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$\Delta_\lambda-$ Statistical convergence of order α on time scales

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ABSTRACT. In this study, we introduce the notions $\Delta_\lambda-$statistical convergence of order α (for $\alpha \in (0,1]$) and $\lambda p-$ summability of order α (for $\alpha \in (0,1]$) on an arbitrary time scale. Moreover, some relations about these notions are obtained. We define $\Delta_\lambda-$statistical boundedness of order α (for $\alpha \in (0,1]$) on a time scale. Furthermore, we give connections between $S_{\mathbb{T}}^{(\lambda,\alpha)}(b)$, $S_{\mathbb{T}}^{(\beta,\theta)}(b)$ and $S_{\mathbb{T}}(b)$ for various sequences $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$ which are determined in class Λ.

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1. Introduction

Zygmund [35] introduced the idea of statistical convergence in the first edition of his monograph published in Warsaw in 1935. Steinhaus [30] and Fast [13] and later Schoenberg [27] introduced the concept of statistical convergence, independently. Later this concept has been generalized in many directions. Fridy [14], Connor [8], Çolak [10], Maddox [20], Nuray [25], Rath and Tripathy [26], Šalát [31], Moricz [22], Miller [21] and others have studied different properties of space of statistically convergent sequences. Recently, the concept of statistical convergence has been applied to many fields of mathematics and statistics.

The statistical convergence depends on density of subsets of $\mathbb{N}$. The natural density of $K \subset \mathbb{N}$ is defined by

$$\delta(K) = \lim_{n \to \infty} \frac{1}{n} |\{k \le n : k \in K\}|,$$

where $|\{k \le n : k \in K\}|$ denotes the number of elements of K not exceeding n Niven and Zuckerman [24]. Any finite subset of $\mathbb{N}$ have zero natural density and $\delta(K^c) = 1 - \delta(K)$.

We give some properties related to the concept of natural density as follows:
$i)$ $0 \le \frac{K(n)}{n} \le 1$, so $0 \le \delta(K) \le 1$.
$ii)$ If K is the set of all non- square positive integers, i.e

$$K = \{2,3,5,6,7,8,10,11,12,13,14,15,17,18,...\},$$

then $K(n) = \frac{n - [[\sqrt{n}]]}{n}$, $\frac{K(n)}{n} = 1 - \frac{[[\sqrt{n}]]}{n}$. Obviously, $\delta(K) = 1$. This shows that $\delta(K) = 1$ does not imply that K contains all natural numbers.

A sequence $x = (x_k)$ of complex numbers is said to be statistically convergent to some number ℓ if, for every positive number ε, $\delta(\{k \in \mathbb{N}: |x_k - \ell| \geq \varepsilon\})$ has natural density zero. The number ℓ is called the statistical limit of (x_k) and written as $S - \lim x_k = \ell$, Fridy [14]. We denote the space of all statistically convergent sequences by S.

Fridy and Orhan [15] introduced the idea of statistically bounded as follows:

A sequence $x = (x_k)$ of complex numbers is said to be statistically bounded if there exists some $M \geq 0$ such that

$$\delta(|k \in \mathbb{N} : |x_k| \geq M|) = 0$$

We denote the linear space of all statistically bounded sequences by $S(b)$.

Gadjiev and Orhan [16], took the initiative to introduce the order of statistical convergence and after Çolak [9] continued this work and termed statistical convergence of order α (for $0 < \alpha \leq 1$).

Leindler [19] defined the generalized de la Vallée-Poussin mean as follows:

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, $\lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$. Throughout this study Λ denotes the set of all such sequences. $x = (x_k)$ is said to be (V, λ)−summable to a number ℓ if $t_n(x) \to \ell$ as $n \to \infty$. (V, λ)−summability reduces to $(C, 1)$ summability when $\lambda_n = n$ (see Leindler [19]). Later, the notions of λ−density and λ−statistical convergence were introduced by Mursaleen [23] as follows: Let $K \subset \mathbb{N}$, λ−density of K is defined by

$$\delta_\lambda(K) = \lim_{n \to \infty} \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : k \in K\}|.$$

$\delta_\lambda(K)$ reduces to the natural density $\delta(K)$ in case of $\lambda_n = n$ for all $n \in \mathbb{N}$ (see Mursaleen [23]). A sequence (x_k) is said to be λ−statistically convergent to a number ℓ if

$$\lim_{n \to \infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - \ell| \geq \varepsilon\}| = 0,$$

for each $\varepsilon > 0$ where $I_n = [n - \lambda_n + 1, n]$ (see Mursaleen [23]). Later, generalizing the concept of λ−statistical convergence, Çolak and Bektaş [11] introduced the concept of λ−statistical convergence of order α. Bhardwaj and Gupta [4] continued these works and defined statistically bounded of order α and λ− statistically bounded of order α (for $0 < \alpha \leq 1$).

First we will give some information about the time scale.

Time scale calculus was introduced by Hilger in his doctoral dissertion in 1988 (see Hilger [18]). However, similar ideas have been used before and go back at least to the introduction of Riemann-Stieltijes integral which unifies sums and integrals. It gives an efficient tool to unify continuous and discrete problems in one theory. During the years many studies appeared in the time scales theory and its applications. One can see basic calculus of time scales in monographs of Bohner and Peterson [5]. For a time scale $\mathbb{T}$, forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

And, the forward stepsize function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$, where $\inf \phi = \sup \mathbb{T}$ and ϕ is empty set. Let A denotes the family of all left closed and right open intervals of $\mathbb{T}$ of the form $[a, b)_\mathbb{T}$. Measure theory on time scales was first constructed by Guseinov [17]. Then further studies were made by Cabada-Vivero [6]. Let $m : A \to [0, \infty)$ be a set function on A such that

$$m\left([a, b)_\mathbb{T}\right) = b - a.$$

Then, it is known that m is a countably additive measure on A. Now, the Caratheodory extension of the set function m associated with family A is said to be the Lebesque $\Delta-$measure on $\mathbb{T}$ and is denoted by μ_Δ. In this case, it is known that If $a \in \mathbb{T} - \{\max \mathbb{T}\}$, then the single point set $\{a\}$ is $\Delta-$measurable and $\mu_\Delta(a) = \sigma(a) - a$. If $a, b \in \mathbb{T}$ and $a \leq b$, then $\mu_\Delta\left((a, b)_\mathbb{T}\right) = b - \sigma(a)$. If $a, b \in \mathbb{T} - \{\max \mathbb{T}\}$, $a \leq b$; $\mu_\Delta\left((a, b]_\mathbb{T}\right) = \sigma(b) - \sigma(a)$ and $\mu_\Delta\left([a, b]_\mathbb{T}\right) = \sigma(b) - a$ (see Deniz [12]).

Let $\mathbb{T}$ be a time scale such that $\mathbb{T} \subset [0, \infty)$ and there exits a subset $\{t_k : k_0 \in \mathbb{N}\} \subset \mathbb{T}$ with $0 = t_0 < t_1 < t_2...$ and $\lim_{k \to \infty} t_k = \infty$.

In the paper by Batit [3], the following space of continuous functions are defined:

$$\ell_\infty\left(\mathbb{T}\right) = \left\{ f/f : \mathbb{T} \to \mathbb{R}, \sup_{t \in \mathbb{T}} |f(t)| < \infty \right\},$$

$$c\left(\mathbb{T}\right) = \left\{ f/f : \mathbb{T} \to \mathbb{R}, \lim_{t \to \infty} f(t) < \infty \right\},$$

$$c_0\left(\mathbb{T}\right) = \left\{ f/f : \mathbb{T} \to \mathbb{R}, \lim_{t \to \infty} f(t) = 0 \right\}.$$

Now we will give some information about the application of time scale to statistical convergence. Seyyidoglu and Tan [28] gave some new notations such as $\Delta-$convergence, $\Delta-$Cauchy by using $\Delta-$density and investigate their relations. Later, Turan and Duman [32] continued to work on this subject. Turan and Duman [33] applied the time scales to the concept of lacunary statistical convergence. Moreover, Cichon and Yantir [7] applied the time scales for some convergent sets.

First we will give an important definition for our study.

The notions of $\lambda-$density and $\lambda-$statistical convergence on time scales have been introduced by Yilmaz $et\ al.$ [34].

Now let Ω be a $\Delta_\lambda-$ measurable subset of $\mathbb{T}$ and $0 < \alpha \leq 1$. Then, $\Omega(t, \lambda)$ is defined by

$$\Omega(t, \lambda) = \left\{ s \in [t - \lambda_t + t_0, t]_\mathbb{T} : s \in \Omega \right\},$$

for $t \in \mathbb{T}$. In this case, $(\lambda, \alpha)-$density of Ω on $\mathbb{T}$ is denoted by $\delta_\mathbb{T}^{(\lambda, \alpha)}(\Omega)$ and defined as follows:

$$\delta_\mathbb{T}^{(\lambda, \alpha)}(\Omega) = \lim_{t \to \infty} \frac{\mu_{\Delta_\lambda}\left(\Omega(t, \lambda)\right)}{\mu_{\Delta_\lambda^\alpha}\left([t - \lambda_t + t_0, t]_\mathbb{T}\right)}, \tag{1.1}$$

provided that the above limit exists. If $\lambda_t = t$ and $\alpha = 1$ in (1.1), we get classical density of Ω on $\mathbb{T}$ which is denoted by $\delta_\mathbb{T}(\Omega)$ and given as follows

$$\delta_\mathbb{T}(\Omega) = \lim_{t \to \infty} \frac{\mu_\Delta\left(\Omega(t)\right)}{\mu_\Delta\left([t_0, t]_\mathbb{T}\right)},$$

provided that the right side limit exists (see Seyyidoglu and Tan [28], Turan and Duman [32]).

2. Statistical Convergence

In this section, we introduce the notions $\Delta_\lambda-$ statistical convergence of order α (for $\alpha \in (0,1]$) and $\lambda p-$ summability of order α (for $\alpha \in (0,1]$) on an arbitrary time scale, using $\Delta_\lambda-$ statistical convergence (Yilmaz *et al.* [34] and Altin [2]) on the time scale. In addition, some relations about these notions are obtained.

Definition 2.1. Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta_\lambda-$ measurable function and $\alpha \in (0,1]$ be any real number. Then f is $\lambda-$ statistically convergent of order α on $\mathbb{T}$ to a number ℓ if

$$\lim_{t \to \infty} \frac{\mu_{\Delta_\lambda} \left(s \in [t - \lambda_t + t_0, t]_\mathbb{T} : |f(s) - \ell| \geq \varepsilon \right)}{\mu_{\Delta_\lambda}^\alpha \left([t - \lambda_t + t_0, t]_\mathbb{T} \right)} = 0, \tag{2.1}$$

for each $\varepsilon > 0$. In this case, we write $\boldsymbol{s}_\mathbb{T}^{(\lambda,\alpha)} - \lim_{t \to \infty} (f(t)) = \ell$. The set of all $\Delta_\lambda-$ statistically convergent of order α functions on $\mathbb{T}$ is denoted by $\boldsymbol{s}_\mathbb{T}^{(\lambda,\alpha)}$. Similarly, by setting $\lambda_t = t$ and $\alpha = 1$ in (2.1), it turns to classical $\Delta-$ statistical convergence on $\mathbb{T}$ as

$$\lim_{t \to \infty} \frac{\mu_\Delta \left(s \in [t_0, t]_\mathbb{T} : |f(s) - \ell| \geq \varepsilon \right)}{\mu_\Delta \left([t_0, t]_\mathbb{T} \right)} = 0,$$

provided that the above limit exists (see Seyyidoglu and Tan [28], Turan and Duman [32]). The set of all $\Delta-$ statistically convergent functions on $\mathbb{T}$ is denoted by $\boldsymbol{s}_\mathbb{T}$.

Proposition 2.1. *If* $f, g : \mathbb{T} \to \mathbb{R}$ *with* $\boldsymbol{s}_\mathbb{T}^{(\lambda,\alpha)} - \lim_{t \to \infty} f(t) = \ell_1$ *and* $\boldsymbol{s}_\mathbb{T}^{(\lambda,\alpha)} - \lim_{t \to \infty} g(t) = \ell_2$ *,* $\Delta_\lambda-$ *measurable function and the following statements hold:*

  i) $\boldsymbol{s}_\mathbb{T}^{(\lambda,\alpha)} - \lim_{t \to \infty} (f(t) + g(t)) = \ell_1 + \ell_2$,
  ii) $\boldsymbol{s}_\mathbb{T}^{(\lambda,\alpha)} - \lim_{t \to \infty} f(t) = \ell$ *and* $\boldsymbol{s}_\mathbb{T}^{(\lambda,\alpha)} - \lim_{t \to \infty} (cf(t)) = c\ell \ (c \in \mathbb{R})$.

Theorem 2.2. *Let* $0 < \alpha \leq 1$, $\boldsymbol{s}_\mathbb{T} \subseteq \boldsymbol{s}_\mathbb{T}^{(\lambda,\alpha)}$ *if*

$$\lim_{t \to \infty} \inf \frac{\mu_{\Delta_\lambda}^\alpha \left([t - \lambda_t + t_0, t]_\mathbb{T} \right)}{\mu_\Delta \left([t_0, t]_\mathbb{T} \right)} > 0 \tag{2.2}$$

Proof. For given $\varepsilon > 0$ we have

$$\mu_\Delta \left(s \in [t_0, t]_\mathbb{T} : |f(s) - \ell| \geq \varepsilon \right) \supset \mu_{\Delta_\lambda} \left(s \in [t - \lambda_t + t_0, t]_\mathbb{T} : |f(s) - \ell| \geq \varepsilon \right).$$

Therefore,

$$\frac{\mu_\Delta \left(s \in [t_0, t]_\mathbb{T} : |f(s) - \ell| \geq \varepsilon \right)}{\mu_\Delta \left([t_0, t]_\mathbb{T} \right)} \geq \frac{\mu_{\Delta_\lambda} \left(s \in [t - \lambda_t + t_0, t]_\mathbb{T} : |f(s) - \ell| \geq \varepsilon \right)}{\mu_\Delta \left([t_0, t]_\mathbb{T} \right)}$$

$$= \frac{\mu_{\Delta_\lambda}^\alpha \left([t - \lambda_t + t_0, t]_\mathbb{T} \right)}{\mu_\Delta \left([t_0, t]_\mathbb{T} \right)} \frac{1}{\mu_{\Delta_\lambda}^\alpha \left([t - \lambda_t + t_0, t]_\mathbb{T} \right)} \mu_{\Delta_\lambda} \left(s \in [t - \lambda_t + t_0, t]_\mathbb{T} : |f(s) - \ell| \geq \varepsilon \right).$$

Hence by using (2.2) and taking the limit as $t \to \infty$, we get $f(s) \to \ell (\boldsymbol{s}_\mathbb{T})$ implies $f(s) \to \ell \left(\boldsymbol{s}_\mathbb{T}^{(\lambda,\alpha)} \right)$. $\qquad \square$

Definition 2.2. Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta_\lambda-$ measurable function, $\lambda \in \Lambda$, $0 < \alpha \leq 1$ and $0 < p < \infty$. We say that f is strongly $\lambda p-$ Cesàro summable of order α on $\mathbb{T}$ if

there exists some $\ell \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{1}{\mu_{\Delta_\lambda}^\alpha \left([t - \lambda_t + t_0, t]_\mathbb{T}\right)} \int\limits_{[t-\lambda_t+t_0,t]_\mathbb{T}} |f(s) - \ell|^p \Delta s = 0.$$

In this case we write $[W, \lambda, p, \alpha]_\mathbb{T} - \lim f(s) = \ell$. The set of all strongly $\lambda p-$

summable functions of order α on $\mathbb{T}$ will be denoted by $[W, \lambda, \alpha]_\mathbb{T}$.

Lemma 2.3. [See Yilmaz et al. [34]] *Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta_\lambda-$ measurable function and let*

$$\Omega(t, \lambda) = \{s \in [t - \lambda_t + t_0, t]_\mathbb{T} : |f(s) - \ell| \geq \varepsilon\}$$

for $\varepsilon > 0$. In this case, we have

$$\mu_{\Delta_\lambda}(\Omega(t, \lambda)) \leq \frac{1}{\varepsilon} \int\limits_{\Omega(t,\lambda)} |f(s) - \ell| \Delta s \leq \frac{1}{\varepsilon} \int\limits_{[t-\lambda_t+t_0,t]_\mathbb{T}} |f(s) - \ell| \Delta s.$$

Theorem 2.4. *Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta_\lambda-$ measurable function, $\lambda \in \Lambda$, $\ell \in \mathbb{R}$, $\alpha \in (0, 1]$ and $0 < p < \infty$. If f is strongly $\lambda p-$ summable of order α to ℓ, then $s_\mathbb{T}^{(\lambda,\alpha)} - \lim_{t\to\infty} (f(t)) = \ell$.*

Proof. Let f is strongly $\lambda p-$ summable of order α to ℓ. For given $\varepsilon > 0$, let $\Omega(t, \lambda) = \{s \in [t - \lambda_t + t_0, t]_\mathbb{T} : |f(s) - \ell| \geq \varepsilon\}$. Then, it follows from Lemma 2.3 that

$$\varepsilon^p \mu_{\Delta_\lambda}(\Omega(t, \lambda)) \leq \int\limits_{[t-\lambda_t+t_0,t]_\mathbb{T}} |f(s) - \ell|^p \Delta s.$$

Dividing both sides of the last equality by $\mu_{\Delta_\lambda}^\alpha \left([t - \lambda_t + t_0, t]_\mathbb{T}\right)$ and taking limit as $t \to \infty$, we obtain that

$$\lim_{t\to\infty} \frac{\mu_{\Delta_\lambda}(\Omega(t, \lambda))}{\mu_{\Delta_\lambda}^\alpha \left([t - \lambda_t + t_0, t]_\mathbb{T}\right)} \leq \frac{1}{\varepsilon^p} \lim_{t\to\infty} \frac{1}{\mu_{\Delta_\lambda}^\alpha \left([t - \lambda_t + t_0, t]_\mathbb{T}\right)} \int\limits_{[t-\lambda_t+t_0,t]_\mathbb{T}} |f(s) - \ell|^p \Delta s = 0,$$

which yields that $s_\mathbb{T}^{(\lambda,\alpha)} - \lim_{t\to\infty} (f(t)) = \ell$. □

Theorem 2.5. *Let $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$ be two sequences in Λ such that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$ and $0 < \alpha \leq \theta \leq 1$.*
i) If

$$\lim_{t\to\infty} \inf \frac{\mu_{\Delta_\lambda}^\alpha \left([t - \lambda_t + t_0, t]_\mathbb{T}\right)}{\mu_{\Delta_\beta}^\theta \left([t - \beta_t + t_0, t]_\mathbb{T}\right)} > 0 \tag{2.5}$$

then $s_\mathbb{T}^{(\beta,\theta)} \subseteq s_\mathbb{T}^{(\lambda,\alpha)}$.
ii) If

$$\lim_{t\to\infty} \frac{\mu_{\Delta_\lambda}^\alpha \left([t - \lambda_t + t_0, t]_\mathbb{T}\right)}{\mu_{\Delta_\beta}^\theta \left([t - \beta_t + t_0, t]_\mathbb{T}\right)} = 1 \ \text{and} \ \lim_{t\to\infty} \frac{\mu_{\Delta_\beta} \left([t - \lambda_t + t_0, t]_\mathbb{T}\right)}{\mu_{\Delta_\beta}^\theta \left([t - \beta_t + t_0, t]_\mathbb{T}\right)} = 1 \tag{2.6}$$

then $s_\mathbb{T}^{(\lambda,\alpha)} \subseteq s_\mathbb{T}^{(\beta,\theta)}$.

Proof. i) Suppose that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$ and let (2.5) be satisfied. Then $I_t \subset J_t$ and so that for $\varepsilon > 0$, we have

$$\mu_{\Delta_\beta}\left(s \in [t-\beta_t+t_0,t]_\mathbb{T} : |f(s)-\ell| \geq \varepsilon\right) \geq \mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_\mathbb{T} : |f(s)-\ell| \geq \varepsilon\right)$$

Therefore, we have

$$\frac{\mu_{\Delta_\beta}\left(s \in [t-j_t+t_0,t]_\mathbb{T} : |f(s)-\ell| \geq \varepsilon\right)}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_\mathbb{T}\right)} \geq \frac{\mu_{\Delta_\lambda}^\alpha\left([t-\lambda_t+t_0,t]_\mathbb{T}\right)}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_\mathbb{T}\right)} \frac{1}{\mu_{\Delta_\lambda}^\alpha\left([t-\lambda_t+t_0,t]_\mathbb{T}\right)}$$

$$\times \;\mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_\mathbb{T} : |f(s)-\ell| \geq \varepsilon\right).$$

for all $t \in \mathbb{T}$, where $J_t = [t-\beta_t+t_0,t]$. Hence by using (2.5) and taking the limit as $t \to \infty$, we get $s_\mathbb{T}^{(\beta,\theta)} \subseteq s_\mathbb{T}^{(\lambda,\alpha)}$.

ii) Let f be a $\Delta_\lambda-$ measurable function and, $s_\mathbb{T}^{(\lambda,\alpha)} - \lim f(s) = \ell$ and (2.6) be satisfied. Since $I_t \subset J_t$ and all $t \in \mathbb{T}$, we can write

$$\frac{\mu_{\Delta_\beta}\left(s \in [t-\beta_t+t_0,t]_\mathbb{T} : |f(s)-\ell| \geq \varepsilon\right)}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_\mathbb{T}\right)}$$

$$= \frac{\mu_{\Delta_\beta}\left(t-\beta_t+t_0 \leq s \leq t-\lambda_t : |f(s)-\ell| \geq \varepsilon\right)}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_\mathbb{T}\right)}$$

$$+ \frac{\mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_\mathbb{T} : |f(s)-\ell| \geq \varepsilon\right)}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_\mathbb{T}\right)}$$

$$\leq \frac{\mu_{\Delta_\beta}^\theta\left(s \in [t-\beta_t+t_0,t]_\mathbb{T}\right) - \mu_{\Delta_\lambda}^\alpha\left(s \in [t-\lambda_t+t_0,t]_\mathbb{T}\right)}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_\mathbb{T}\right)}$$

$$+ \frac{\mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_\mathbb{T} : |f(s)-\ell| \geq \varepsilon\right)}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_\mathbb{T}\right)}$$

$$\leq \left(1 - \frac{\mu_{\Delta_\lambda}^\alpha\left([t-\lambda_t+t_0,t]_\mathbb{T}\right)}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_\mathbb{T}\right)}\right)$$

$$+ \frac{\mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_\mathbb{T} : |f(s)-\ell| \geq \varepsilon\right)}{\mu_{\Delta_\lambda}^\alpha\left([t-\lambda_t+t_0,t]_\mathbb{T}\right)}$$

for all $t \in \mathbb{T}$. Since

$$\lim_{t \to \infty} \frac{\mu_{\Delta_\lambda}^\alpha\left([t-\lambda_t+t_0,t]_\mathbb{T}\right)}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_\mathbb{T}\right)} = 1 \text{ and } \lim_{t \to \infty} \frac{\mu_{\Delta_\beta}\left([t-\beta_t+t_0,t]_\mathbb{T}\right)}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_\mathbb{T}\right)} = 1$$

by ii), the term in above inequality tends to 0 as $t \to \infty$. Furthermore, since $s_\mathbb{T}^{(\lambda,\alpha)} - \lim f(s) = \ell$, the second term of the right hand side of the above inequailty goes to 0 as $t \to \infty$.

$$\frac{\mu_{\Delta_\beta}\left(t-\beta_t+t_0 \leq s \leq t : |f(s)-\ell| \geq \varepsilon\right)}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_\mathbb{T}\right)} \to 0$$

as $t \to \infty$. Therefore $s_\mathbb{T}^{(\lambda,\alpha)} \subseteq s_\mathbb{T}^{(\beta,\theta)}$. $\square$

From Theorem 2.5, we have the following result.

Corollary 2.6. *Let $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$ be two sequences in Λ such that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$ and $0 < \alpha \leq \theta \leq 1$. If (2.6) holds, then $s_\mathbb{T}^{(\lambda,\alpha)} = s_\mathbb{T}^{(\beta,\theta)}$.*

Theorem 2.7. Let $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$ be two sequences in Λ such that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$ and $0 < \alpha \leq \theta \leq 1$. Then we get:
i) If (2.5) holds then $[W, \beta, \alpha]_{\mathbb{T}} \subseteq [W, \lambda, \theta]_{\mathbb{T}}$.
ii) If (2.6) holds then $\ell_\infty(\mathbb{T}) \cap [W, \lambda, \alpha]_{\mathbb{T}} \subseteq [W, \beta, \theta]_{\mathbb{T}}$.

Proof. i) Suppose that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$. Then $I_t \subset J_t$ for all $t \in \mathbb{T}$ so that we may write

$$\frac{1}{\mu_{\Delta_\beta^\theta}([t - \beta_t + t_0, t]_{\mathbb{T}})} \int\limits_{[t-\beta_t+t_0,t]_{\mathbb{T}}} |f(s) - \ell| \Delta s$$

$$\geq \frac{1}{\mu_{\Delta_\lambda^\alpha}([t - \lambda_t + t_0, t]_{\mathbb{T}})} \int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} |f(s) - \ell| \Delta s$$

for all $t \in \mathbb{T}$. This gives that

$$\frac{1}{\mu_{\Delta_\beta}^\theta([t - \beta_t + t_0, t]_{\mathbb{T}})} \int\limits_{[t-\beta_t+t_0,t]_{\mathbb{T}}} |f(s) - \ell| \Delta s$$

$$\geq \frac{\mu_{\Delta_\lambda}^\alpha([t - \lambda_t + t_0, t]_{\mathbb{T}})}{\mu_{\Delta_\beta}^\theta([t - \beta_t + t_0, t]_{\mathbb{T}})} \frac{1}{\mu_{\Delta_\lambda}^\alpha([t - \lambda_t + t_0, t]_{\mathbb{T}})} \int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} |f(s) - \ell| \Delta s.$$

Then taking limit $t \to \infty$ in the last inequality and using (2.5) we obtain $[W, \beta, \alpha]_{\mathbb{T}} \subseteq [W, \lambda, \theta]_{\mathbb{T}}$.

ii) Let $f \in \ell_\infty(\mathbb{T}) \cap [W, \lambda, \alpha]_{\mathbb{T}}$ and suppose that (2.6) holds. Since $f \in \ell_\infty(\mathbb{T})$ then there exits some a positive number M such that $|f(s)| \leq M$ for all $s \in \mathbb{T}$ and also now, since $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ and so that $\frac{1}{\mu_{\Delta_{\beta(t)}}} \leq \frac{1}{\mu_{\Delta_{\lambda(t)}}}$ and $I_t \subset J_t$ for all $t \in \mathbb{T}$, we may write

$$\frac{1}{\mu_{\Delta_\beta}^\theta([t - \beta_t + t_0, t]_{\mathbb{T}})} \int\limits_{[t-\beta_t+t_0,t]_{\mathbb{T}}} |f(s) - \ell| \Delta s$$

$$\leq \frac{1}{\mu_{\Delta_\beta}^\theta([t - \beta_t + t_0, t]_{\mathbb{T}})} \int\limits_{J_t/I_t} |f(s) - \ell| \Delta s + \frac{1}{\mu_{\Delta_\beta}^\theta([t - \beta_t + t_0, t]_{\mathbb{T}})} \int\limits_{I_t} |f(s) - \ell| \Delta s$$

$$\leq \left(\frac{\mu_{\Delta_\beta}^\theta([t - \beta_t + t_0, t]_{\mathbb{T}}) - \mu_{\Delta_\lambda}^\alpha([t - \lambda_t + t_0, t]_{\mathbb{T}})}{\mu_{\Delta_\beta}^\theta([t - \beta_t + t_0, t]_{\mathbb{T}})} \right) M$$

$$+ \frac{1}{\mu_{\Delta_\lambda}^\alpha([t - \lambda_t + t_0, t]_{\mathbb{T}})} \int\limits_{I_t} |f(s) - \ell| \Delta s$$

for all $t \in \mathbb{T}$. Since $\lim\limits_{t \to \infty} \frac{\mu_{\Delta_\lambda}^\alpha([t-\lambda_t+t_0,t]_{\mathbb{T}})}{\mu_{\Delta_\beta}^\theta([t-\beta_t+t_0,t]_{\mathbb{T}})} = 1$ by (2.6) the first term and since $f \in [W, \lambda, \alpha]_{\mathbb{T}}$ the second term of right hand side of above inequality tend to 0 as $t \to \infty$. This implies that $\ell_\infty(\mathbb{T}) \cap [W, \lambda, \alpha]_{\mathbb{T}} \subseteq [W, \beta, \theta]_{\mathbb{T}}$ and so that $\ell_\infty(\mathbb{T}) \cap [W, \lambda, \alpha]_{\mathbb{T}} \subseteq \ell_\infty(\mathbb{T}) \cap [W, \beta, \theta]_{\mathbb{T}}$. $\qquad \square$

From Theorem 2.7 we have the following result.

Corollary 2.8. *Let $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$ be two sequences in Λ such that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$ and $0 < \alpha \leq \theta \leq 1$. If (2.6) holds then $\ell_\infty(\mathbb{T}) \cap [W, \lambda, \alpha]_{\mathbb{T}} = \ell_\infty(\mathbb{T}) \cap [W, \beta, \theta]_{\mathbb{T}}$.*

3. Statistical boundedness

In this section, we use the concepts of statistical boundedness on time scales defined by Seyyidoğlu and Tan [29] and $\Delta_\lambda-$statistical boundedness defined by Altin *et al.* [1], we define $\Delta_\lambda-$statistical boundedness of order α (for $0 < \alpha \leq 1$) on time scale. Furthermore, the connections between $S_{\mathbb{T}}^{(\lambda,\alpha)}(b)$, $\boldsymbol{S}_{\mathbb{T}}^{(\beta,\theta)}(b)$ and $\boldsymbol{S}_{\mathbb{T}}(b)$ for various sequences are determined class Λ, according to in $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$.

Definition 3.1. Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta_\lambda-$measurable function and $\alpha \in (0, 1]$. f is $\Delta_\lambda-$statistically bounded of order α on $\mathbb{T}$ if there exists some $M > 0$ such that

$$\lim_{t\to\infty} \frac{\mu_{\Delta_\lambda}(s \in [t - \lambda_t + t_0, t]_{\mathbb{T}} : |f(s)| \geq M)}{\mu_{\Delta_\lambda}^\alpha([t - \lambda_t + t_0, t]_{\mathbb{T}})} = 0, \tag{3.1}$$

for $t \in \mathbb{T}$. The set of all $\Delta_\lambda-$statistically bounded functions of order α on $\mathbb{T}$ is denoted by $S_{\mathbb{T}}^{(\lambda,\alpha)}(b)$. If $\alpha = 1$ and $\lambda_t = t$, then we obtain $S_{\mathbb{T}}(b)$ which is the set of all $\Delta-$statistically bounded functions [29].

Theorem 3.1. *Let $\mu_{\Delta_{\lambda(t)}}$, $\mu_{\Delta_{\beta(t)}} \in \Lambda$ such that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$ and $0 < \alpha \leq \theta \leq 1$.*
i) If

$$\lim_{t\to\infty} \inf \frac{\mu_{\Delta_{\lambda(t)}}^\alpha([t - \lambda_t + t_0, t]_{\mathbb{T}})}{\mu_{\Delta_{\beta(t)}}^\theta([t - \beta_t + t_0, t]_{\mathbb{T}})} > 0, \tag{3.2}$$

then $\boldsymbol{S}_{\mathbb{T}}^{(\beta,\theta)}(b) \subset \boldsymbol{S}_{\mathbb{T}}^{(\lambda,\alpha)}(b)$.
ii) If

$$\lim_{t\to\infty} \frac{\mu_{\Delta_{\lambda(t)}}^\alpha([t - \lambda_t + t_0, t]_{\mathbb{T}})}{\mu_{\Delta_{\beta(t)}}^\theta([t - \beta_t + t_0, t]_{\mathbb{T}})} = 1, \tag{3.3}$$

then $\boldsymbol{S}_{\mathbb{T}}^{(\lambda,\alpha)}(b) \subseteq \boldsymbol{S}_{\mathbb{T}}^{(\beta,\theta)}(b)$.
iii) If

$$\lim_{t\to\infty} \frac{\mu_{\Delta_{\beta(t)}}([t - \beta_t + t_0, t]_{\mathbb{T}})}{\mu_{\Delta_{\beta(t)}}^\theta([t - \beta_t + t_0, t]_{\mathbb{T}})} = 1,$$

then $\boldsymbol{S}_{\mathbb{T}}^{(\lambda,\alpha)}(b) = \boldsymbol{S}_{\mathbb{T}}^{(\beta,\theta)}(b)$.

Proof. i) Suppose that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$ and (3.3) is satisfied. Then, $I_t \subset J_t$ and so for $\varepsilon > 0$, we have

$$\mu_{\Delta_{\beta(t)}}(\{s \in [t - \beta_t + t_0, t]_{\mathbb{T}} : |f(s)| \geq M)$$
$$\geq \mu_{\Delta_{\lambda(t)}}(\{s \in [t - \lambda_t + t_0, t]_{\mathbb{T}} : |f(s)| \geq M\}).$$

Therefore,

$$\frac{\mu_{\Delta_{\beta(t)}}\left(\{s \in [t-j_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\}\right)}{\mu^\theta_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}$$

$$\geq \frac{\mu^\alpha_{\Delta_{\lambda(t)}}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu^\theta_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)\mu^\alpha_{\Delta_{\lambda(t)}}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}$$

$$\times \mu_{\Delta_{\lambda(t)}}\left(\{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\}\right),$$

for $\forall t \in \mathbb{T}$ where $J_t = [t-\beta_t+t_0,t]$. Hence by using (3.3) and taking the limit as $t \to \infty$, we get $\boldsymbol{S}_{\mathbb{T}}^{(\beta,\theta)}(b) \subseteq \boldsymbol{S}_{\mathbb{T}}^{(\lambda,\alpha)}(b)$.

ii) Let f be a $\Delta_\lambda-$ measurable and $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)} - \lim f(s) = \ell$. Since $I_t \subset J_t$, we can write

$$\frac{\mu_{\Delta_{\beta(t)}}\left(\{s \in [t-\beta_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\}\right)}{\mu^\theta_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}$$

$$= \frac{\mu_{\Delta_{\beta(t)}}\left(\{t-\beta_t+t_0 \leq s \leq t-\lambda_t : |f(s)| \geq M\}\right)}{\mu^\theta_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}$$

$$+ \frac{\mu_{\Delta_{\lambda(t)}}\left(\{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\}\right)}{\mu^\theta_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}$$

$$\leq \frac{\mu^\theta_{\Delta_{\beta(t)}}\left(s \in [t-\beta_t+t_0,t]_{\mathbb{T}}\right) - \mu^\alpha_{\Delta_{\lambda(t)}}\left(s \in [t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu^\theta_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}$$

$$+ \frac{\mu_{\Delta_{\lambda(t)}}\left(\{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\}\right)}{\mu^\theta_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}$$

$$\leq \left(1 - \frac{\mu^\alpha_{\Delta_{\lambda(t)}}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu^\theta_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}\right)$$

$$+ \frac{\mu_{\Delta_{\lambda(t)}}\left(\{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\}\right)}{\mu^\alpha_{\Delta_{\lambda(t)}}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}$$

for all $t \in \mathbb{T}$. Since $\lim_{t\to\infty} \frac{\mu^\alpha_{\Delta_{\lambda(t)}}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu^\theta_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} = 1$ by ii), the term in above inequality tends to 0 as $t \to \infty$. Furthermore, since $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)} - \lim f(s) = \ell$, the second term of the right hand side of the above inequality goes to 0 as $t \to \infty$. Therefore $\boldsymbol{S}_{\mathbb{T}}^{(\lambda,\alpha)}(b) \subseteq \boldsymbol{S}_{\mathbb{T}}^{(\beta,\theta)}(b)$.

iii) The proof is clear. $\qquad\square$

Theorem 3.2. *Let $\mu_{\Delta_{\lambda(t)}} \in \Lambda$ for all $t \in \mathbb{T}$ and $0 < \alpha \leq 1$.*
i) If

$$\lim_{t\to\infty} \inf \frac{\mu^\alpha_{\Delta_{\lambda(t)}}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)} > 0,$$

then $S_{\mathbb{T}}(b) \subset \boldsymbol{S}_{\mathbb{T}}^{(\lambda,\alpha)}(b)$.

ii) If

$$\lim_{n \to \infty} \frac{\mu_{\Delta_{\lambda(t)}}^{\alpha}\left([t - \lambda_t + t_0, t]_{\mathbb{T}}\right)}{\mu_{\Delta}\left([t_0, t]_{\mathbb{T}}\right)} = 1,$$

then $\boldsymbol{S}_{\mathbb{T}}^{(\lambda,\alpha)}(b) \subset \boldsymbol{S}_{\mathbb{T}}(b)$ *and hence* $\boldsymbol{S}_{\mathbb{T}}^{(\lambda,\alpha)}(b) = \boldsymbol{S}_{\mathbb{T}}(b)$.

Proof. The proof is obvious. □

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Δ-wavy probability distributions and Potts model

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Abstract. We define the wavy probability distributions on a subset and Δ-wavy probability distributions — two generalizations of the wavy probability distributions. A classification on the Δ-waviness is given. For the Δ-wavy probability distributions having normalization constant, we give a formula for this constant, to compute this constant. We show that the Potts model is a Δ-wavy probability distribution, where Δ is a partition which will be specified. For the normalization constant of Potts model, we give formulas and bounds. As to the formulas for this constant, we give two general formulas, one of them is simple while the other is more complicated, and based on independent sets, a formula for the Potts model on connected separable graphs — closed-form expressions are then obtained in several cases —, and a formula for the Potts model on graphs with a vertex of degree 2 — a recurrence relation is then obtained for the normalization constant of Potts model on $\mathcal{C}_n$, the cycle graph with n vertices; the normalization constant of Ising model on $\mathcal{C}_n$ is computed using this relation. As to the bounds for the normalization constant, we present two ways to obtain such bounds; we illustrate these ways giving a general lower bound, and a lower bound and an upper one when the model is the Potts model on $\mathcal{G}_{n,n}$, the square grid graph, $n = 6k$, $k \geq 1$ — two upper bounds for the free energy per site of this model are then obtained, one of them being in the limit. A sampling method for the Δ-wavy probability distributions is given and, as a result, a sampling method for the Potts is given. This method — that for the Potts model too — has two steps, Step 1 and Step 2, when $|\Delta| > 1$ and one step, Step 2 only, when $|\Delta| = 1$. For the Potts model, Step 1 is, in general, difficult. As to Step 2, for the Potts model too, using the Gibbs sampler in a generalized sense, we obtain an exact (not approximate) sampling method having $p + 1$ steps ($p + 1$ substeps of Step 2), where $p = |I|$, I is an independent set, better, a maximal independent set, best, a maximum independent set — for the Potts model on $\mathcal{G}_{n_1,n_2,...,n_d}$, the d-dimensional grid graph, $d \geq 1$, $n_1, n_2, ..., n_d \geq 1$, $n_1 n_2 ... n_d \geq 2$, we obtain an exact sampling method for half or half+1 vertices.

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1. Δ-wavy probability distributions

In this section, we present some basic things on nonnegative matrices, products of stochastic matrices, the hybrid Metropolis-Hastings chain(s), the Gibbs sampler(s) in a generalized sense, the wavy probability distributions, the wavy probability distributions on subsets, and the Δ-wavy probability distributions. The notions of wavy probability distribution on a subset and of Δ-wavy probability distribution together with the things concerning them are new — the most important things obtained are

for the Δ-wavy probability distributions: 1) a formula for the normalization constant for the Δ-wavy probability distributions which have normalization constant; 2) a sampling method. Moreover, two results, one on our hybrid Metropolis-Hastings chain and the other on our Gibbs sampler in a generalized sense, are improved and a classification on the Δ-waviness is given.

Set
$$\mathrm{Par}\,(E) = \{\Delta \mid \Delta \text{ is a partition of } E\}\,,$$
where E is a nonempty set. We shall agree that the partitions do not contain the empty set. $(E) \in \mathrm{Par}(E)\,;$ (E) is the improper (degenerate) partition of E.

Definition 1.1. Let $\Delta_1, \Delta_2 \in \mathrm{Par}(E)\,.$ We say that Δ_1 *is finer than* Δ_2 if $\forall V \in \Delta_1$, $\exists W \in \Delta_2$ such that $V \subseteq W$.

Write $\Delta_1 \preceq \Delta_2$ when Δ_1 is finer than Δ_2.

In this article, a vector is a row vector and a stochastic matrix is a row stochastic matrix.

The entry (i, j) of a matrix Z will be denoted Z_{ij} or, if confusion can arise, $Z_{i \to j}$.

Set
$$\langle m \rangle = \{1, 2, ..., m\}\,(m \in \mathbb{N},\ m \geq 1),$$
$$\langle\langle m \rangle\rangle = \{0, 1, ..., m\}\ (m \in \mathbb{N}),$$
$$N_{m,n} = \{P \,|\, P \text{ is a nonnegative } m \times n \text{ matrix}\}\,,$$
$$S_{m,n} = \{P \,|\, P \text{ is a stochastic } m \times n \text{ matrix}\}\,,$$
$$N_n = N_{n,n},$$
$$S_n = S_{n,n}.$$
Let $P = (P_{ij}) \in N_{m,n}.$ Let $\emptyset \neq U \subseteq \langle m \rangle$ and $\emptyset \neq V \subseteq \langle n \rangle$. Set the matrices
$$P_U = (P_{ij})_{i \in U, j \in \langle n \rangle}\,,\ P^V = (P_{ij})_{i \in \langle m \rangle, j \in V}\,,\ \text{and}\ P_U^V = (P_{ij})_{i \in U, j \in V}\,.$$

Set
$$(\{i\})_{i \in \{s_1, s_2, ..., s_t\}} = (\{s_1\}, \{s_2\}, ..., \{s_t\})\,;$$
$$(\{i\})_{i \in \{s_1, s_2, ..., s_t\}} \in \mathrm{Par}\,(\{s_1, s_2, ..., s_t\})\ (t \geq 1).$$
E.g.,
$$(\{i\})_{i \in \langle\langle n \rangle\rangle} = (\{0\}, \{1\}, ..., \{n\})\,.$$

Definition 1.2. Let $P \in N_{m,n}.$ We say that P is a *generalized stochastic matrix* if $\exists a \geq 0,$ $\exists Q \in S_{m,n}$ such that $P = aQ$.

Definition 1.3. ([13].) Let $P \in N_{m,n}.$ Let $\Delta \in \mathrm{Par}(\langle m \rangle)$ and $\Sigma \in \mathrm{Par}(\langle n \rangle)$. We say that P is a $[\Delta]$-*stable matrix on* Σ if P_K^L is a generalized stochastic matrix, $\forall K \in \Delta, \forall L \in \Sigma$. In particular, a $[\Delta]$-stable matrix on $(\{i\})_{i \in \langle n \rangle}$ is called $[\Delta]$-*stable* for short.

Definition 1.4. ([13].) Let $P \in N_{m,n}.$ Let $\Delta \in \mathrm{Par}(\langle m \rangle)$ and $\Sigma \in \mathrm{Par}(\langle n \rangle)$. We say that P is a Δ-*stable matrix on* Σ if Δ is the least fine partition for which P is a $[\Delta]$-stable matrix on Σ. In particular, a Δ-stable matrix on $(\{i\})_{i \in \langle n \rangle}$ is called Δ-*stable* while a $(\langle m \rangle)$-stable matrix on Σ is called *stable on* Σ for short. A stable matrix on $(\{i\})_{i \in \langle n \rangle}$ is called *stable* for short.

Let $\Delta_1 \in \mathrm{Par}(\langle m \rangle)$ and $\Delta_2 \in \mathrm{Par}(\langle n \rangle)$. Set (see [13] for G_{Δ_1, Δ_2} and [14] for $\overline{G}_{\Delta_1, \Delta_2}$)

$$G_{\Delta_1, \Delta_2} = \{P \mid P \in S_{m,n} \text{ and } P \text{ is a } [\Delta_1]\text{-stable matrix on } \Delta_2\}$$

and

$$\overline{G}_{\Delta_1, \Delta_2} = \{P \mid P \in N_{m,n} \text{ and } P \text{ is a } [\Delta_1]\text{-stable matrix on } \Delta_2\}.$$

When we study or even when we construct products of nonnegative matrices (in particular, products of stochastic matrices) using G_{Δ_1, Δ_2} or $\overline{G}_{\Delta_1, \Delta_2}$, we shall refer this as the *G method*. *G* comes from the verb *to group* and its derivatives.

Below we give an important result — a beautiful result — on products of stochastic matrices.

Theorem 1.1. ([13].) *Let* $P_1 \in G_{(\langle m_1 \rangle), \Delta_2} \subseteq S_{m_1, m_2}$, $P_2 \in G_{\Delta_2, \Delta_3} \subseteq S_{m_2, m_3}, ...,$ $P_{n-1} \in G_{\Delta_{n-1}, \Delta_n} \subseteq S_{m_{n-1}, m_n}$, $P_n \in G_{\Delta_n, (\{i\})_{i \in \langle m_{n+1} \rangle}} \subseteq S_{m_n, m_{n+1}}$. *Then*

$$P_1 P_2 ... P_n$$

is a stable matrix (i.e., a matrix with identical rows, see Definition 1.4).

Proof. See [13]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Definition 1.5. (See, e.g., [21, p. 80].) Let $P \in N_{m,n}$. We say that P is a *row-allowable matrix* if it has at least one positive entry in each row.

Let $P \in N_{m,n}$. Set

$$\overline{P} = \left(\overline{P}_{ij}\right) \in N_{m,n}, \ \overline{P}_{ij} = \begin{cases} 1 \text{ if } P_{ij} > 0, \\ 0 \text{ if } P_{ij} = 0, \end{cases}$$

$\forall i \in \langle m \rangle, \forall j \in \langle n \rangle$. We call $\overline{P}$ the *incidence matrix of* P (see, *e.g.*, [8, p. 222]).

In this article, the transpose of a vector x is denoted x'. Set $e = e(n) = (1, 1, ..., 1) \in \mathbb{R}^n$, $\forall n \geq 1$.

In this article, some statements on the matrices hold eventually by permutation of rows and columns. For simplification, further, we omit to specify this fact.

Warning! In this article, if a Markov chain has the transition matrix $P = P_1 P_2 ... P_s$, where $s \geq 1$ and P_1, P_2, ..., P_s are stochastic matrices, then any 1-step transition of this chain is performed via P_1, P_2, ..., P_s, *i.e.*, doing s transitions: one using P_1, one using P_2, ..., one using P_s.

Let S be a finite set with $|S| = r$, where $r \geq 2$ ($|\cdot|$ is the cardinal; for "$r \geq 2$", see below). Let $\pi = (\pi_i)_{i \in S}$ be a positive probability distribution on S. One way to sample approximately or, at best, exactly from S is by means of our hybrid Metropolis-Hastings chain from [14]. Below we define this chain.

Let E be a nonempty set. Set $\Delta \succ \Delta'$ if $\Delta' \preceq \Delta$ and $\Delta' \neq \Delta$, where Δ, $\Delta' \in \mathrm{Par}(E)$.

Let Δ_1, Δ_2, ..., $\Delta_{t+1} \in \mathrm{Par}(S)$ with $\Delta_1 = (S) \succ \Delta_2 \succ ... \succ \Delta_{t+1} = (\{i\})_{i \in S}$, where $t \geq 1$. ($\Delta_1 \succ \Delta_2$ implies $r \geq 2$.) Let $Q_1, Q_2, ..., Q_t \in S_r$, $Q_1 = \left((Q_1)_{ij}\right)_{i,j \in S}$, $Q_2 = \left((Q_2)_{ij}\right)_{i,j \in S}$, ..., $Q_t = \left((Q_t)_{ij}\right)_{i,j \in S}$, such that

(C1) $\overline{Q}_1, \overline{Q}_2, ..., \overline{Q}_t$ are symmetric matrices;

(C2) $(Q_l)_K^L = 0, \forall l \in \langle t \rangle - \{1\}, \forall K, L \in \Delta_l, K \neq L$ (this condition implies that Q_l is a block diagonal matrix and Δ_l-stable matrix on $\Delta_l, \forall l \in \langle t \rangle - \{1\}$);

(C3) $(Q_l)_K^U$ is a row-allowable matrix, $\forall l \in \langle t \rangle, \forall K \in \Delta_l, \forall U \in \Delta_{l+1}, U \subseteq K$.