication equations that is a similar model derived in [European J. Applied Mathematics 24(6) (2013), 803-833]. The model considered is represented by the two superposed immiscible fluids. Under an hypothesis about the unknowns, we show the existence of global weak solution in time with a periodic domain.

2010 Mathematics Subject Classification. 35Q30; 76B15.
Key words and phrases. shallow water equations, bilayer models, viscosity, friction, capillarity.

---

1. Introduction

In this paper, we study the existence of global weak solutions in time for the following one dimensional model of transport of pollutant derived in [5] :

$$\begin{cases} \partial_t h_1 + \partial_x(h_1 u_1) = 0, \\ \partial_t(h_1 u_1) + \partial_x(h_1 {u_1}^2) + \dfrac{1}{2} g \partial_x h_1^2 - 4\nu_1 \partial_x(h_1 \partial_x u_1) + \dfrac{\alpha}{\rho_1}\gamma(h_1)u_1 - \dfrac{\delta_\xi}{\rho_1} h_1 \partial_x^3 h_1 \\ +r_1 h_1 |u|^2 u + rgh_1 \partial_x h_2 + rgh_2 \partial_x(h_1 + h_2) = 0, \\ \partial_t h_2 + \partial_x(h_2 u_1) + \partial_x \left(-{h_2}^2 \dfrac{1}{\rho_2}\Big(\dfrac{1}{c} + \dfrac{1}{3\nu_2} h_2\Big)\partial_x p_2 \right) = 0, \end{cases} \tag{1}$$

with

$$\partial_x p_2 = \rho_2 g \partial_x(h_1 + h_2) \quad \text{and} \quad \gamma(h_1) = \left(1 + \frac{\alpha}{3\nu_1} h_1\right)^{-1}. \tag{2}$$

Subscript 1 will correspond to the layer located below and subscript 2 to that located on the top. In this model, we denote by h_1, h_2 respectively, the water and the pollutant heights, u_1 is the water velocity, ρ_1 and ρ_2 the densities of each layer of fluid (we also introduce the ratio of densities $r = \dfrac{\rho_2}{\rho_1}$), ν_i is the kinematic viscosity, p_2 the pressure of the pollutant layer and g is the constant gravity. The coefficients δ_ξ, α, r_1, c, are respectively the coefficients of the intrefaz tension, friction at the bottom, quadratic friction and friction at the interfaz. This model is derived from a two-dimensional Navier-Stokes bilayer equations with capillary and friction effects at the interfaz. It is used to simulate the evolution of a thin viscous pollutant over water (see [5]). Let us recall some results about the existence of weak solution for a

---

Received January 07, 2017. Accepted May, 2017.

system composed by three equations (Shallow-water and transport equations). The case with viscosity term of the form $-\nu\Delta u$ was investigated in [8] in which existence of weak solutions for a viscous sedimentation model is obtained by assuming smallness of the data. In their analysis the authors considered a transport equation with Grass model of the form $q_b = hu$ and used Brower fixed point theorem to get the result. In [14], the authors studied the stability of global weak solutions for a sediment transport model in two- dimensional case. In this model, the viscosity coefficient is of the form $-\nu\mathtt{div}(hD(u))$ and the sediment transport equation considered is $\partial_t z + \mathtt{div}(h|u|^k u) - \frac{\nu}{2}\Delta u = 0$. The stability result is obtained without any restriction on the data and by using a mathematical entropy introduced firstly in [4] namely BD entropy . We note that it's the BD entropy inequality which allows the authors in [1, 3, 4, 6, 7] to get existence results of global weak solutions for Shallow-Water and viscous compressible Navier-Stokes equations.

In [12], the authors obtained a result of existence of global weak solution of similar model in a two dimensional case. To have this result, the authors needed of some additional regularizing terms such as a quadratic friction term $h_1|u|^2u$, a cold pressure $h_1^{1-\alpha}$ with $\alpha > 1$ and a capillarity term of the form $h_1\nabla\Delta h_1$. They used a transport equation of the form $\partial_t h_2 + \text{div}(h_2 u) - g\nabla\cdot\left((1+\dfrac{h_2}{h_1})\nabla(h_1+h_2)\right) = 0$. The key point with the BD entropy is that, with the structure of the diffusive term, we get an extra regularity for the water height. In our analysis, we consider in one-dimensional, a periodic domain $\Omega = (0,1)$ to simplify. We assume that the pollutant layer is smaller than that of the water:

$$h_2 \le h_1. \tag{3}$$

Notice that, to deduce the model, we make this hypothesis for the caracteristic heights (see [5]). We will intend in the future to study the present model without this condition. We complete system (1) with initial conditions :

$$h_1(0,x) = h_{1_0}(x), \quad h_2(0,x) = h_{2_0}(x), \quad (h_1u_1)(0,x) = \mathbf{m}_0(x) \quad \text{in } (0,1). \tag{4}$$

$$\begin{gathered} h_{1_0} \in L^2(0,1), \quad h_{1_0} + h_{2_0} \in L^2(0,1), \quad \partial_x(h_{1_0} \in L^2(0,1), \\ \partial_x\mathbf{m}_0 \in L^1(0,1), \quad \mathbf{m}_0 = 0 \quad \text{if } h_{1_0} = 0, \\ \frac{|\mathbf{m}_0|^2}{h_{1_0}} \in L^1(0,1), \quad f(h_{1_0}) \in L^1(0,1), \end{gathered} \tag{5}$$

where f will be defined later on (see (16)).

The paper is organized as follows : in the Section 2, we will start by giving the definition of global weak solutions, then we will establish a classical energy equality and the "mathematical BD entropy", which give some regularities on the unknowns. We will also give an existence theorem of global weak solutions. In section 3, we will give the proof of the existence theorem.

2. Main results

Definition 2.1. We shall say that (h_1, h_2, u_1) is a weak solution on $(0,T)$ of (1), with initial conditions (4) if the following conditions are satisfied :

- (4) holds in $\mathcal{D}'(\Omega)$;

- (h_1, h_2, u_1) verified the energy inequalities (2.1) and (2.2) for a.e. non negative t;
- for all smooth test function $\varphi = \varphi(t, x)$ with $\varphi(T,) = 0$, we have:

$$h_{1_0}\varphi(0,.) - \int_0^T \int_0^1 h_1 \partial_t \varphi - \int_0^T \int_0^1 h_1 u_1 \partial_x \varphi = 0, \tag{6}$$

$$\begin{aligned} -h_{2_0}\varphi(0,.) &- \int_0^T \int_0^1 h_2 \partial_t \varphi - \int_0^T \int_0^1 h_2 u_1 \partial_x \varphi \\ &+ \int_0^T \int_0^1 h_2{}^2 \frac{1}{\rho_2}\Big(\frac{1}{c} + \frac{1}{3\nu_2} h_2\Big) \partial_x p_2 \partial_x \varphi = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} &h_{1_0}u_{1_0}\varphi(0,.) - \int_0^T \int_0^1 h_1 u_1 \partial_t \varphi - \int_0^T \int_0^1 h_1 u_1{}^2 \partial_x \varphi - \frac{1}{2} g \int_0^T \int_0^1 h_1^2 \partial_x \varphi \\ &+ 4\nu_1 \int_0^T \int_0^1 h_1 \partial_x u_1 \partial_x \varphi + \frac{\alpha}{\rho_1} \int_0^T \int_0^1 \gamma(h_1) u_1 \varphi + \frac{\delta_\xi}{\rho_1} \int_0^T \int_0^1 h_1 \partial_x^2 h_1 \partial_x \varphi \\ &+ \frac{\delta_\xi}{\rho_1} \int_0^T \int_0^1 \partial_x h_1 \partial_x^2 h_1 \varphi - rg \int_0^T \int_0^1 h_2 \partial_x h_1 \varphi - rg \int_0^T \int_0^1 h_1 h_2 \partial_x \varphi \\ &- rg \int_0^T \int_0^1 (h_1 + h_2) h_2 \varphi - rg \int_0^T \int_0^1 (h_1 + h_2) \partial_x h_2 \varphi + r_1 \int_0^T \int_0^1 |u_1|^2 u_1 \varphi = 0. \end{aligned} \tag{8}$$

Before giving the main theorem, we give the following two important lemmas. We firstly give the classical energy associated with system (1) and secondly the mathematical BD entropy.

Lemma 2.1. The model defined by (1) admits an entropy equality

$$\begin{aligned} &\int_0^1 \Big[\frac{1}{2} h_1 |u_1|^2 + \frac{1}{2} g(1 - r)|h_1|^2 + \frac{1}{2} rg|h_1 + h_2|^2 + \frac{1}{2}\frac{\delta_\xi}{\rho_1} |\partial_x h_1|^2\Big] \\ &+ r_1 \int_0^T \int_0^1 h_1 |u_1|^4 + 4\nu_1 \int_0^T \int_0^1 h_1 |\partial_x u_1|^2 + \frac{\alpha}{\rho_1} \int_0^T \int_0^1 \gamma(h_1)|u_1|^2 \\ &\qquad + rg^2 \int_0^T \int_0^1 h_2^2\Big(\frac{1}{c} + \frac{1}{3\nu_2} h_2\Big)\Big(\partial_x(h_1 + h_2)\Big)^2 \\ &\quad = \int_0^1 \Big[\frac{1}{2} h_{1_0} |u_{1_0}|^2 + \frac{1}{2} g(1 - r)|h_{1_0}|^2 + \frac{1}{2} rg|h_{1_0} + h_{2_0}|^2 + \frac{1}{2}\frac{\delta_\xi}{\rho_1} |\partial_x h_{1_0}|^2\Big]. \end{aligned} \tag{9}$$

Proof. Firstly, we multiply the momentum equation by u_1 and we integrate from 0 to 1. We use the mass conservation equation of the first layer for simplification. Then, we obtain

$$\begin{aligned} &\frac{d}{dt} \int_0^1 \Big[\frac{1}{2}(h_1 u_1^2 + g h_1^2)\Big] - \frac{\delta_\xi}{\rho_1} \int_0^1 \partial_t h_1 \partial_x^2 h_1 + r_1 \int_0^1 h_1 |u_1|^4 + rg \int_0^1 h_2 \partial_t h_1 \\ &\quad + rg \int_0^1 h_2 u_1 \partial_x (h_1 + h_2) + 4\nu_1 \int_0^1 h_1 (\partial_x u_1)^2 + \frac{\alpha}{\rho_1} \int_0^1 \gamma(h_1) u_1^2 = 0. \end{aligned} \tag{10}$$

Secondly, we multiply the equation for the thin film flow by $\rho_2 g(h_1 + h_2)$ and integrate to obtain

$$\frac{1}{2} rg \frac{d}{dt} \int_0^1 h_2^2 + rg \int_0^1 h_1 \partial_t h_2 + rg \int_0^1 (h_1 + h_2) \partial_x (h_2 u_1)$$

$$= rg^2 \int_0^1 h_2^2(\frac{1}{c} + \frac{1}{3\nu_2}h_2)\Big(\partial_x(h_1+h_2)\Big)^2. \tag{11}$$

We use the mass conservation equation to write

$$\int_0^1 h_2\partial_t h_1 + \int_0^1 h_1\partial_t h_2 = \frac{d}{dt}\int_0^1 h_1h_2, \tag{12}$$

and to develop the following product affecting the terms with δ_ξ

$$\int_0^1 \partial_x(h_1u_1)\partial_x^2 h_1 = \int_0^1 h_1\partial_t(h_1)\partial_x^2 h_1 = -\frac{1}{2}\frac{d}{dt}\int_0^1 |\partial_x h_1|^2. \tag{13}$$

By adding (10) and (11), and taking into account (12) and (13), we obtain

$$\frac{d}{dt}\int_0^1 \left[\frac{1}{2}h_1u_1^2 + \frac{1}{2}gh_1^2 + rgh_2(h_1 + \frac{h_2}{2})\right] + \frac{1}{2}\frac{\delta_\xi}{\rho_1}\frac{d}{dt}\int_0^1 (\partial_x h_1)^2 + r_1\int_0^1 h_1|u_1|^4$$

$$+4\nu_1\int_0^1 h_1(\partial_x u_1)^2 + rg^2\int_0^1 h_2^2(\frac{1}{c}+\frac{1}{3\nu_2}h_2)\Big(\partial_x(h_1+h_2)\Big)^2 + \frac{\alpha}{\rho_1}\int_0^1 \gamma(h_1)u_1^2 = 0. \tag{14}$$

To end, we integrate from 0 to t to have the equality (9). ☐

Corollary 2.1. Let (h_1, h_2, u_1) be a solution of model (1). Then, thanks to Lemma 2.1 we have:

h_1 is bounded in $L^\infty(0,T;L^2(0,1))$,
h_2 is bounded in $L^\infty(0,T;L^2(0,1))$,
$\partial_x h_1$ is bounded in $L^\infty(0,T;L^2(0,1))$,
$\sqrt{h_1}u_1$ is bounded in $L^\infty(0,T;L^2(0,1))$,
$\sqrt{h_1}\partial_x u_1$ is bounded in $L^2(0,T;L^2(0,1))$,
u_1 is bounded in $L^2(0,T;L^2(0,1))$,
$h_1^{\frac{1}{4}}u_1$ is bounded in $L^2(0,T;L^2(0,1))$,
$h_2\sqrt{\frac{1}{c}+\frac{1}{3\nu_2}h_2}\Big(\partial_x(h_1+h_2)\Big)$ is bounded in $L^2(0,T;L^2(0,1))$.

Remark 2.1. (1) In the Corollary 2.1, the estimate

$$\sqrt{h_1}u_1 \quad \text{is bounded in} \quad L^\infty(0,T;L^2(0,1))$$

implies,

$$h_1u_1 \quad \text{is bounded in} \quad L^\infty(0,T;L^2(0,1))$$

this leads us

$$\partial_t h_1 \quad \text{is bounded in} \quad L^\infty(0,T;W^{-1,2}(0,1)).$$

(2) We have the additional regularities thanks to **Corollary** 2.1:
  (a) h_1 is bounded in $L^2(0,T;H^1(0,1))$,
  (b) h_1u_1 is bounded in $L^3(0,T;L^3(0,1)) \cap L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;W^{1,1}(0,1))$,
  (c) $\gamma(h_1)$ is bounded in $L^\infty(0,T;H^1(0,1)) \cap L^\infty(0,T;L^\infty(0,1))$.

Remark 2.2. We have the following additional regularities:
(1) h_2 is bounded in $L^\infty(0,T;L^\infty(0,1))$,
(2) $\partial_x(h_1+h_2)$ is bounded in $L^2(0,T;L^2(0,1))$.

We will need in the following some additional regularity on h_1 and this will be achieved through an additional BD entropy inequality presented in the next lemma.

Lemma 2.2. For smooth solutions (h_1, h_2, u_1) of model (1) satisfying the classical energy equality of the **Lemma** 2.1, we have the following mathematical BD entropy inequality:

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left[h_1|u_1 + 4\nu_1\partial_x \log h_1|^2 + rg|h_1+h_2|^2 + g(1-r)|h_1|^2| - 8\nu_1 f(h_1) + \frac{\delta_\xi}{\rho_1}|\partial_x h_1|^2\right] \\
&\qquad + \frac{\alpha}{\rho_1}\int_0^T\int_0^1 \gamma(h_1)|u_1|^2 + r_1\int_0^T\int_0^1 h_1|u_1|^4 + 4\nu_1 r_1\int_0^T\int_0^1 |u_1|^2 u_1\partial_x h_1 \\
&\qquad + 2g\nu_1\int_0^T\int_0^1 (1+2r\frac{h_2}{h_1})|\partial_x h_1|^2 + 4rg\nu_1\int_0^1 (1+\frac{h_2}{h_1})\partial_x h_1\partial_x h_2 + \frac{\delta_\xi}{\rho_1}\int_0^T\int_0^1 |\partial_x^2 h_1|^2 \\
&\qquad + rg^2\int_0^T\int_0^1 h_2^2(\frac{1}{c} + \frac{1}{3\nu_2}h_2)\Big(\partial_x(h_1+h_2)\Big)^2 + 4\frac{\nu_1\alpha}{\rho_1}\int_0^T\int_0^1 \gamma'(h_1)u_1\partial_x h_1 \\
&\qquad \leqslant 4\nu_1\int_0^1 f(h_{1_0}) + \int_0^1 \left[h_{1_0}|u_{1_0}|^2 + 128\nu_1^2|\partial_x\sqrt{h_{1_0}}|^2 + \frac{1}{2}g(1-r)|h_{1_0}|^2\right] \\
&\qquad\qquad + \int_0^1 \left[\frac{1}{2}rg|h_{1_0} + h_{2_0}|^2 + \frac{1}{2}\frac{\delta_\xi}{\rho_1}|\partial_x h_{1_0}|^2\right],
\end{aligned} \tag{15}$$

where

$$f(h_1) = \alpha\log\left(\frac{h_1}{3+\alpha\nu_1^{-1}h_1}\right). \tag{16}$$

Proof. Let us consider the mass equation

$$\partial_t h_1 + \partial_x h_1 u_1 = 0.$$

When we use both the transport equation and the renormalized technical, we get:

$$\partial_t(\partial_x h_1) + \partial_x(h_1\partial_x u_1) + \partial_x(u_1\partial_x h_1) = 0.$$

Replacing $\partial_x h_1$ by $h_1\partial_x \log h_1$ and introducing the viscosity $4\nu_1$, this becomes

$$4\nu_1\partial_t(h_1\partial_x\log h_1) + 4\nu_1\partial_x(h_1\partial_x u_1) + 4\nu_1\partial_x(h_1 u_1\partial_x\log h_1) = 0.$$

Then, we add the momentum equation to obtain

$$\begin{aligned}
&\partial_t[h_1(u_1+4\nu_1\partial_x\log h_1)] + \partial_x[h_1u_1(u_1+4\nu_1\partial_x\log h_1)] + \tfrac{1}{2}g\partial_x h_1^2 + \tfrac{\alpha}{\rho_1}\gamma(h_1)u_1 \\
&\qquad -h_1\frac{\delta_\xi}{\rho_1}\partial_x^3 h_1 + r_1h_1|u_1|^2u_1 + rgh_1\partial_x h_2 + rgh_2\partial_x(h_1+h_2) = 0.
\end{aligned}$$

We multiply this equation by $(u_1 + 4\nu_1\partial_x\log h_1)$ and we integrate between 0 and 1. Now, we transform each term of the resulting identity separately

$$\begin{aligned}
&\int_0^1 [\partial_t[h_1(u_1+4\nu_1\partial_x\log h_1)] + \partial_x[h_1u_1(u_1+4\nu_1\partial_x\log h_1)]](u_1+4\nu_1\partial_x\log h_1) \\
&\qquad\qquad = \frac{1}{2}\frac{d}{dt}\int_0^1 h_1|u_1+4\nu_1\partial_x\log h_1|^2.
\end{aligned}$$

Next, we only study the terms which do not appear in (9).
The pressure terms become:

$$\frac{1}{2}g\int_0^1 \partial_x h_1^2(4\nu_1\partial_x\log h_1) = 2g\nu_1\int_0^1 |\partial_x h_1|^2,$$

$$rg\int_0^1 [h_1\partial_x h_2 + h_2\partial_x(h_1+h_2)](4\nu_1\partial_x \log h_1)$$
$$= 4rg\nu_1\int_0^1 \frac{h_2}{h_1}|\partial_x h_1|^2 + 4rg\nu_1\int_0^1 (1+\frac{h_2}{h_1})\partial_x h_1\partial_x h_2.$$

Adding these two terms, we have:

$$\frac{1}{2}g\int_0^1 \partial_x h_1^2(4\nu_1\partial_x\log h_1 + rg\int_0^1 [h_1\partial_x h_2 + h_2\partial_x(h_1+h_2)](4\nu_1\partial_x\log h_1)$$
$$= 2g\nu_1\int_0^1 (1+2r\frac{h_2}{h_1})|\partial_x h_1|^2 + 4rg\nu_1\int_0^1 (1+\frac{h_2}{h_1})\partial_x h_1\partial_x h_2.$$

For the friction term at the bottom, we have

$$\frac{\alpha}{\rho_1}\int_0^1 \gamma(h_1)u_1(4\nu_1\partial_x\log h_1) = \frac{4\nu_1}{\rho_1}\int_0^1 \frac{3\nu_1\alpha}{3\nu_1+\alpha h_1}u_1\partial_x\log h_1$$
$$= -\frac{4\nu_1}{\rho_1}\int_0^1 \frac{3\nu_1\alpha}{3\nu_1+\alpha h_1}\Big(\frac{\partial_t h_1}{h_1} + \partial_x u_1\Big).$$

Considering that Lemma 2.2 gives $f'(h_1) = \dfrac{3\nu_1\alpha}{3\nu_1+\alpha h_1}\dfrac{1}{h_1}$,

therefore,

$$4\frac{\nu_1\alpha}{\rho_1}\int_0^1 \gamma(h_1)u_1\partial_x\log h_1 = -4\frac{\nu_1}{\rho_1}\frac{d}{dt}\int_0^1 f(h_1) + 4\frac{\nu_1\alpha}{\rho_1}\int_0^1 \gamma'(h_1)u_1\partial_x h_1. \qquad \square$$

Remark 2.3. (1) The term including $\log\left(\dfrac{h_1}{3+\alpha\nu_1^{-1}h_1}\right)$ is bounded, see [12].

(2) In Lemma 2.2 all the terms, except $-\displaystyle\int_0^T\int_0^1 |u_1|^2u_1\partial_x h_1$ and $\displaystyle\int_0^T\int_0^1 (1+\frac{h_2}{h_1})\partial_x h_1\partial_x h_2$ are controlled since they have the good sign. But the control of the both terms takes inspiration in [12].

(3) If (h_1,h_2,u_1) is solution of the model (1), then, thanks to Lemma 2.2, we have that:

$$\partial_x\sqrt{h_1} \text{ is bounded in } L^\infty(0,T;L^2(0,1)) \text{ and } \partial_x^2 h_1 \text{ is bounded in } L^2(0,T;L^2(0,1)).$$

Theorem 2.1. There exists global weak solutions to system (1) with initial data (4), (5) and satisfying energy equality (9) and energy inequality (15).

3. Convergences

This section is devoted to the proof of Theorem 2.1. Let (h_1^k, h_2^k, u_1^k) be a sequence of weak solutions with initial data

$$h^k_{1|t=0} = h^k_{1_0}, \quad h^k_{2|t=0} = h^k_{2_0}, \quad (h_1^k u_1^k)_{|t=0} = m_0^k$$

such as

$$h^k_{1_0} \longrightarrow h_{1_0} \text{ in } L^1(0,1), \quad h^k_{2_0} \longrightarrow h_{2_0} \text{ in } L^1(0,1), \quad m_0^k \longrightarrow m_0 \text{ in } L^1(0,1),$$

and satisfies

$$4\nu_1\int_0^1 f(h_{1_0}) + \int_0^1 \Big[h_{1_0}|u_{1_0}|^2 + 128\nu_1^2|\partial_x\sqrt{h_{1_0}}|^2 + \frac{1}{2}g(1-r)|h_{1_0}|^2 + \frac{1}{2}rg|h_{1_0}+h_{2_0}|^2\Big]$$

$$+\frac{1}{2}\frac{\delta_\xi}{\rho_1}\int_0^1 |\partial_x h_{1_0}|^2 \leq C.$$

Such approximate solutions can be built by a regularization of capillary effect.

3.1. Strong convergence of $\left(\sqrt{h_1^k}\right)_k$. Here, we are going to establish the spaces in which $\left(\sqrt{h_1^k}\right)_k$ is bounded.

In this sense we are going to integrate the mass equation and we directly get $\sqrt{h_1^k}$ in $L^\infty(0,T;L^2(0,1))$, the Remark 2.3 gives us $\left|\partial_x\sqrt{h_1^k}\right|$ in $L^\infty(0,T;L^2(0,1))$. So we obtain:

$$\sqrt{h_1^k} \quad \text{is bounded in } L^\infty(0,T;H^1(0,1)). \qquad (*)$$

Moreover, we use the mass equation again to have the following equality:

$$\partial_t\sqrt{h_1^k} = \frac{1}{2}\sqrt{h_1^k}\partial_x u^k - \partial_x(\sqrt{h_1^k}u^k),$$

which gives that $\partial_t\sqrt{h_1^k}$ is bounded in $L^2(0,T;H^{-1}(0,1))$.
Applying Aubin-Simon lemma ([9, 13]), we can extract a subsequence, still denoted $(h_1^k)_{1\leq k}$, such as

$$\left(\sqrt{h_1^k}\right)_k \quad \text{strongly converges to} \quad \sqrt{h_1} \text{ in } L^2(0,T;L^2(0,1)).$$

3.2. Strong convergence of h_1 and h_2. Let now study the subsequence $(h_1^k)_k$. According to the property $(*)$ and Sobolev embeddings, we know that, for any finite s,

$$(h_1^k)_k \quad \text{is bounded in} \quad L^\infty(0,T;L^s(0,1)).$$

In the following, we will assume that $4 \leq s$ in order to simplify our expressions and ensure that

$$(h_1^k)_k \quad \text{is bounded in} \quad L^\infty(0,T;L^2(0,1)).$$

The equality $\partial_x h_1^k = 2\sqrt{h_1^k}\partial_x\sqrt{h_1^k}$ enables us to bound the sequence $\partial_x h_1^k$ in $L^\infty(0,T;(L^{\frac{2s}{s+2}}(0,1))^2)$ and consequently the sequence

$$(h_1^k)_k \quad \text{is bounded in} \quad L^\infty(0,T;W^{1,\frac{2s}{s+2}}(0,1)).$$

Moreover, we have some properties on the time derivative of (h_1^k); actually the mass equation can be written as: $\partial_t h_1^k = -\partial_x(h_1^k u_1^k)$. Splitting the product $h_1^k u_1^k$ into $h_1^k u_1^k = \sqrt{h_1^k}\sqrt{h_1^k}u_1^k$, we get

$$h_1^k u_1^k \quad \text{in} \quad L^\infty(0,T;(L^{\frac{2s}{s+2}}(0,1))^2) \quad \text{and} \quad \partial_t h_1^k \quad \text{in} \quad L^\infty(0,T;W^{-1,\frac{2s}{s+2}}(0,1)).$$

Thanks to Aubin-Simon lemma again, we find:

$$h_1^k \longrightarrow h_1 \quad \text{in} \quad C^0(0,T;L^{\frac{2s}{s+2}}(0,1))$$

We have $h_2^k \in L^2(0,T;H^1(0,1))$.
Moreover, we have $\partial_t h_2^k = -\partial_x(h_2^k u_1^k) + g\partial_x\left[-{h_2^k}^2(\frac{1}{c}+\frac{1}{3\nu_1}h_2^k)\partial_x(h_1^k+h_2^k)\right]$.
According to the Sobolev embeddings, we show that the first term is in $W^{-1,1}(0,1)$,

since
$h_2^k \in L^2(0,1)$ and $u_1^k \in L^2(0,1)$. By analogy we prove that the last term is in the same space and we get $\partial_t h_2^k$ also in this space. Thanks to the Aubin-Simon lemma, we find:

$$(h_2^k)_k \quad \text{converges strongly to} \quad h_2 \quad \text{in} \quad L^2(0,T;W^{-1,\frac{2s}{s+2}}(0,1)).$$

3.3. Strong convergence of $(h_1^k u_1^k)_k$. Let us write $h_1^k u_1^k$ as follow:
$h_1^k u_1^k = \sqrt{h_1^k}\sqrt{h_1^k}u_1^k$, we have

$$\left(\sqrt{h_1^k}\right)_k \text{ bounded in } L^\infty(0,T;L^4(0,1))$$

and

$$\left(\sqrt{h_1^k}u_1^k\right)_k \text{ bounded in } L^\infty(0,T;L^2(0,1)).$$

Thus we have:

$$(h_1^k u_1^k)_k \text{ bounded in } L^\infty(0,T;L^{\frac{4}{3}}(0,1)).$$

Let's write the gradient as follows:

$$\partial_x(h_1^k u_1^k) = h_1^k \partial_x u_1^k + u_1^k \partial_x h_1^k = \sqrt{h_1^k}\sqrt{h_1^k}\partial_x u_1^k + u_1^k \partial_x h_1^k,$$

since the first term is in $L^2(0,T;L^{\frac{4}{3}}(0,1))$ and thanks to the Corollary 2.1, second one belongs to $L^\infty(0,T;W^{-1,\frac{4}{3}}(0,1)) \cap L^2(0,T;L^1(0,1))$, we have

$$(h_1^k u_1^k)_k \text{ bounded in } L^2(0,T;W^{1,1}(0,1)).$$

Moreover, the momentum equation of (1) enables us to write the time derivation of the water discharge:
$\partial_t(h_1^k u_1^k) = -\partial_x(h_1^k {u_1^k}^2)) - \frac{1}{2}g\partial_x[(h_1^k)^2] - 4\nu_1\partial_x(h_1^k\partial_x u_1^k) - \frac{\alpha}{\rho_1}\gamma(h_1^k)u_1^k + \alpha(h_1^k)h_1^k u_1^k|u_1^k|^2$

$$+\frac{\delta_\xi}{\rho_1}h_1^k\partial_x^3 h_1^k - rgh_1^k\partial_x h_2^k - rgh_2^k\partial_x(h_1^k + h_2^k) = 0.$$

we then study each term:
- $\partial_x(h_1^k(u_1^k)^2) = \partial_x(\sqrt{h_1^k}\sqrt{h_1^k}(u_1^k)^2)$ which is in $L^2(0,T;W^{-1,\frac{4}{3}}(0,1))$.
- as $(h_1^k)_k$ is bounded in $L^\infty(0,T;W^{1,1}(0,1))$, it is also bounded in $L^\infty(0,T;L^2(0,1))$ and we can write the following relation:

$$\left(\partial_x[(h_1^k)^2]\right)_k \text{ is bounded in } L^\infty(0,T;W^{-1,1}(0,1)).$$

- $\left(\partial_x(h_1^k\partial_x u_1^k)\right)_k$ is bounded in $L^2(0,T;W^{-1,\frac{4}{3}}(0,1))$.
- Let us write $h_1^k u_1^k(u_1^k)^2 = \sqrt{h_1^k}u_1^k\sqrt{h_1^k}(u_1^k)^2$, which is in $L^2(0,T;W^{1,1}(0,1))$.
- The last three terms are bounded in $L^\infty(0,T;W^{-1,2}(0,1))$.

Then, applying Aubin-Simon lemma, we obtain,

$$(h_1^k u_1^k)_k \text{ converges stongly to } \mathbf{m} \text{ in } C^0(0,T;W^{-1,1}(0,1)).$$

3.4. Strong convergence of $\left(\sqrt{h_1^k}u_1^k\right)_k$. Setting $\mathbf{m}^k = h_1^k u_1^k$, so, we have $\sqrt{h_1^k}u^k = \dfrac{\mathbf{m}^k}{\sqrt{h_1^k}}$. We want to prove the strong convergence for this term. We know that

$$\left(\frac{\mathbf{m}^k}{\sqrt{h_1^k}}\right)_k \text{ is bounded in } L^\infty(0,T;(L^2(0,1))^2);$$

consequently Fatou lemma reads:

$$\int_0^1 \liminf \frac{(\mathbf{m}^k)^2}{h_1^k} \leq \liminf \int_0^1 \frac{(\mathbf{m}^k)^2}{h_1^k} < +\infty.$$

In particular, $\mathbf{m}$ is equal to zero for almost every x where $h_1(t,x)$ vanishes. Then, we can define the limit velocity taking $u_1(t,x) = \frac{\mathbf{m}(t,x)}{h_1(t,x)}$ if $h_1(t,x) \neq 0$ or else $u_1(t,x) = 0$. So we have a link between the limits $\mathbf{m}(t,x) = h_1(t,x)u_1(t,x)$ and:

$$\int_0^1 \frac{(\mathbf{m})^2}{h_1} = \int_0^1 h_1|u_1|^2 < +\infty.$$

Moreover, we can use Fatou lemma again to write

$$\int_0^T \int_0^1 h_1|u_1|^4 \leq \int_0^T \int_{]0,1[} \liminf h_1|u_1|^4 \leq \liminf \int_0^T \int_0^1 h_1|u_1|^4$$
$$= \liminf \int_0^T \int_0^1 \sqrt{h_1}|u_1|^2 \sqrt{h_1}|u_1|^2,$$

which gives $\sqrt{h_1}|u_1|^2$ in $L^2(0,T;L^2(0,1))$.

As $\mathbf{m}^k$ and h_1^k converge almost everywhere, the sequence of $\sqrt{h_1^k}u_1^k = \dfrac{\mathbf{m}^k}{\sqrt{h_1^k}}$ converges almost everywhere to $\sqrt{h_1}u_1 = \dfrac{\mathbf{m}}{\sqrt{h_1}}$. Moreover, for all M positive $\sqrt{h_1^k}u_1^k 1_{|u_1^k|\leq M}$ converges to $\sqrt{h_1}u_1 1_{|u|\leq M}$ (still assuming that h_1^k does not vanish). If h_1 vanishes, we can write $\sqrt{h_1^k}u^k_{1|u_1^k|\leq M} \leq M\sqrt{h_1^k}$ and then have convergence towards zero. Then, almost everywhere, we obtain the convergence of $(\sqrt{h_1^k}u_1^k 1_{|u_1^k|\leq M})_k$.

Finally, let us consider the following norm:

$$\int_0^T \int_0^1 \left|\sqrt{h_1^k}u_1^k - \sqrt{h_1}u_1\right|^2 \leq$$
$$\int_0^T \int_0^1 \left(\left|\sqrt{h_1^k}u_1^k 1_{|u_1^k|\leq M} - \sqrt{h_1}u_1 1_{|u_1|\leq M}\right| + \left|\sqrt{h_1^k}u_1^k 1_{|u_1^k|>M}\right| + \left|\sqrt{h_1}u_1 1_{|u_1|>M}\right|\right)^2$$
$$\leq 3\int_0^T \int_0^1 \left|\sqrt{h_1^k}u_1^k 1_{|u_1^k|\leq M} - \sqrt{h_1}u_1 1_{|u_1|\leq M}\right|^2 + 3\int_0^T \int_0^1 \left|\sqrt{h_1^k}u_1^k 1_{|u_1^k|>M}\right|^2$$
$$+3\int_0^T \int_0^1 \left|\sqrt{h_1^k}u_1^k 1_{|u_1^k|>M}\right|^2.$$

Since $\left(\sqrt{h_1^k}\right)_k$ is bounded in $L^2(0,T;L^4(0,1))$, it follows

$$\left(\sqrt{h_1^k}u_1^k 1_{|u_1^k|\leq M}\right)_k \text{ is bounded in this space.}$$

So, as we have seen previously, the first integral tends to zero. Let us study the other two terms:

$$\int_0^1 \left|\sqrt{h_1^k}u_1^k 1_{|u_1^k|>M}\right|^2 \leq \frac{1}{M^2}\int_0^1 h_1^k(u_1^k)^4 \leq \frac{c}{M^2}$$

and

$$\int_0^1 \left|\sqrt{h_1}u_1 1_{|u_1|>M}\right|^2 \leq \frac{1}{M^2}\int_0^1 h_1 u_1^4 \leq \frac{c}{M^2},$$

for all $M>0$. When M tends to the infinity, our two integrals tend to zero. Then

$$\left(\sqrt{h_1^k}u_1^k\right)_k \quad \text{converges strongly to} \quad \sqrt{h_1}u_1 \quad \text{in} \quad L^2(0,T;(L^2(]0,1[))^2).$$

3.5. Convergence of $(\partial_x h_1^k)_k$, $(h_1^k\partial_x h_1^k)_k$, $(h_2^k\partial_x h_1^k)_k$, $(\partial_x^2 h_1^k)_k$, $(h_1^k\partial_x^2 h_1)_k$ and $(\partial_x h_1^h \partial_x^2 h_1^k)_k$. • We have $(\partial_x h_1^k)_k$ bounded in $L^2(0,T;H^1(0,1))$ and $(\partial_t\partial_x h_1^k)_k$ is bounded in $L^\infty(0,T;H^{-2}(0,1))$ since $(\partial_t h_1^k)_k$ is bounded in $L^\infty(0,T;H^{-1}(0,1))$. Thanks to compact injection of $H^1(0,1)$ in $L^2(0,1)$ in one dimension, we have:

$$(\partial_x h_1^k)_k \quad \text{converges strongly to} \quad \partial_x h_1 \quad \text{in } L^2(0,T;L^2(0,1))$$

• The bound of $\partial_x^2 h_1^k$ in $L^2(0,T;L^2(0,1))$ and $\partial_x h_2^k$ in $L^2(0,T;L^2(0,1))$ gives us:

$$(\partial_x^2 h_1^k)_k \quad \text{converges weakly to} \quad \partial_x^2 h_1 \quad \text{in } L^2(0,T;L^2(0,1)),$$

$$(\partial_x h_2^k)_k \quad \text{converges weakly to} \quad \partial_x h_2 \quad \text{in } L^2(0,T;L^2(0,1)).$$

• Thanks to the strong convergence of $(h_1^k)_k$, $(h_2^k)_k$, $(\partial_x h_1^k)_k$ and the weak convergence of $(\partial_x^2 h_1^k)_k$, we have:

$$(h_1^k\partial_x h_1^k)_k \quad \text{converges strongly to} \quad h_1\partial_x h_1 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$(h_2^k\partial_x h_1^k)_k \quad \text{converges strongly to} \quad h_2\partial_x h_1 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$(h_1^k\partial_x^2 h_1^k)_k \quad \text{converges weakly to} \quad h_1\partial_x^2 h_1 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$(\partial_x h_1^k\partial_x^2 h_1^k)_k \quad \text{converges weakly to} \quad \partial_x h_1\partial_x^2 h_1 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$(h_1^k\partial_x h_2^k)_k \quad \text{converges strongly to} \quad h_1\partial_x h_2 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$(h_2^k\partial_x h_2^k)_k \quad \text{converges strongly to} \quad h_2\partial_x h_2 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$\left((h_1^k)^2\right)_k \quad \text{converges strongly to} \quad {h_1}^2 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$\left((h_2^k)^2\right)_k \quad \text{converges strongly to} \quad {h_2}^2 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$(h_1^k h_2^k)_k \quad \text{converges strongly to} \quad h_1 h_2 \quad \text{in } L^1(0,T;L^1(0,1)).$$

3.6. Convergence of $(h_1^k\partial_x u_1^k)_k$, $(\gamma(h_1^k)u_1^k)_k$ and $(h_1^k|u_1^k|^2u_1^k)_k$. As $(u_1^k)_k$ is bounded in $L^2(0,T;L^2(0,1))$, then $(\partial_x u_1^k)_k$ is bounded in $L^2(0,T;W^{-1,2}(0,1))$. Moreover, we have $(\gamma(h_1^k))_k$ bounded in $L^\infty(0,T;H^1(0,1))$.

Then,

$$(\gamma(h_1^k))_k \quad \text{converges strongly to} \quad \gamma(h_1) \quad \text{in } C^0(0,T;L^2(0,1)),$$

$$(u_1^k)_k \quad \text{converges weakly to} \quad u_1 \quad \text{in } L^2(0,T;L^2(0,1)).$$

So,

$$(\gamma(h_1^k)u_1^k)_k \quad \text{converges weakly to} \quad \gamma(h_1)u_1 \text{ in } L^2(0,T;L^2(0,1)).$$

However, the function $(h_1^k,\partial_x h_1^k) \longmapsto h_1^k\partial_x h_1^k$ is a continuous in $L^\infty(0,T;H^1(0,1)) \times L^2(0,T;W^{-1,2}(0,1))$ to $L^2(0,T;W^{-1,2}(0,1))$.
So,

$$(h_1^k\partial_x u_1^k)_k \quad \text{converges weakly to} \quad h_1\partial_x u_1 \quad \text{in } L^2(0,T;H^{-1}(0,1)).$$

Finally, thanks to the strong convergence of $\left(\sqrt{h_1^k}u_1^k\right)_k$ to $\sqrt{h_1}u_1$ in $L^2(0,T;L^2(0,1))$ and the weak convergence of $(u_1^k)_k$ to u_1 mentioned above, we have :

$$(h_1^k|u_1^k|^2u_1^k)_k \quad \text{converges weakly to} \quad h_1|u_1|^2u_1 \quad \text{in } L^1(0,T;L^1(0,1)).$$

3.7. Convergences of $(h_2^ku_1^k)_k$ and $\left((h_2^k)^2(\frac{1}{c}+\frac{1}{3\nu_2}h_2^k)\partial_x(h_1^k+h_2^k)\right)_k$. We know that $(\partial_x(h_1^k+h_2^k))_k$ converges weakly to $\partial_x(h_1+h_2)$ in $L^2(0,T;L^2(0,1))$ and $\left((h_2^k)^2(\frac{1}{c}+\frac{1}{3\nu_2}h_2^k)\right)_k$ converges strongly to $h_2^2(\frac{1}{c}+\frac{1}{3\nu_2})h_2$ in $L^1(0,T;L^1(0,1))$. So,

$$\left((h_2^k)^2(\frac{1}{c}+\frac{1}{3\nu_2}h_2^k)\partial_x(h_1^k+h_2^k)\right)_k \quad \text{converges weakly to } (h_2)^2(\frac{1}{c}+\frac{1}{3\nu_2}h_2)\partial_x(h_1+h_2)$$

in $L^1(0,T;L^1(0,1))$. To conclude, we have:

$$(u_1^k)_k \quad \text{converges weakly to} \quad u_1 \text{ in } L^2(0,T;L^2(0,1))$$

and the strong convergence of$(h_2^k)_k$ to h_2, both give us:

$$(h_2^ku_1^k)_k \quad \text{converges weakly to} \quad h_2u_1 \quad \text{in } L^1(0,T;L^1(0,1)).$$

References

[1] D. Bresch, B. Desjardins, Existence of global weak solution for 2D viscous shallow water equations and convergence to the quasi-geostrophic model, *Comm. Math. Phys.* **238**(1-3) (2003), 211–223.

[2] D. Bresch, B. Desjardins, Some diffusive capillary model of Korteweg type, *C. R. Acad. Sci. Paris, Section Mécanique* **332**(11) (2004), 881–886.

[3] D. Bresch, B. Desjardins, D. Gérard-Varet, On compressible Navier-Stokes equations with density dependent viscosities in bounded domains, *J. Math. Pures Appl.* **87**(9) (2007), 227–235.

[4] D. Bresch, B. Desjardins, C.K. Lin, On some compressible fluid models: Korteweg, lubrication and shallow water systems, *Communications in partial differential equations* **28**(3,4) (2003), 843–868.

[5] E.D. Fernández-Nieto, G. Narbona-Reina, J.D. Zabsonré, Formal derivation of a bilayer model coupling shallow water and Reynolds lubrication equations: evolution of a thin pollutant layer over water, *J. Nonlinear-Anal Mech.* **24**(6): (2013), 803–833.
[6] F. Marche, Theoretical and Numerical Study of Shallow Water Models. Applications to Nearshore Hydrodynamics, PhD Thesis, University of Bordeaux.
[7] A. Mellet, A. Vasseur, On the barotropic compressible Navier-Stokes equations, *Comm. Partial Differential Equations* **32**(3) (2007), 431–452.
[8] B. Toumbou, D. Le Roux, A. Sene, An existence theorem for a 2-D coupled sedimentation shallow-water model, *C. R. Math. Acad. Sci. Paris* **344** (2007) 443–446.
[9] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, 1969.
[10] A. Oron, S.H. Davis, S.G. Bankoff, Long scale evolution of thin films, *Rev. Phys.* **69**(3) (1997), 931–980.
[11] O. Reynolds, On the theory of lubrication and its application to Mr. Beauchamp tower's experiment, *Phil. Trans. R. Soc. London Part I* **52** (1886), 228–310.
[12] B. Roamba, J.D.D. Zabsonré, S. Traoré, Formal derivation and existence of global weak solutions of a two-dimensional bilayer model coupling shallow water and Reynolds lubrication equations, *Asymptotic Analysis* **99** (2016) 207–239.
[13] J. Simon, Compact set in the space $L^p(0;t;B)$, *Ann. Mat. Pura appl.* **146**(4), 1987, 65–96.
[14] J.D.D. Zabsonré, C. Lucas, E. Fernàndez-Nieto, An energetically consistent viscous sedimentation model, *Math. Models Methods Appl. Sci.* **19**(3) (2009), 477–499.

(Brahima Roamba) UFR/ST-IUT, Université Nazi Boni, 10 BP 1091 Bobo-Dioulasso, Burkina Faso
E-mail address: `braroamba@gmail.com`

(Jean De Dieu Zabsonré) IUT, Université Nazi Boni, 10 BP 1091 Bobo-Dioulasso, Burkina Faso
E-mail address: `jzabsonre@gmail.com`

(Yacouba Zongo) UFR/ST, Université Nazi Boni, 10 BP 1091 Bobo-Dioulasso, Burkina Faso
E-mail address: `yaczehn10@gmail.com`

Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 149–155
ISSN: 1223-6934

Ideals with linear resolution in Segre products

Gioia Failla

Abstract. We consider a homogeneous graded algebra on a field K, which is the Segre product of a $K-$polynomial ring in m variables and the second squarefree Veronese subalgebra of a $K-$polynomial ring in n variables, generated over K by elements of degree 1. We describe a class of graded ideals of the Segre product with a linear resolution, provided that the minimal system of generators satisfies a suitable condition of combinatorial kind.

2010 Mathematics Subject Classification. Primary, 13A30; Secondary, 13D45.
Key words and phrases. Monomial algebras, graded ideals, linear resolutions.

---

1. Introduction

Let A and B be two homogeneous graded algebras and let $A * B$ be their Segre product $K[u_1, \dots, u_n]$, where all generators have degree 1. In [14] the notion of strongly Koszul algebra is introduced and the main consequence is that the maximal graded ideal has linear quotients, hence a linear resolution. In particular if $A = K[x_1, \dots, x_n]$ and $B = K[y_1, \dots, y_m]$ are polynomial rings, the graded maximal ideal $(x_1y_1, \dots, x_ny_m)$ of $A * B$ has linear quotients and a linear resolution. For the significant applications in combinatorics, the case where A and B are monomial algebras received a lot of attention from algebrists. In this case, note that the generators $u_1, \dots, u_n$ are monomials and the subtended affine semigroup reflects properties of the algebra. The problem to yield monomial ideals with linear quotients and having linear resolution is particularly interesting for homogeneous semigroup rings. The aim of this paper is to investigate if the class of monomial ideals of the semigroup ring studied in [12], and with linear quotients, has a linear resolution.
More precisely, in Section 1, we consider two polynomials rings $A = K[x_1, \dots, x_n]$ and $B = K[y_1, \dots, y_m]$ with the standard graduation and the Segre product $B * A^{(2)}$ between B and the second squarefree Veronese ring $A^{(2)}$ generated over K by all squarefree monomials of degree 2 of A. We recall in particular the property P considered in [12], on ordered subsets of the generators of C, that has an interpretation in algebraic combinatorics. In Section 2, we focus our attention to monomial ideals of $B * A^{(2)}$, that admit quotient ideals linearly generated and, as a consequence, they have a linear resolution, being linear modules, following the definition given in [3]. We examine a class of ideals, generated by a suitable subset of the set of the generators of the K-algebra $B * A^{(2)}$, studied in [12] and with linear quotients. The main point is to require that the set of generators satisfies a property able to guarantee that a family of colon ideals of the ideal has linear quotients.

2. Preliminaries and known results

Let $A = K[x_1, \ldots, x_n]$ and $B = K[y_1, \ldots, y_m]$ be two polynomial rings in n and m variables respectively with coefficients in any field K. Let $A^{(2)} \subset A$ be the 2nd squarefree Veronese algebra of A and let $C = B * A^{(2)}$ be the Segre product of B and $A^{(2)}$. We consider C as a standard $K-$algebra generated in degree 1 by the monomials $y_\alpha x_i x_j$, with $1 \leq \alpha \leq m$, $1 \leq i < j \leq n$. For convenience, we will indicate such a monomial by αij.

In [12] we computed all quotient ideals of principal ideals of C, generated by generators of the graded maximal ideal m^* of C in order to obtain the intersection degree of this algebra [13], [14]. The description of the generators of the colon ideals will be used in the following.

Theorem 2.1. [12, Theorem 1.1] *Let $C = B * A^{(2)}$ be the Segre product and let $m^* = (u_1, \ldots, u_N)$, $N = m\binom{n}{2}$ the maximal ideal of C. Let $(u_r) : (u_s)$, $1 \leq r, s \leq N, r \neq s$, a colon ideal of generators of m^*, in the lexicographic order. Then we have:*

1. $(\alpha\, ij_1) : (\alpha\, ij_2) = (\beta k j_1, k \neq j_1, j_2,\, \beta \in \{1, \ldots, m\})$
2. $(\alpha_1 i j_1) : (\alpha_2 i j_2) = (\alpha_1 k j_1,\, k \neq j_1, j_2)$
3. $(\alpha\, i_1 j) : (\alpha\, i_2 j) = (\beta i_1 k, k \neq i_1, i_2,\, \beta \in \{1, \ldots, m\})$
4. $(\alpha_1 i_1 j) : (\alpha_2 i_2 j) = (\alpha_1 i_1 k,\, k \neq i_1, i_2)$
5. $(\alpha\, i_1 j) : (\alpha\, j j_2) = (\beta i_1 k,\, k \neq i_1, j_2,\, \beta \in \{1, \ldots, m\})$
6. $(\alpha\, ij_1) : (\alpha\, i_2 i) = (\beta k j_1,\, k \neq j_1, i_2,\, \beta \in \{1, \ldots, m\})$
7. $(\alpha_1 i_1 j) : (\alpha_2 j j_2) = (\alpha_1 i_1 k,\, k \neq i_1, j_2)$
8. $(\alpha_1 i j_1) : (\alpha_2 i_2 i) = (\alpha_1 k j_1,\, k \neq i_2, j_1)$
9. $(\alpha_1 i_1 j_1) : (\alpha_2 i_2 j_2) = (\alpha_1 i_1 j_1, (\alpha_1 i_1 s)(\beta j_1 s), \beta \in \{1, \ldots, m\}, s \neq i_1, j_1, i_2, j_2)$

10 $(\alpha_1 ij) : (\alpha_2 ij) = (\alpha_1 kl, k \neq l)$

Corollary 2.2. [12, Corollary 1.2] *Let $B * A^{(2)}$ be the Segre product as in Theorem 2.1, where all generators are of degree 1. Then the intersection degree of the monomial algebra $B * A^{(2)}$ is equal to 3 for $n > 4$.*

The fact that there are colon ideals not generated in degree 1 can to not be a problem for special classes of monomial ideals. In particular the strong condition that we consider monomial ideals generated by subsets of generators that verify the property P implies that a family of associated quotients ideals are generated in degree 1, provided a suitable order on the generators.

For this end, we introduce in the set of monomials of $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ the lexicographic order with the order on the variables $y_1 > \ldots > y_m > x_1 > \ldots > x_n$. Moreover, following [12], we call "bad pair" a pair of monomials ij, kl in $A^{(2)}$ or $\alpha\, ij, \beta\, kl$ in C, with $i \neq k$ and $j \neq l$.

Definition 2.1. Let $(u_1, \ldots, u_t)$ be an ideal of $C = B * A^{(2)}$ generated by a sequence $\mathcal{L} = \{\alpha_1 i_1 j_1, \ldots, \alpha_t i_t j_t\}$ of generators of C, with $u_1 > \ldots > u_t$. Fixed $\alpha kl \in \mathcal{L}$, let $\mathcal{L}_{\alpha kl} = \{\beta rs \in \mathcal{L} / \beta rs > \alpha kl$ and $rs > kl\}$ and $\mathcal{L}'_{\alpha kl} = \{\beta rs \in \mathcal{L} / \beta rs < \alpha kl$ and $rs > kl\}$ be. We say that the sequence $\mathcal{L}$ satisfies the property P if:

(1) for each bad pair $\alpha ij > \alpha kl$ in $\mathcal{L}$, $\alpha ik \in \mathcal{L}_{\alpha kl}$ or $\alpha il \in \mathcal{L}_{\alpha kl}$ or $\alpha kl \in \mathcal{L}_{\alpha jk}$ or $\alpha jl \in \mathcal{L}_{\alpha kl}$

(2) for each bad pair $\alpha ij > \beta kl$ in $\mathcal{L}$, with $ij > kl$, $\alpha ik \in \mathcal{L}_{\beta kl}$ or $\alpha il \in \mathcal{L}_{\beta kl}$ or $\alpha jk \in \mathcal{L}_{\beta kl}$ or $\alpha jl \in \mathcal{L}_{\beta kl}$

(3) for each bad pair $\alpha ij > \beta kl$ in $\mathcal{L}$, with $ij < kl$, or $\beta ki \in \mathcal{L}^{'}_{\alpha ij}$ or $\beta kj \in \mathcal{L}^{'}_{\alpha ij}$ or $\beta il \in \mathcal{L}^{'}_{\alpha ij}$ or $\beta jl \in \mathcal{L}^{'}_{\alpha ij}$.

By using this definition, we have:

Theorem 2.3. *Let $(u_1, \ldots, u_t)$ be the ideal of $B * A^{(2)}$ generated by a sequence $\mathcal{L} = \{\alpha_1 i_1 j_1, \ldots, \alpha_t i_t j_t\}$ of generators of M, with $u_1 > \ldots > u_t$. Fixed $\alpha kl \in \mathcal{L}$, let $\mathcal{L}_{\alpha kl} = \{\beta rs \in \mathcal{L}/\beta rs > \alpha kl$ and $rs > kl\}$ and $\mathcal{L}^{'}_{\alpha kl} = \{\beta rs \in \mathcal{L}/\beta rs < \alpha kl$ and $rs > kl\}$ be. Suppose that the sequence $\mathcal{L}$ satisfies the property P. Then $(u_1, \ldots, u_t)$ has linear quotients.*

Proof. See [12, Theorem 2.3]. □

Example 2.1. For $n = 2$, $m = 5$, consider $C = K[y_1, y_2] * K[x_1, x_2, x_3, x_4, x_5]$. The sequences $\mathcal{L}_1 = \{112, 113, 114, \ldots, 145, 212, 213, 214, \ldots, 245\}$ and $\mathcal{L}_2 = \{112, 113, 123, 125, 135, 212, 213, 223, 225, 235\}$ satisfy the property P. For $\mathcal{L}_1$ the result is obvious, since $\mathcal{L}_1$ is the generating sequence of the maximal irrelevant ideal of C. For $\mathcal{L}_2$, we observe that it comes from the colon ideal $(112, 113) : (114) = (112, 113, 123, 125, 135, 212, 213, 223, 225, 235)$. Consider the bad pair $112 > 135$, with $12 > 35$. Then $\mathcal{L}_{135} = \{112, 113, 123, 125\}$. We have $113 \in \mathcal{L}_{135}, 123 \in \mathcal{L}_{135}, 125 \in \mathcal{L}_{135}, 115 \notin \mathcal{L}_{135}$. Consider the bad pair $125 > 213$, with $25 < 13$, $\mathcal{L}^{'}_{125} = \{212, 213, 223\}$. We have $212 \in \mathcal{L}^{'}, 215 \notin \mathcal{L}^{'}, 223 \in \mathcal{L}^{'}, 235 \notin \mathcal{L}^{'}$. Then the property P is satisfied.

3. Monomial Ideals with linear quotients

The aim of this section is to prove that the class of monomial ideals of the Segre product $C = B * A^{(2)}$ described in [12], having linear quotients, has a linear resolution on C. For this we need the following;

Theorem 3.1. *Let $(u_1, \ldots u_t)$ be the ideal of $C = B * A^{(2)}$ generated by the sequence $\mathcal{L}$ as in Theorem 2.1 and $I_{q-1} = (\alpha_1 i_1 j_1, \ldots, \alpha_{q-1} i_{q-1} j_{q-1}), q \leq t-1$. Then the colon ideal $I : \alpha_{q-1} i_q j_q$ satisfies condition P.*

Proof. Note that the monomial ideal $I : \alpha_q i_q j_q$ is generated by all colon ideals $\alpha_p i_p j_p : \alpha_q i_q j_q$ such that each pair $\alpha_p i_p j_p, \alpha_q i_q j_q$ is not a bad pair for $p < q$ (see [12, Theorem 2.3]). Set $i = i_p$ and $j = j_p$. Assume $i < j$. Consider a bad pair $a, b \in I : \alpha_q ij$:

I case: $a \in \alpha\, i_s j_s : \alpha\, ij$, $\quad i_s j_s, ij$ is not a bad pair
$b \in \alpha\, i_t j_t : \alpha\, ij$, $\quad i_t j_t, ij$ is not a bad pair

II case: $a \in \alpha\, i_s j_s : \beta\, ij$, $\quad i_s j_s, ij$ is not a bad pair
$b \in \beta\, i_s j_s : \beta\, ij$, $\quad i_s j_s, ij$ is not a bad pair

III case: $a \in \alpha\, i_s j_s : \beta\, ij$, $\quad i_s j_s, ij$ is not a bad pair
$b \in \gamma\, i_s j_s : \beta\, ij, \gamma \neq \alpha$, $\quad i_s j_s, ij$ is not a bad pair

IV case: $a \in \alpha\, i_s j_s : \alpha\, ij$, $\quad i_s j_s, ij$ is not a bad pair
$b \in \beta\, i_t j_t : \alpha\, ij$, $\quad i_t j_t, ij$ is not a bad pair

I case: Note that $i_sj_s > ij, i_tj_t > ij$ and $i_s < j_s, i < j, i_t < j_t$. Write $i = i_t$ and $j = j_s$, the colon ideals to be considered are $\alpha i_sj : \alpha ij$ and $\alpha ij_t : \alpha ij$. Let $a \in \alpha\, i_sj : \alpha\, ij$ and $b \in \alpha\, ij_t : \alpha\, ij$ be. Suppose $a = \alpha\, i_sk, k \neq i_s, j$ and $b = \alpha\, lj_t, l \neq j_t, j$.

We look to the following cases:

i) $i_s < k,\ l < j_t$. If $\alpha\, i_sk > \alpha\, lj_t$ (that is $a > b$) then $i_sk > lj_t,\ i_s < l < j_t$, hence $i_s < j_t$. Since $i < j_t$, it follows that $\alpha\, i_sj_t$ is a generator of the colon ideal $I : \alpha ij$ (since $j_t \neq i_s, i$). It follows that $\alpha\, i_sj_t > \alpha\, lj_t$, that is $\alpha\, i_sj_t \in \mathcal{L}_{\alpha\, lj_t}$. If $a < b$, $i_sk < lj_t, l < i_s$, $\alpha i_sk < \alpha li_s$ and $i_sk < li_s$. It follows $\alpha li_s \in \mathcal{L}_{\alpha\, i_sk}$ (that is the property P).

ii) $i_s < k,\ l > j_t,\ a = \alpha\, i_sk,\ b = \alpha j_tl$. If $a > b$, $\alpha\, i_sk > \alpha\, j_tl$ and $i_s < j_t < l$. Since $i < j_t$, it follows that αi_sj_t is a generator of the colon ideal $I : \alpha\, ij$ (since $j_t \neq i_s, i$) hence $\alpha\, i_sj_t > \alpha j_tl, \alpha\, i_sj_t \in \mathcal{L}_{\alpha\, j_tl}$(that is the property P). If $a < b$, $i_sk < j_tl$. Then $j_t < i_s < k$, hence $j_ti_s > i_sk$ and $\alpha j_ti_s \in \mathcal{L}_{\alpha i_sk}$.

iii) $i_s > k, l < j_t$. If $a > b$, $ki_s > lj_t$, then $k < l$ and so $k < l < j_t$. Since $k < j$, the element αkj_t is a generator of $I : \alpha ij$ and $\alpha kj_t > \alpha lj_t$, so $\alpha kj_t \in \mathcal{L}_{\alpha lj_t}$. If $ki_s < lj_t$, $l < k < i_s$ and, since $l < i$, it follows that αli_s is a generator of $I : \alpha ij$ and $\alpha li_s > \alpha ki_s$, $\alpha li_s \in \mathcal{L}_{\alpha\, ki_s}$ (that is the property P).

iv) $i_s > k, l > j_t$. If $a > b$, write $a = \alpha ki_s,\ b = \alpha j_tl$. If $ki_s > j_tl$, $k < j_t < l$ and $kj_t > j_tl$. Since $k < i_s < i < j$, αkj_t is a generator of $I : \alpha ij$. It follows $\alpha kj_t \in \mathcal{L}_{\alpha\, j_tl}$ (that is the property P). If $ki_s < j_tl$, $j_t < k < i_s$ and $j_t < i_s < i$. Hence αj_ti_s is a generator of $I : \alpha ij$, $\alpha j_ti_s > \alpha ki_s$, and $\alpha j_ti_s \in \mathcal{L}_{\alpha\, ki_s}$.

Indeed, we have to achieve the property P for the remaining cases. In synthesis, we can suppose:

$$a = \beta\, i_sk,\ k \neq i_s, i \text{ and } \beta \neq \alpha, \beta > \alpha$$

$$b = \gamma\, lj_t,\ l \neq j_t, j,\ \beta\, i_sk > \gamma lj_t \ \text{ then } \beta > \gamma.$$

We can have:

a) $i_sk > lj_t$

b) $i_sk < lj_t$.

We look to the following cases:

i') $i_s < k,\ l < j_t$, ii') $i_s < k,\ l > j_t$, iii') $i_s > k, l < j_t$, iv') $i_s > k, l > j_t$:

i') For a), $i_s < k$ and $l < j_t$, hence $i_s < l < j_t$ and $j_t \neq i_s$. Since $j_t > i,\ j_t \neq i$, it follows that βi_sj_t is an element of $I : \alpha ij$ and $\beta i_sj_t > \gamma lj_t$ (that is the property P). For b), $i_s > l$. Since $j_t > i, j_t \neq i$. It follows that the monomial βi_sj_t is an element of $I : \alpha ij$ and $\beta i_sj_t > \gamma lj_t$, $\gamma lj_t \in \mathcal{L}'_{\beta i_sj_t}$ (that is the property P).

ii') For a), write $i_sk > j_tl$. Then $i_s < j_t < l$. Since $i < j_t$, it follows that βi_sj_t is a generator of the colon ideal $I : \alpha\, ij$ (since $j_t \neq i_s, i$) hence $\beta\, i_sj_t > \gamma j_tl, \beta\, i_sj_t \in \mathcal{L}_{\gamma j_tl}$(that is the property P). For b), $i_sk < j_tl$. Then $j_t < i_s < k$, hence $j_ti_s > i_sk$,$\beta j_ti_s > \gamma i_sk$ and $\beta j_ti_s \in \mathcal{L}_{\gamma i_sk}$.

iii') For a) $ki_s > lj_t$, then $k < l$ and so $k < l < j_t$. Since $k < j$, the element βkj_t is a generator of $I : \alpha ij$ and $\beta kj_t > \gamma lj_t$, so $\beta kj_t \in \mathcal{L}_{\gamma lj_t}$. For b), $l < k < i_s$ and, since $l < i$, it follows that $\gamma li_s < \beta ki_s$, $\gamma li_s \in \mathcal{L}'_{\beta ki_s}$.

iv') For a), write $ki_s > j_tl$, $k < j_t < l$ and $kj_t > j_tl$. Since $k < i_s < i < j$, γkj_t is a generator of $I : \alpha ij$, and $\gamma kj_t < \beta j_tl$, it follows $\gamma kj_t \in \mathcal{L}^{'}_{\beta j_t l}$, that is the property P. For b), write $ki_s < j_tl$, $j_t < k < i_s$ and $j_t < i_s < i$. Hence $\beta j_t i_s$ is a generator of $I : \alpha ij$, $\beta j_t i_s > \gamma k i_s$, and $\beta j_t i_s \in \mathcal{L}_{\gamma k i_s}$.

The proof of cases II, III, IV is analogous. ☐

Now we recall the definition of linear module, as found in [3].

Definition 3.1. Let $R = K[u_1, \dots, u_n]$ be a homogeneous $K-$algebra, K a field, finitely generated over K by elements of degree 1, and let M a graded $R-$module. M is said to be linear if it has a system of generators $m_1, \dots, m_t$ all of the same degree, such that for $j = 1, \dots, t$ the colon ideals:

$$(Rm_1 + \dots Rm_{j-1}) : m_j$$

is generated by a subset of $\{u_1, \dots, u_n\}$.

Proposition 3.2. [14, Theorem 1.2] *Suppose R a strongly Koszul K-algebra. Let $I \subset R$ be a homogeneous ideal generated by a subset of generators of the maximal irrelevant ideal of R. Then I has linear quotients and a linear resolution on R.*

Proposition 3.3. *Let C be the monomial algebra $B * A^{(2)}$ and let I be a monomial ideal $(u_1, \dots, u_t)$ generated by a sequence $\mathcal{L}$ of generators of the algebra that satisfies the property P. Then I has a linear resolution.*

Proof. By Definition 3.1, I is a linear module. Hence the statement will be true if we show that I has linear relations and its first syzygy module is again a linear module. For the first assertion, if $a_1u_1 + \dots + a_ru_r, 1 \leq r \leq t$, is a homogeneous generating relation of I, let a_j be the last non zero coefficient of that relation, then a_j is a generator of the colon ideal $(u_1, \dots, u_{j-1}) : u_j$. Hence a_j is a generator of the algebra of degree 1, and the relation is linear. Let $Syz_1(I)$ be the first syzygy module of I. We will prove that $Syz_1(I)$ is a linear module by induction on the number of generators. If the ideal I is principal, then $Syz_1(I) = \{0\}$. Suppose $Syz_1(I)$ generated by r elements $g_1, \dots, g_s, s > 1$,such that with respect to them $Syz_1(I)$ is a linear module. Consider the submodule $D = Cg_1 + \dots + Cg_{s-1}$ that is linear by induction and so its $Syz_1(D)$ module, with respect to a system of minimal generators $l_1, \dots, l_u$. By the exact sequence

$$0 \to Syz_1(D) \to Syz_1(Syz_1(I)) \to Syz_1(Syz_1(I)/Syz_1(D)) \to 0,$$

the module $Syz_1(I)/Syz_1(D)$ is cyclic with annihilator ideal $Cg_1 + \dots + Cg_{s-1} : Cg_s$,then $Syz_1(Syz_1(I)/Syz_1(D)) \cong (u_{i_1}, \dots, u_{i_v})$, $1 \leq i_1 < \dots < i_v \leq t$, that verifies the Property P by induction and then it is a linear module. Now we can complete the set $l_1, \dots, l_u$ in $Syz_1(D)$, hence in $Syz_1(Syz_1(I))$, choosing homogeneous elements $h_1, h_2, \dots, h_v$ of $Syz_1(Syz_1(I))$, such that they can be mapped onto in the set $u_{i_1}, \dots, u_{i_v}$. We claim that the module $Syz_1(Syz_1(I))$, generated by the set $l_1, \dots, l_u$, $h_1, h_2, \dots, h_v$ is a linear module with respect to these generators. In fact the quotient ideals $Cl_1 + \dots + Cl_{j-1} : Cl_j$, $1 \leq j \leq s$,are generated by a subset of generators. By induction, each colon ideal $Cl_{i_j} : Ch_{j_k} = (0)$, $1 \leq i_j \leq u$, $1 \leq i_k \leq v$, and $Ch_1 + Ch_{k-1} : Ch_k, 1 \leq k \leq v$, are generated by a subset of variables. For this, let m be a monomial generator, then $mh_k = b_1h_1 + \dots + b_{k-1}h_{k-1}$ and mapping onto in

$Syz_1(Syz_1/Syz_1(D))$, we obtain the relation $mu_{i_k} = b_1u_{i_1} + b_k - 1u_{i_{k-1}}$ in C. So m is a generator of the quotient ideal $(u_{i_1}, \dots, u_{i_{k-1}}) : u_{i_k}$, hence of degree 1. □

Corollary 3.4. *Let* $I = (u_1, \dots, u_t)$ *be an ideal of* $B * A^{(2)}$ *as in Theorem* 2.1. *Let* I_r *be any colon ideal* $(u_1, \dots, u_r) : (u_{r+1})$ *of* I, $r = 1, \dots, t-1$.*Then we have:*

(1) I_r *has linear quotients*
(2) I_r *has a linear resolution.*

Proof. (1) By Theorem 3.1 and (2) by Proposition 3.3. □

Remark 3.1. We proved in Theorem 3.1 that any colon ideal I_r of I verifies the property P. In the same way any colon ideal of I_r verifies P and so on. The previous condition characterizes the sequentially Koszul algebras, as defined in [1].

Remark 3.2. For $n = 4$, $A^{(2)}$ is a strongly Koszul algebra and consequently the Segre product $B * A^{(2)}$ [14]. As a consequence any ideal generated by a subset of generators has a linear resolution.

Remark 3.3. For homogeneous semigroup rings arising from Grassmann varieties, Hankel varieties of $\mathbb{P}^n$ and their subvarieties [7], [8], [9], [10], [15], the problem is more difficult. For $G(1,3) = H(1,3)$ its toric ring is strongly Koszul, being a quotient of the polynomial ring $K[[12],[13],[14],[23],[24],[34]]$ for the ideal generated by the binomial relation $[14][23] - [13][24]$, where $[i,j]$ is the variable corresponding to the minor with columns i, j, $i < j$, of a 2×4 generic matrix. The semigroup ring of $\mathbb{G}(1,4)$ is a subring of $K[t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}]$, t_{ij} the generic entry of a 2×5- matrix

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} \end{pmatrix}$$

and it is generated by the diagonal initial terms of ten 2×2 minors of the matrix. The semigroup of $H(1,4)$ is a subring of $K[t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{16}]$, generated by the diagonal initial terms of ten 2×2 minors of the Hankel matrix

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} \\ t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \end{pmatrix}.$$

Both rings have a toric ideal generated by a Gröbner basis of degree 2 [15], [8] and they are Koszul. The problem to find monomial ideals generated by subsets of generators of the semigroup ring with linear resolution is open, for $n > 4$.

Remark 3.4. Segre products between polynomial rings on any field K and squarefree Veronese rings have been employed for algebraic models in statistic, in graphs theory, in transportation problems [4], [5], [6]. In particular, if I_r and J_s are respectively the rth squarefree Veronese ideal of $K[x_1, \dots, x_n]$ and the sth squarefree Veronese ideal of $K[y_1, \dots, y_m]$, we can consider the sum $I_r + J_s$ or the product I_rJ_s in the ring $K[x_1, \dots, x_n; y_1, \dots, y_m]$ that describe particular simple graphs and the semigroup rings $K[I_r]$, $K[I_r, J_s]$, $K[I_rJ_s]$, respectively subrings of $K[x_1, \dots, x_n]$, $K[x_1, \dots, x_m; y_1, \dots, y_n]$ generated by the minimal system of generators of I_r,$I_r + J_s$ and I_rJ_s. Observe that we have that $C = K[J_1I_2]$. Since the sorted Gröbner basis of the defining ideals of the previous semigroup rings is quadratic [15], initial simplicial complexes with respect the a total order received a lot of attention in several articles. Indeed the subtended affine semigroup presents easy triangulations [11],[15]. Alternately, one studied classify the simplicial complexes defined by the squarefree monomial ideals $I_r + J_s$ and $I_r J_s$ to obtain combinatorial statements [16].

In this paper we referred to the excellent books whose in [2], [17].

References

[1] A. Aramova, J. Herzog, T. Hibi, Finite lattices and lexicographic Gröbner bases, *Europ. J. Combin.* **21** (2000), no. 4, 431–439.

[2] W. Bruns, J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, 1993.

[3] W. Bruns, J. Herzog, U. Vetter, Syzygies and Walks in: *ICTP Proceedings Commutative Algebra*, A. Simia, N.V. Trung and G.Valla, Eds., World Scientific (1994), 36–57.

[4] G. Failla, Projective Toric Varietes and associated fibers, *Rend. Circ. Mat. Palermo (2) Suppl.)* **77** (2006), 267–280.

[5] G. Failla, R. Utano, Connected graphs arising from products of Veronese varieties, *Algebra Colloq.* **23** (2016), 281–292.

[6] G. Failla, On the (1,1)-Segre model for business, *Applied Mathematical Sciences* **8** (2014), 8329–8336.

[7] G. Failla, Quadratic Plücker relations for Hankel Varieties, *Rend. Circ. Mat. Palermo (2) Suppl.* **81** (2009), 171–180.

[8] G. Failla, On the defining equations of the Hankel varieties $H(2, n)$, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **104** (2013), Tome 56, 403–418.

[9] G. Failla, Combinatorics of Hankel relations, *Ann. Acad. Rom. Sci.Ser. Math. Appl.* **9** (2017), no.2.

[10] G. Failla, On Certain Loci of Hankel $r-$planes of $\mathbb{P}^m$, *Mathematical Notes* **92** (2012), no. 4, 544–553.

[11] G. Failla, Linear Triangulations of Polytopes, *Aplimat 2017 16th Conference on Applied Mathematics, Proceedings* (2017), 499–509.

[12] G.Failla, Ideals with linear quotients in Segre products, *Opuscola Mathematica* **37**, no. 6 (2017).

[13] G. Restuccia, G. Rinaldo, Intersection degree and bipartite graphs, *ADJM* **8** (2008), no. 2, 114–124.

[14] J. Herzog, T. Hibi, G. Restuccia, *Strongly Koszul algebras*, *Math. Scand.* **86** (2000), 161–178.

[15] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, Univ. Lect. Series **8**, Amer. Math. Soc., 1995.

[16] R. Utano, Mixed Product ideals and simplicial complexes, *Aplimat 2017, 16th Conference on Applied Mathematics, Proceedings* (2017), 1586–1591.

[17] R.H. Villarreal, *Monomial Algebras*, Monographs and Textbooks in Pure and Applied Mathematics **238**, Marcel and Dekker Inc., New york, 2001.

(Gioia Failla) DEPARTMENT DIIES, UNIVERSITY OF REGGIO CALABRIA, VIA GRAZIELLA, SALITA FEO DI VITO, REGGIO CALABRIA
E-mail address: `gioia.failla@unirc.it`

Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 156–168
ISSN: 1223-6934

Instantaneous shrinking of compact support of solutions of semi-linear parabolic equations with singular absorption

Anh Nguyen Dao

Abstract. We prove an existence of weak solutions of semi-linear parabolic equations with a strong singular absorption term. Moreover, we study the qualitative behavior of solutions such as the quenching phenomenon, the finite speed of propagation and the instantaneous shrinking of compact support.

2010 Mathematics Subject Classification. 35K55, 35K67, 35K65.
Key words and phrases. Singular parabolic equation, quenching phenomenon, instantaneous shrinking of compact support.

1. Introduction

In this paper, we are interested in nonnegative solutions of the following equation:

$$\begin{cases} \partial_t u - \Delta u + u^{-\beta}\chi_{\{u>0\}} + f(u) = 0 & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\beta \in (0,1)$, and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x,t) where $u(x,t) > 0$, i.e:

$$\chi_{\{u>0\}} = \begin{cases} 1, & \text{if } u > 0, \\ 0, & \text{if } u \leq 0. \end{cases}$$

Note that the absorption term $u^{-\beta}\chi_{\{u>0\}}$ becomes singular when u is near to 0, and we impose tactically $u^{-\beta}\chi_{\{u>0\}} = 0$ whenever $u = 0$. Through this paper, $f : [0,\infty) \longrightarrow \mathbb{R}$ is a nondecreasing continuous function such that $f(0) = 0$.

Problem (1) can be considered as a limit of mathematical models describing enzymatic kinetics (see [1]), or the Langmuir-Hinshelwood model of the heterogeneous chemical catalyst (see, e.g. [20] p. 68, [11], [18]). This problem has been studied by the authors in [18], [14], [15], [17], [10], [7], [21], and references therein. These authors have considered the existence and uniqueness, and the qualitative behavior of these solutions. For example, when $f = 0$, D. Phillips [18] proved the existence of solution for the Cauchy problem associating to equation (1). A partial uniqueness of

---

Received June 13, 2017.
Communicated by Jesus Ildefonso Diaz.

solution of equation (1) was proved by J. Davila and M. Montenegro, [10] for a class of solutions with initial data u_0 satisfying

$$u_0(x) \geq Cdist(x, \partial\Omega)^{\mu}, \quad \text{for } \mu \in (1, \frac{2}{1+\beta}),$$

see also [9] the uniqueness in a different class of solutions. Moreover, M. Winkler, [21] proved that the uniqueness of solution fails in general. One of the most striking phenomenon of solutions of equation (1) is the extinction that any solution vanishes after a finite time even beginning with a positive initial data, see [18], [14] (see also [7] for a quasilinear equation of this type). It is known that this phenomenon occurs according to the presence of the nonlinear singular absorption $u^{-\beta}\chi_{\{u>0\}}$. The same situation happens for the nonlinear absorption u^{β}, for $\beta \in (0,1)$, see [2] and references therein. Furthermore, equation (1) with source term $f(u)$ satisfying the sublinear condition, i.e: $f(u) \leq C(u+1)$, was considered by J. Davila and M. Montenegro, [10]. The authors proved the existence of solution and showed that the measure of the set $\{(x,t) \in \Omega \times (0,\infty) : u(x,t) = 0\}$ is positive (see also a more general statement in [12]). In other words, the solution may exhibit the quenching behavior.

To prove the existence of solutions of equation (1), we must prove the following gradient estimate:

$$|\nabla u(x,t)|^2 \leq Cu^{1-\beta}(x,t), \quad \text{for } (x,t) \in \Omega \times (0,T),$$

where the constant C depends on the f', f, see [10]. Thus, it requires the nonlinear $f \in \mathcal{C}^1([0,\infty))$. In this paper, we show that if f is a nondecreasing function then constant C above is independent of f', so we can remove the regularity $f \in \mathcal{C}^1([0,\infty))$.

Before establishing the existence of solutions of equation (1), it is necessary to introduce a notion of weak solution.

Definition 1.1. Let $u_0 \in L^{\infty}(\Omega)$. A nonnegative function $u(x,t)$ is called a weak solution of equation (1) if $u^{-\beta}\chi_{\{u>0\}} \in L^1(\Omega \times (0,T))$, and $u \in L^2(0,T; W_0^{1,2}(\Omega)) \cap L^{\infty}(\Omega \times (0,T))$ satisfies equation (1) in the sense of distributions $\mathcal{D}'(\Omega \times (0,T))$, i.e:

$$\int_0^T \int_{\Omega} \left(-u\phi_t + \nabla u.\nabla\phi + u^{-\beta}\chi_{\{u>0\}}\phi + f(u)\phi\right) dxdt = 0, \quad \forall\phi \in \mathcal{C}_c^{\infty}(\Omega \times (0,T)). \tag{2}$$

Then, our existence result is as follows:

Theorem 1.1. *Let $u_0 \in L^{\infty}(\Omega)$, and $\beta \in (0,1)$. Then, equation (1) has a maximal weak solution u satisfying*

$$|\nabla u(x,t)|^2 \leq Cu^{1-\beta}(x,t)\left(t^{-1}+1\right), \quad \text{for a.e } (x,t) \in \Omega \times (0,\infty), \tag{3}$$

where constant $C = C(f, \|u_0\|_{\infty}) > 0$.
Furthermore, if $\nabla(u_0^{\frac{1}{\gamma}}) \in L^{\infty}(\Omega)$, then there is a constant $C = C(f, \|u_0\|_{\infty}, \|\nabla(u_0^{\frac{1}{\gamma}})\|_{\infty}) > 0$ such that

$$|\nabla u(x,t)|^2 \leq Cu^{1-\beta}(x,t), \quad \text{for a.e } (x,t) \in \Omega \times (0,\infty). \tag{4}$$

Besides, we also study behaviors of solutions of the Cauchy problem associating to equation (1):

$$\begin{cases} \partial_t u - \Delta u + u^{-\beta}\chi_{\{u>0\}} + f(u) = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{5}$$

In [18], Phillips showed that the quenching phenomenon, and the finite speed of propagation hold for the solutions of the Cauchy problem. In this paper, we show that if initial data u_0 satisfies a certain growth condition at infinity, then any weak solution has the instantaneous shrinking of compact support (in short ISS), namely, if initial data u_0 goes to 0 uniformly as $|x| \to \infty$, then the support of any weak solution is bounded for any $t > 0$. This property was first proved in the literature in the study of variational inequalities by Brezis and Friedman, see [5]. After that this phenomenon has been considered for quasilinear parabolic equations, see [4], [13], and references therein. Then, our main result of the Cauchy problem is as follows:

Theorem 1.2. *Let $0 \le u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and $\beta \in (0,1)$. Then, there exists a weak bounded solution $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}^N)) \cap L^2(0,T; W^{1,2}(\mathbb{R}^N))$, satisfying equation (5) in $\mathcal{D}'(\mathbb{R}^N \times (0,\infty))$.*
In addition, if $u_0(x) \to 0$ uniformly as $|x| \to \infty$, then such a weak solution of problem (5) has ISS property.

The paper is organized as follows: In the next section, we prove some gradient estimates for the approximating solutions. In Section 3, we shall prove Theorem 1.1. The last section is devoted to study the Cauchy problem (5) and the instantaneous shrinking of compact support.

Several notations which will be used through this paper are the following: we denote by C a general positive constant, possibly varying from line to line. Furthermore, the constants which depend on parameters will be emphasized by using parentheses. For example, $C = C(\beta, f)$ means that C depends on β, f.

2. Gradient estimate for the approximate solutions

In this section, we consider a regularized equation of (1):

$$(P_{\varepsilon,\eta}) \begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon + g_\varepsilon(u_\varepsilon) + f(u_\varepsilon) = 0 & \text{in } \Omega \times (0,\infty), \\ u_\varepsilon = \eta & \text{on } \partial\Omega \times (0,\infty), \\ u_\varepsilon(0) = u_0 + \eta & \text{on } \Omega \end{cases}$$

for any $0 < \eta < \varepsilon$, with $g_\varepsilon(s) = \psi_\varepsilon(s)s^{-\beta}$, $\psi_\varepsilon(s) = \psi(\frac{s}{\varepsilon})$, and $\psi \in \mathcal{C}^\infty(\mathbb{R})$ is a non-decreasing function on $\mathbb{R}$ such that $\psi(s) = 0$ for $s \le 1$, and $\psi(s) = 1$ for $s \ge 2$. Note that g_ε is a globally Lipschitz function for any $\varepsilon > 0$. We will show that solution $u_{\varepsilon,\eta}$ of equation $(P_{\varepsilon,\eta})$ tends to a solution of equation (1) as $\eta, \varepsilon \to 0$. In passing to the limit, we need to derive some gradient estimates for solution $u_{\varepsilon,\eta}$. Then, we have the following result:

Lemma 2.1. *Let $0 \le u_0 \in \mathcal{C}_c^\infty(\Omega)$, $u_0 \ne 0$. There exists a classical unique solution $u_{\varepsilon,\eta}$ of $(P_{\varepsilon,\eta})$ in $\Omega \times (0,\infty)$.*
i) Moreover, there is a constant $C > 0$ only depending on $\beta, f, \|u_0\|_\infty$ such that

$$|\nabla u_{\varepsilon,\eta}(x,\tau)|^2 \le C u_{\varepsilon,\eta}^{1-\beta}(x,\tau)\big(\tau^{-1} + 1\big), \quad \textit{for any } (x,\tau) \in \Omega \times (0,\infty). \tag{6}$$

ii) If $\nabla(u_0^{\frac{1}{\gamma}}) \in L^\infty(\Omega)$*, then we get*

$$|\nabla u_{\varepsilon,\eta}(x,\tau)|^2 \leq C u_{\varepsilon,\eta}^{1-\beta}(x,\tau), \quad \textit{for any } (x,\tau) \in \Omega \times (0,\infty), \tag{7}$$

with $C > 0$ *merely depends on* $\beta, f, \|u_0\|_\infty, \|\nabla(u_0^{\frac{1}{\gamma}})\|_\infty$.

Proof. We first prove *i)*.
Fixed $\varepsilon \in (0, \|u_0\|_\infty)$. For any $\eta \in (0,\varepsilon)$, there exists a unique classical solution $u_{\varepsilon,\eta}$ of problem $(P_{\varepsilon,\eta})$ (see [16]). We denote by $u = u_{\varepsilon,\eta}$ for short. It follows from the comparison principle that

$$\eta \leq u(x,t) \leq \|u_0\|_\infty + \eta, \quad \forall (x,t) \in \Omega \times (0,\infty).$$

We can assume $f \in \mathcal{C}^1([0,\infty))$ if not we regularize f by a standard sequence f_n. Note that since f is nondecreasing so is f_n.
Put $u = \phi(v) = v^\gamma$, with $\gamma = 2/(1+\beta)$. Then,

$$v_t - \Delta v = \frac{\phi''}{\phi'}|\nabla v|^2 - \frac{1}{\phi'}\big(g_\varepsilon(\phi(v)) + f(\phi(v))\big). \tag{8}$$

For any $\tau \in (0,T)$, let us consider a cut-off function $\xi(t) \in \mathcal{C}^1(0,\infty)$, $0 \leq \xi(t) \leq 1$, such that

$$\xi(t) = \begin{cases} 1, & \text{on } [\tau, T], \\ 0, & \text{outside } (\frac{\tau}{2}, T + \frac{\tau}{2}), \end{cases}$$

and $|\xi_t| \leq \dfrac{1}{\tau}$.
Next, we set $w = \xi(t)|\nabla v|^2$.
If $\max\limits_{\Omega\times[0,T]} w = 0$, then $\nabla v(\tau) = 0$, so estimate (6) is trivial.
If not, there is a point $(x_0,t_0) \in \Omega \times (\tau/2, T+\tau/2)$ such that $\max\limits_{\Omega\times[0,T]} w = w(x_0,t_0)$.
Thus, we have at (x_0,t_0)

$$w_t = \nabla w = 0, \quad \Delta w \leq 0. \tag{9}$$

This implies

$$0 \leq w_t - \Delta w = \xi_t|\nabla v|^2 + 2\xi(t)\big(\nabla v.\nabla v_t - \nabla v.\nabla(\Delta v)\big) - 2\xi(t)|D^2 v|^2.$$

Or,

$$0 \leq \xi_t|\nabla v|^2 + 2\xi(t)\nabla v.\nabla(v_t - \Delta v). \tag{10}$$

A combination of (8) and (10) provides us

$$0 \leq \xi_t|\nabla v|^2 + 2\xi(t)\nabla v.\nabla\Big(\frac{\phi''}{\phi'}|\nabla v|^2 - \frac{g_\varepsilon(\phi(v)) + f(\phi(v))}{\phi'}\Big).$$

Since $\xi(t_0) > 0$, we get

$$0 \leq \frac{1}{2}\xi^{-1}\xi_t|\nabla v|^2 + \nabla v.\nabla\Big(\frac{\phi''}{\phi'}|\nabla v|^2 - \frac{g_\varepsilon(\phi(v)) + f(\phi(v))}{\phi'}\Big). \tag{11}$$

At the moment, we estimate the terms on the right hand side of (11). First of all, we have from (9) that $\nabla(|\nabla v(x_0,t_0)|^2) = 0$, so

$$\nabla v.\nabla\Big(\frac{\phi''}{\phi'}|\nabla v|^2\Big) = \nabla v.\nabla\Big(\frac{\phi''}{\phi'}\Big)|\nabla v|^2 = (\gamma-1)(2\gamma-3)v^{-2}|\nabla v|^4. \tag{12}$$

Next, we have

$$\nabla v.\nabla\left(\frac{f(\phi)}{\phi'}\right) = f'(\phi)|\nabla v|^2 - f(\phi)\frac{\phi''}{\phi'^2}|\nabla v|^2$$
$$= f'(\phi)|\nabla v|^2 - (\frac{\gamma-1}{\gamma})f(\phi)v^{-\gamma}|\nabla v|^2. \qquad (13)$$

Since $f, f' \geq 0$, and $\gamma > 1$, it follows from (13) that

$$-\nabla v.\nabla\left(\frac{f(\phi)}{\phi'}\right) \leq (\frac{\gamma-1}{\gamma})f(\phi)v^{-\gamma}|\nabla v|^2. \qquad (14)$$

After that, we have

$$\nabla v.\nabla\left(\frac{g_\varepsilon(\phi)}{\phi'}\right) = (g'_\varepsilon - g_\varepsilon\frac{\phi''}{\phi'^2})|\nabla v|^2 = \left(\psi'_\varepsilon(\phi)v^{-\beta} - (\beta + \frac{\gamma-1}{\gamma})\psi_\varepsilon(\phi)v^{-(1+\beta)\gamma}\right)|\nabla v|^2.$$

Since $\psi'_\varepsilon \geq 0$, and $0 \leq \psi_\varepsilon \leq 1$, we obtain

$$-\nabla v.\nabla\left(\frac{g(\phi)}{\phi'}\right) \leq (\beta + \frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}|\nabla v|^2. \qquad (15)$$

By inserting (12), (14) and (15) into (11), we obtain

$$(\gamma-1)(2\gamma-3)v^{-2}|\nabla v|^4 \leq \frac{1}{2}\xi^{-1}\xi_t|\nabla v|^2 + (\beta+1-\frac{1}{\gamma})v^{-(1+\beta)\gamma}|\nabla v|^2$$
$$+(\frac{\gamma-1}{\gamma})f(\phi)v^{-\gamma}|\nabla v|^2. \qquad (16)$$

Now, we multiply both sides of (16) with v^2 to get

$$(\gamma-1)(2\gamma-3)|\nabla v|^4 \leq \frac{1}{2}\xi^{-1}|\xi_t|v^2|\nabla v|^2 + (\beta+1-\frac{1}{\gamma})|\nabla v|^2 + (\frac{\gamma-1}{\gamma})f(\phi)v^{2-\gamma}|\nabla v|^2. \qquad (17)$$

by noting that $(1+\beta)\gamma = 2$.
By simplifying the term $|\nabla v|^2$ both sides of the last inequality, we obtain

$$(\gamma-1)(2\gamma-3)|\nabla v|^2 \leq \frac{1}{2}\xi^{-1}|\xi_t|v^2 + (\beta+1-\frac{1}{\gamma}) + (\frac{\gamma-1}{\gamma})f(\phi)v^{2-\gamma}.$$

Multiplying both sides of the last inequality with $\xi(t_0)$ yields

$$(\gamma-1)(2\gamma-3)\xi(t_0)|\nabla v|^2 \leq \frac{1}{2}|\xi_t|v^2 + \xi(t_0)\left((\beta+1-\frac{1}{\gamma}) + (\frac{\gamma-1}{\gamma})f(\phi)v^{2-\gamma}\right). \qquad (18)$$

Note that $w(x_0,t_0) = \xi(t_0)|\nabla v(x_0,t_0)|^2$, $0 \leq \xi(t) \leq 1$, and $|\xi_t| \leq \tau^{-1}$. It follows from (18) that there is a constant $C = C(\beta) > 0$ such that

$$w(x_0,t_0) \leq C\big(\tau^{-1}v^2 + f(\phi)v^{2-\gamma} + 1\big).$$

Since $w(x_0,t_0) \geq w(x,\tau) = |\nabla v(x,\tau)|^2$, we obtain

$$|\nabla v(x,\tau)|^2 \leq C\big(\tau^{-1}v^2 + f(\phi)v^{2-\gamma} + 1\big).$$

Moreover, we have

$$v^\gamma(x,t) = u(x,t) \leq 2\|u_0\|_\infty, \quad \text{for any } (x,t) \in \Omega \times (0,\infty).$$

Then,

$$|\nabla v(x,\tau)|^2 \leq C\left(\tau^{-1}\|u_0\|_\infty^{1+\beta} + \|u_0\|_\infty^{\beta}M_f + 1\right),$$

with $M_f = \max\limits_{0\leq s\leq \|u_0\|_\infty}\{|f(s)|\}$.
Thus,

$$|\nabla u(x,\tau)|^2 \leq C_1 u^{1-\beta}\left(\tau^{-1}\|u_0\|_\infty^{1+\beta} + \|u_0\|_\infty^\beta M_f + 1\right). \tag{19}$$

This completes the proof of i).

Now, we prove ii).
The proof of estimate (7) is similar to the one of estimate (6). We just make a slight change by considering a cut-off function, still denoted by $\xi(t) \in \mathcal{C}^1(\mathbb{R})$ such that $0 \leq \xi(t) \leq 1$, $\xi_t(t) \leq 0$, and $\xi(t) = \begin{cases} 1, & \text{if } t \leq T, \\ 0, & \text{if } t \geq 2T. \end{cases}$

Then, either $w(x,t)$ attains its maximum at the initial data, i.e:

$$\max_{(x,t)\in I\times[0,2T]} w(x,t) = w(x_0,0) = \overline{\xi}(0)|\nabla v(x_0,0)|^2 \leq \|\nabla(u_0^{\frac{1}{\gamma}})\|_\infty^2, \quad \text{for some } x_0 \in \Omega,$$

which implies

$$|\nabla u(x,\tau)|^2 \leq \gamma^2\|\nabla(u_0^{\frac{1}{\gamma}})\|_\infty^2 u^{1-\beta}(x,\tau), \quad \text{for any } x \in \Omega. \tag{20}$$

Thus, we get estimate (7) immediately.

Or there is a point $(x_0,t_0) \in \Omega \times (0,2T)$ such that

$$\max_{(x,t)\in\Omega\times[0,2T]} w(x,t) = w(x_0,t_0)$$

Then, we mimic the proof of i) to get an estimate like estimate (16).

$$(\gamma-1)(2\gamma-3)v^{-2}|\nabla v|^4 \leq \frac{1}{2}\xi^{-1}\xi_t|\nabla v|^2 + (\beta+1-\frac{1}{\gamma})v^{-(1+\beta)\gamma}|\nabla v|^2 + (\frac{\gamma-1}{\gamma})f(\phi)v^{-\gamma}|\nabla v|^2.$$

Since $\xi_t(t) \leq 0$, we get from the above inequality

$$(\gamma-1)(2\gamma-3)v^{-2}|\nabla v|^4 \leq (\beta+1-\frac{1}{\gamma})v^{-(1+\beta)\gamma}|\nabla v|^2 + (\frac{\gamma-1}{\gamma})f(\phi)v^{-\gamma}|\nabla v|^2.$$

By repeating the proof of i) after this inequality, we obtain

$$|\nabla u(x,\tau)|^2 \leq Cu^{1-\beta}(x,\tau)\left(\|u_0\|_\infty^\beta M_f + 1\right), \tag{21}$$

for some constant $C = C(\beta) > 0$.
A combination of (20) and (21) yields estimate (7). Or we complete the proof of Lemma 2.1. □

Remark 2.1. Note that gradient estimates (19) and (21) are independent of f'.

As a consequence of Lemma 2.1, we have the following regularity results.

Proposition 2.2. *Let u be a solution of $(P_{\varepsilon,\eta})$. Then, we have*

$$|u(x,t)-u(y,s)| \leq C\left(|x-y| + |t-s|^{\frac{1}{3}}\right), \quad \forall (x,t),(y,s) \in \Omega\times(\tau,\infty), \tag{22}$$

for any $\tau > 0$, where $C > 0$ depends on β, τ, $\|u_0\|_\infty$, f.
Moreover, if $\nabla(u_0^{\frac{1}{\gamma}}) \in L^\infty(\Omega)$, then inequality (22) holds for any $(x,t),(y,s) \in \Omega\times(0,\infty)$, and C depends on $\beta, f, \|u_0\|_\infty, \|\nabla(u_0^{\frac{1}{\gamma}})\|_\infty$.

Proof. We refer the proof to Proposition 14, [7] (see also [18]). ☐

It is obvious that the estimates in Lemma 2.1 are independent of ε, η. Thus, a classical argument allows us to pass to the limit as $\eta \to 0$ in order to obtain $u_{\varepsilon,\eta} \to u_\varepsilon$ (resp. $\nabla u_{\varepsilon,\eta} \to \nabla u_\varepsilon$) uniformly on $\overline{\Omega} \times (0, \infty)$, in that u_ε is a unique classical solution of the following equation:

$$(P_\varepsilon) \begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon + g_\varepsilon(u_\varepsilon) = f(u_\varepsilon) & \text{in } \Omega \times (0,\infty), \\ u_\varepsilon = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u_\varepsilon(0) = u_0 & \text{on } \Omega \end{cases}$$

Remark 2.2. The above gradient estimates also hold for u_ε.

Next, we will pass $\varepsilon \to 0$ to obtain an existence of solution of equation (1).

3. Proof of Theorem 1.1

Let u_ε be a unique solution of equation (P_ε) in $\Omega \times (0, \infty)$. Then, we show that $\{u_\varepsilon\}_{\varepsilon>0}$ is a non-decreasing sequence. Indeed, we have

$$g_{\varepsilon_1}(s) \geq g_{\varepsilon_2}(s), \quad \text{for any } 0 < \varepsilon_1 < \varepsilon_2.$$

This implies that u_{ε_1} is a sub-solution of the equation satisfied by u_{ε_2}. Therefore, the comparison principle yields

$$u_{\varepsilon_1} \leq u_{\varepsilon_2}, \quad \text{in } \Omega \times (0,\infty), \quad \forall \varepsilon_1 < \varepsilon_2,$$

so the conclusion follows. Consequently, there is a nonnegative function u such that $u_\varepsilon \downarrow u$ as $\varepsilon \to 0^+$.

Integrating equation (P_ε) on $\Omega \times (0, T)$ yields

$$\int_\Omega u_\varepsilon(x,T)dx - \int_0^T \int_{\partial\Omega} \nabla u_\varepsilon.\mathbf{n}\, d\sigma ds + \int_0^T \int_\Omega g_\varepsilon(u_\varepsilon) dxds + \int_0^T \int_\Omega f(u_\varepsilon) dxds = \int_\Omega u_\varepsilon(x,0)dx,$$

where $\mathbf{n}$ is the unit outward normal vector of $\partial\Omega$.

Since $\nabla u_\varepsilon.\mathbf{n} \leq 0$, we obtain

$$\int_0^T \int_\Omega g_\varepsilon(u_\varepsilon) dxds + \int_0^T \int_\Omega f(u_\varepsilon) dxds \leq \int_\Omega u_0(x)dx.$$

This implies that $\|g_\varepsilon(u_\varepsilon)\|_{L^1(\Omega\times(0,T))}$, and $\|f(u_\varepsilon)\|_{L^1(\Omega\times(0,T))}$ are bounded by a constant not depending on ε.

Thanks to Fatou's lemma, there is a function $\Upsilon \in L^1(\Omega \times (0, T))$ such that

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon) = \Upsilon, \quad \text{in } L^1(\Omega \times (0,T)). \tag{23}$$

Next, the monotonicity of $\{u_\varepsilon\}_{\varepsilon>0}$ deduces

$$g_\varepsilon(u_\varepsilon)(x,t) \geq g_\varepsilon(u_\varepsilon)\chi_{\{u>0\}}(x,t), \quad \forall (x,t) \in \Omega \times (0,T),$$

so

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon)(x,t) = \Upsilon(x,t) \geq u^{-\beta}\chi_{\{u>0\}}(x,t), \quad \text{for } (x,t) \in \Omega \times (0,T), \tag{24}$$

which implies that $u^{-\beta}\chi_{\{u>0\}}$ is integrable on $\Omega\times(0,T)$.
In fact, we shall prove

$$\Upsilon = u^{-\beta}\chi_{\{u>0\}}, \quad \text{in } L^1(\Omega\times(0,T)). \tag{25}$$

On the other hand, we can use a result of gradient convergence of Boccardo et al., [3] in order to obtain

$$\nabla u_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \nabla u, \quad \text{for a.e } (x,t)\in\Omega\times(0,T), \tag{26}$$

see the detail of its proof in [9].
As a result, ∇u fulfills estimate (3) for a.e $(x,t)\in\Omega\times(0,T)$, and then for any $\tau\in(0,T)$,

$$\nabla u_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \nabla u, \quad \text{in } L^r(\Omega\times(\tau,T)), \quad \forall r\in[1,\infty). \tag{27}$$

Now, it suffices to demonstrate that u satisfies equation (1) in the sense of distribution. For any $\eta>0$ fixed, we use the test function $\psi_\eta(u_\varepsilon)\phi$, for any $\phi\in\mathcal{C}_c^\infty(\Omega\times(0,T))$, to the equation satisfied by u_ε. Then, using integration by parts yields

$$\int_{Supp(\phi)}\Big(-\Psi_\eta(u_\varepsilon)\phi_t+\frac{1}{\eta}|\nabla u_\varepsilon|^2\psi'(\frac{u_\varepsilon}{\eta})\phi+\nabla u.\nabla\phi\psi_\eta(u_\varepsilon)+g_\varepsilon(u_\varepsilon)\psi_\eta(u_\varepsilon)\phi+ f(u_\varepsilon)\psi_\eta(u_\varepsilon)\phi\Big)dxds=0,$$

with $\Psi_\eta(u)=\displaystyle\int_0^u\psi_\eta(s)ds$.
Note that the role of the function $\psi_\eta(.)$ is to avoid the singularity of the term $u^{-\beta}\chi_{\{u>0\}}$ as u is near 0. Thus, there is no problem of passing to the limit as $\varepsilon\to 0$ in the indicated equation in order to get

$$\int_{Supp(\phi)}\Big(-\Psi_\eta(u)\phi_t+\frac{1}{\eta}|\nabla u|^2\psi'(\frac{u}{\eta})\phi+\nabla u.\nabla\phi\psi_\eta(u)+u^{-\beta}\psi_\eta(u)\phi+f(u)\psi_\eta(u)\phi\Big)dxds=0.$$

Next, we go to the limit as $\eta\to 0$ in the last equation.
By (26), (27), and the integration of $u^{-\beta}\chi_{\{u>0\}}$ in $\Omega\times(0,T)$, it is not difficult to verify

$$\begin{cases}\displaystyle\lim_{\eta\to 0}\int_{Supp(\phi)}\Psi_\eta(u)\phi_t dxds=\int_{Supp(\phi)}u\phi_t dxds,\\[2ex] \displaystyle\lim_{\eta\to 0}\int_{Supp(\phi)}\nabla u.\nabla\phi\psi_\eta(u)dxds=\int_{Supp(\phi)}\nabla u.\nabla\phi dxds,\\[2ex] \displaystyle\lim_{\eta\to 0}\int_{Supp(\phi)}u^{-\beta}\psi_\eta(u)\phi dxds=\int_{Supp(\phi)}u^{-\beta}\chi_{\{u>0\}}\phi dxds,\\[2ex] \displaystyle\lim_{\eta\to 0}\int_{Supp(\phi)}f(u)\psi_\eta(u)\phi dxds=\int_{Supp(\phi)}f(u)\phi dxds.\end{cases} \tag{28}$$

(Note that the assumption $f(0)=0$ is used in the final limit of (28)).
Next, we show that

$$\lim_{\eta\to 0}\int_{Supp(\phi)}\frac{1}{\eta}|\nabla u|^2\psi'(\frac{u}{\eta})\phi\ dxds=0. \tag{29}$$

In fact, since u satisfies estimate (3), we have

$$\frac{1}{\eta}\int_{Supp(\phi)}|\nabla u|^2|\psi'(\frac{u}{\eta})\phi|dxds \leq C\frac{1}{\eta}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}}u^{1-\beta}dxds$$
$$\leq 2C\int_{Supp(\phi)\cap\{\eta<u<2\eta\}}u^{-\beta}dxds,$$

where $Supp(\phi)$ means the support compact of ϕ, and the constant $C>0$ is independent of η. Since $u^{-\beta}\chi_{\{u>0\}}$ is integrable on $\Omega\times(0,T)$, we obtain

$$\lim_{\eta\to 0}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}}u^{-\beta}dxds = 0,$$

which implies the conclusion (29). A combination of (28) and (29) deduces

$$\int_{Supp(\phi)}\big(-u\phi_t+\nabla u.\nabla\phi+u^{-\beta}\chi_{\{u>0\}}\phi+f(u)\phi\big)dxds = 0. \tag{30}$$

In other words, u satisfies equation (1) in $\mathcal{D}'(\Omega\times(0,T))$.
As mentioned above, we prove (25) now. From equation (P_ε), we have

$$\int_{Supp(\phi)}\big(-u_\varepsilon\phi_t+\nabla u_\varepsilon.\nabla\phi+g_\varepsilon(u_\varepsilon)\phi+f(u_\varepsilon)\phi\big)dxds = 0,$$

for any $\phi\in\mathcal{C}_c^\infty(\Omega\times(0,T))$, $\phi\geq 0$.
Then, letting $\varepsilon\to 0$ deduces

$$\int_{Supp(\phi)}(-u\phi_t+\nabla u.\nabla\phi)\,dxds+\lim_{\varepsilon\to 0}\int_{Supp(\phi)}g_\varepsilon(u_\varepsilon)\phi\,dxds+\int_{Supp(\phi)}f(u)\phi\,dxds = 0. \tag{31}$$

By (30) and (31), we get

$$\lim_{\varepsilon\to 0}\int_{Supp(\phi)}g_\varepsilon(u_\varepsilon)\phi\,dxds = \int_{Supp(\phi)}u^{-\beta}\chi_{\{u>0\}}\phi\,dxds. \tag{32}$$

According to (23), (32) and Fatou's lemma, we obtain

$$\int_{Supp(\phi)}u^{-\beta}\chi_{\{u>0\}}\phi dxds \geq \int_{Supp(\phi)}\Upsilon\phi dxds,\quad \forall\phi\in\mathcal{C}_c^\infty(\Omega\times(0,T)),\phi\geq 0.$$

The last inequality and (24) yield conclusion (25).
The conclusion $u\in\mathcal{C}([0,T];L^1(\Omega))$ is well known, so we skip its proof and refer to the compactness result of J. Simon, [19]. Thus, u is a weak solution of equation (1).

To complete the proof of Theorem 1.1, it remains to show that u is the maximal solution of equation (1).

Proposition 3.1. *Let v be any weak solution of equation* (1) *on* $\Omega\times(0,\infty)$. *Then, we have*

$$v(x,t)\leq u(x,t),\quad \textit{for a.e } (x,t)\in\Omega\times(0,\infty).$$

In fact, we observe that

$$g_\varepsilon(v)\leq v^{-\beta}\chi_{\{v>0\}},\quad \forall\varepsilon>0.$$

Thus,

$$\partial_t v-\Delta v+g_\varepsilon(v)+f(v)\leq 0,\quad \text{in } \mathcal{D}'(\Omega\times(0,\infty)),$$

which implies that v is a sub-solution of equation (P_ε).
By the comparison principle, we get

$$v(x,t) \leq u_\varepsilon(x,t), \quad \text{for a.e } (x,t) \in \Omega \times (0,\infty).$$

Letting $\varepsilon \to 0$ yields the result.

Next, it is known that the quenching phenomenon holds for any weak solution of equation (1), see e.g., [18], [9], [7], [8]. By this fact, we show that the condition $f(0) = 0$ is a necessary condition for the existence of a solution of equation (1).

Theorem 3.2. *Assume that $f(0) > 0$. Then equation* (1) *has no nonnegative solution.*

Proof. We assume a contradiction that there is a weak solution u of equation (1). Then, we have the following result:

Lemma 3.3. *Let $0 \leq u_0 \in L^\infty(\Omega)$, and $\beta \in (0,1)$. Then, there is a finite time $T_0 > 0$ such that $u(x,t) = 0$, for any $(x,t) \in \Omega \times (T_0,\infty)$.*

We skip the proof of the above lemma, and refer its proof to [18], [9].

Thanks to this lemma, there is a finite time $T_0 > 0$ such that

$$u(x,t) = 0, \quad \forall (x,t) \in \Omega \times (T_0,\infty).$$

This implies that $f(0) = 0$. Then, we get the above theorem. □

4. The instantaneous shrinking of compact support of solutions of the Cauchy problem

4.1. Existence of a weak solution.

Proof. Let u_r be the maximal solution of the following equation

$$\begin{cases} \partial_t u - \Delta u + u^{-\beta}\chi_{\{u>0\}} + f(u) = 0 & \text{in } B_r \times (0,\infty), \\ u = 0, & \partial B_R \times (0,\infty), \\ u(x,0) = u_0(x), & \text{in } B_r, \end{cases} \tag{33}$$

see Theorem 1.1. Obviously, $\{u_r\}_{r>0}$ is a nondecreasing sequence. Moreover, the strong comparison principle deduces

$$u_r(x,t) \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}, \quad \text{for } (x,t) \in B_r \times (0,\infty). \tag{34}$$

Thus, there exists a function u such that $u_r \uparrow u$ as $r \to \infty$. We will show that u is a solution of problem (5).

By integrating both sides of (33), we get

$$\begin{cases} \|u_r(.,t)\|_{L^1(B_r)} \leq \|u_0\|_{L^1(\mathbb{R}^N)}, \quad \text{for any } t \in (0,\infty), \\ \|f(u_r)\|_{L^1(B_r\times(0,\infty))},\ \|u_r^{-\beta}\chi_{\{u_r>0\}}\|_{L^1(B_r\times(0,\infty))} \leq \|u_0\|_{L^1(\mathbb{R}^N)}. \end{cases} \tag{35}$$

It follows immediately from the Monotone Convergence Theorem that $u_r(t)$ converges to $u(t)$ in $L^1(\mathbb{R})$, and $f(u_r)$ converges to $f(u)$ in $L^1(\mathbb{R}^N \times (0,\infty))$ as $r \to \infty$, likewise

$$\begin{cases} \|u(.,t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0\|_{L^1(\mathbb{R}^N)}, \quad \text{for any } t \in (0,\infty), \\ \|f(u)\|_{L^1(\mathbb{R}^N\times(0,\infty))} \leq \|u_0\|_{L^1(\mathbb{R}^N)}. \end{cases} \tag{36}$$

On the other hand, we have from Lemma 2.1

$$|\nabla u_r(x,t)|^2 \leq C u_r^{1-\beta}(x,t)\left(t^{-1}+1\right), \quad \text{for a.e } (x,t) \in B_r \times (0,\infty), \tag{37}$$

for any $r > 0$. By using again a result of [3] (almost everywhere convergence of the gradients), there is a subsequence of $\{u_r\}_{r>0}$ (still denoted as $\{u_r\}_{r>0}$) such that

$$\nabla u_r \overset{r\to\infty}{\longrightarrow} \nabla u, \quad \text{for a.e } (x,t) \in \mathbb{R}^N \times (0,\infty).$$

Thus,

$$|\nabla u(x,t)|^2 \leq C u^{1-\beta}(x,t)\left(t^{-1}+1\right), \quad \text{for a.e } (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{38}$$

and

$$\nabla u_r \overset{r\to\infty}{\longrightarrow} \nabla u, \quad \text{in } L^q_{loc}(\mathbb{R} \times (0,\infty)), \quad \forall q \geq 1. \tag{39}$$

Now, we show that u satisfies equation (5) in the sense of distribution. Indeed, using the test function $\psi_\eta(u_r)\phi$ for the equation satisfied by u_r gives us

$$\int_{Supp(\phi)} \Big(-\Psi_\eta(u_r)\phi_t + \nabla u_r.\nabla\phi\psi_\eta(u_r) + |\nabla u_r|^2\phi\psi'_\eta(u_r) + u_r^{-\beta}\chi_{\{u_r>0\}}\psi_\eta(u_r)\phi + \\ f(u_r)\psi_\eta(u_r)\phi\Big)\, dsdx = 0, \quad \forall\phi \in \mathcal{C}^\infty_c(\mathbb{R}^N \times (0,\infty)).$$

We first take care of the term $u_r^{-\beta}\chi_{\{u_r>0\}}\psi_\eta(u_r)\phi$ in passing $r \to \infty$ and $\eta \to 0$. It is not difficult to see that $u_r^{-\beta}\chi_{\{u_r>0\}}\psi_\eta(u_r) = u_r^{-\beta}\psi_\eta(u_r)$ is bounded by $\eta^{-\beta}$. Then for any $\eta > 0$, the Dominated Convergence Theorem yields $u_r^{-\beta}\psi_\eta(u_r) \overset{r\to\infty}{\longrightarrow} u^{-\beta}\psi_\eta(u)$ in $L^1_{loc}(\mathbb{R}^N \times (0,\infty))$, which implies

$$\|u^{-\beta}\psi_\eta(u)\|_{L^1(\mathbb{R}^N\times(0,\infty))} \overset{(35)}{\leq} \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Next, using the Monotone Convergence Theorem deduces $u^{-\beta}\psi_\eta(u) \uparrow u^{-\beta}\chi_{\{u>0\}}$ in $L^1(\mathbb{R}^N \times (0,\infty))$, as $\eta \to 0$, thereby proves

$$\|u^{-\beta}\chi_{\{u>0\}}\|_{L^1(\mathbb{R}^N\times(0,\infty))} \leq \|u_0\|_{L^1(\mathbb{R}^N)}. \tag{40}$$

Thanks to (39), (35) and (34), there is no problem of passing to the limit as $r \to \infty$ in the indicated variational equation in order to get

$$\int_{Supp(\phi)} \Big(-\Psi_\eta(u)\phi_t + \nabla u.\nabla\phi\psi_\eta(u) + |\nabla u|^2\phi\psi'_\eta(u) \\ + u^{-\beta}\psi_\eta(u)\phi + f(u)\psi_\eta(u)\phi\Big)\, dsdx = 0, \quad \forall\phi \in \mathcal{C}^\infty_c(\mathbb{R}^N \times (0,\infty)).$$

By (36), (38), and (40), we can proceed similarly as in the proof of Theorem 1.1 to obtain after letting $\eta \to 0$

$$\int_{Supp(\phi)} \left(-u\phi_t + \nabla u.\nabla\phi + u^{-\beta}\chi_{\{u>0\}}\phi + f(u)\phi\right) dxds = 0, \quad \forall\phi \in \mathcal{C}^\infty_c(\mathbb{R}^N\times(0,\infty)). \tag{41}$$

Or u satisfies equation (5) in the sense of distribution.
The conclusion $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R})^N)$ is classical, so we leave it to the reader. ☐

4.2. Instantaneous shrinking of compact support of solutions.

Proof. Let u be a solution of equation (1). Since $f(u) \geq 0$, we have for some $q \in (0,1)$

$$f(u) + u^{-\beta}\chi_{\{u>0\}} \geq c_0 u^q,$$

with $c_0 = \dfrac{1}{\|u_0\|^{\beta+q}_{L^\infty(\mathbb{R}^N)}}$. This implies that u is a sub-solution of the following equation:

$$\begin{cases} \partial_t w - \Delta w + c_0 w^q = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ w(x,0) = u_0(x), & \text{in } \mathbb{R}^N. \end{cases} \tag{42}$$

Since equation (42) has a unique solution w, then the comparison principle yields

$$u(x,t) \leq w(x,t), \quad \text{in } \mathbb{R}^N \times (0,\infty).$$

Thanks to the result of Evans et al. [13], w has an instantaneous shrinking of compact support, so does u.

Thus, we obtain the conclusion. □

Acknowledgement. The author would like to thank Professor J. I. Diaz for his comments and encouragement.

References

[1] H. T. Banks, *Modeling and control in the biomedical sciences. Lecture Notes in Biomathematics*, Vol. 6. Springer-Verlag, Berlin-New York, 1975.

[2] Y. Belaud, J.I. Díaz, Abstract results on the finite extinction time property: application to a singular parabolic equation, *Journal of Convex Analysis* **17** (2010), no. 3-4, 827–860.

[3] L. Boccardo, F. Murat, Almost everywhere convergence of the gradients of solutions to Elliptic and Parabolic equations, *Nonlinear Anal. TMA*, **19** (1992), no. 6, 581-597.

[4] M. Borelli, M. Ughi, The fast diffusion equation with strong absorption: the instantaneous shrinking phenomenon, *Rend. Istit. Mat. Univ. Trieste* **26** (1994), 109–140.

[5] H. Brézis, A. Friedman, Estimates on the support of solutions of parabolic variational inequalities, *Illinois J. Math.* **20** (1976), 82–97.

[6] E. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[7] A.N. Dao, J. I. Díaz, A gradient estimate to a degenerate parabolic equation with a singular absorption term: global and local quenching phenomena, *J. Math. Anal. Appl.* **437** (2016), 445–473.

[8] A.N. Dao, J.I. Díaz, The extinction versus the blow-up: Global and non-global existence of solutions of source types of degenerate parabolic equations with a singular absorption, *Submitted.*

[9] A.N. Dao, J.I. Díaz, P. Sauvy, Quenching phenomenon of singular parabolic problem with L^1 initial data, *Elec. Jour. Diff. Equa.* **2016** (2016), no. 136, 1–16.

[10] J. Dávila, M. Montenegro, Existence and asymptotic behavior for a singular parabolic equation, *Transactions of the AMS* **357** (2004) 1801–1828.

[11] J.I. Díaz, *Nonlinear partial differential equations and free boundaries, Research Notes in Mathematics*, vol. 106, Pitman, London, 1985.

[12] J.I. Díaz, On the free boundary for quenching type parabolic problems via local energy methods, *Communications on Pure and Applied Analysis* **13** (2014), 1799–1814.

[13] L.C. Evans, B.F. Knerr, Instantaneous shrinking of the support of nonnegative solutions to certain nonlinear parabolic equations and variational inequalities, *Illinois J. of Math.* **23** (1979), no. 1, 153–166.

[14] H. Kawarada, On solutions of initial-boundary problem for $u_t = u_{xx} + \frac{1}{1-u}$, *Publ. Res. Inst. Math. Sci.* **10** (1974/75), 729–736.

[15] B. Kawohl, Remarks on Quenching, *Doc. Math., J. DMV* **1** (1996), 199–208.

[16] O. A. Ladyzenskaja, V.A. Solonnikov, N.N. Uralceva, *Linear and Quasi-Linear Equations of Parabolic Type*, AMS 23, 1988.
[17] H. A. Levine, Quenching and beyond: a survey of recent results, *GAKUTO Internat. Ser. Math. Sci. Appl.* 2 (1993) , Nonlinear mathematical problems in industry II, Gakkotosho, Tokyo, 501–512.
[18] D. Phillips, Existence of solutions of quenching problems, *Applicable Anal.* **24** (1987), 253–264.
[19] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.* **196** (1987), 65–96.
[20] W. Strieder, R. Aris, *Variational Methods Applied to Problems of Diffusion and Reaction*, Berlin: Springer-Verlag, 1973.
[21] M. Winkler, Nonuniqueness in the quenching problem, *Math. Ann.* **339** (2007), 559–597.

(Anh Nguyen Dao) FACULTY OF MATHEMATICS AND STATISTICS, TON DUC THANG UNIVERSITY, HO CHI MINH CITY, VIETNAM
E-mail address: `daonguyenanh@tdt.edu.vn`