COMPLEMENTS
OF ***HYPERBOLIC MATHEMATICS***

Monographs in Applied Mathematics

Trandafir T. BĂLAN

COMPLEMENTS
OF *HYPERBOLIC MATHEMATICS*

from SUPER – ADDITIVITY
to STRUCTURAL DISCRETENESS

EDITURA UNIVERSITARIA
CRAIOVA, 2016

Referenți științifici:
Prof.univ.dr. Radu CONSTANTINESCU – Universitatea din Craiova
Prof.univ.dr. Nicolae BOJA – Universitatea Politehnică din Timișoara

Descrierea CIP a Bibliotecii Naţionale a României
BĂLAN, TRANDAFIR
Complements of hyperbolic mathematics from super-additivity to structural discreteness / Bălan T. Trandafir. - Craiova : Universitaria, 2016
Conţine bibliografie
ISBN 978-606-14-1100-9

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Preface

The Universe is made up of stories,
not of atoms.
Muriel Rukeyser

The main purpose of the present book is to support the ideas that just like continuity, whose proper structures are topologies, discreteness has its own structures. When involving usual metrics, the topologies essentially refine the property of sub-additivity, known as *rule of a triangle*: each side is less than the sum of the others. The fact is that most people strongly renounce this rule and its consequence "*The shortest route between two points is a straight line*". However, the reversed inequality is realistic and present in a lot of aspects of our existence. The book highlights several cases of super-additive phenomena, with the aim to increase the interest in finding unifier structures of discreteness.

Most frequently, when we are looking for super-additivity, we arrive at chapters of hyperbolic mathematics; this remark justifies the title. The word *Complements* in the title suggests that the book brings only a few additional facts to the vast field of hyperbolic mathematics, particularly, in connection with super-additivity and related topics. Such complements are still useful since traditionally, most works on fields naturally carrying super-additivity (e.g. indefinite inner products), avoid discussion on this topic.

The book has a didactical character. From the huge volume of existing literature on the discussed topics, it extracts the most representative results in the study of the super-additivity. The incipient aspect of the problematic and the simplicity of the text make the book accessible to young students, starting at the high school level. It does not aim to be facile popular book, but a rigorous one, which could offer a solid base for further studies. Of course, the experienced may directly enter Chapter V, where *horistologies* play the role of structures of discreteness.

The initial educational curriculum usually regards the knowledge of the material, palpable and visible aspects of the reality. This is a natural order of things, but a time may come when we ask about immaterial and invisible entities of the same reality, like events, causality, evolution etc. Then, we have to turn, *become like children*, and look for a book like this.

The book contains a lot of unexpected properties of a universe of events. Thus, there exists no triangle in the classical sense of an *angle*, but often, the Euclidean properties have peers in terms of events (see Chrono-geometry in Section IV.1); two events in spatial relation have neither a closest cause, nor a closest effect (see Multi-lattices in Section IV.2); in incomplete universes of events (like our ?!) there are not e-emergent sequences of events because these universes lack necessary e-germs, hence we cannot find the essential cause of such sequences (see Section V.4).

From a meta-theoretical point of view, the present book is a combination of several physical and mathematical theories, which contain fundamental arguments in favor of structural discreteness. The greatest physical theories that operate with discreteness are Relativity and Quantum Physics: The twin's paradox means super-additivity of proper time, while quanta and principles of uncertainty are features of discreteness. In mathematics, the theory of Indefinite Inner Product Spaces is most interfacing with discreteness, via the super-additivity of the intrinsic norms. Among these theories, Einsteinian Relativity has the greatest involvement in the text, with examples, physical interpretations and even with terminology. The fact that the universe of relativist events, presented in real variables, is a natural example of indefinite inner product space, assures the integrated study of these two theories.

Einstein himself remarked the discrete character of the universe of events. In a letter to Walter Dallenbach (1916), he wrote: *The problem seems to me how one can formulate statements about a discontinuum without calling upon continuum space-time as an aid; the latter should be banned from the theory as a supplementary construction not justified by the essence of the problem, which corresponds to nothing "real". But we still lack the mathematical structure unfortunately.*

The remaining question is to what degree do horistologies satisfy Einstein's request. Of course, the answer shall come latter, from specialists, in so far as they consider that it deserves the effort.

* * *

Undoubtedly, I could never have realize this book without the support of the *Alexander von Humboldt Stiftung*. During 1974, under Professor Werner Heisenberg's presidency, this foundation granted a Stipendium to me, with the plan to search for the proper structures of super-additivity and discreteness. Therefore, I address my sincere gratitude to all those who gave that opportunity to me.

I acknowledge all my professors who have guided me on this strange and unfriendly road *from super-additivity to discreteness*. I am also indebted to the scientific reviewers for their kindness to read the manuscript and give a lot of valuable suggestions.

Last but not least, I am grateful to all my colleagues who were interested in my work and have brought their own contributions to horistology.

The author,
Craiova, November 2016

CHAPTER I
MATHEMATICAL BACKGROUND

I.1. Logic and Sets

Generally speaking, *LOGIC* is the science of reason; its objects are our own judgments, expressed by *assertions* (frequently called *propositions*, *sentences*, *logic formulas*). The main purpose of the *logic* is to decide about *truth*, which is that quality of a sentence of being conformal to reality. In the beginning, getting the truth was a result of dialogues (Socrates, Plato). Latter, developing the ancient idea about the duality of the world (Lao Tzeu, Confucius), the concept of truth received a dialectical character (Kant, Hegel), based on antinomies, respectively theses and antitheses.

There exist more methods of evaluating the truth, which lead to different types of logics. In *binary logic*, each sentence can be either *true*, or *false*. This means that we deal with a *truth function* Truth, which may take only two values, say Truth $(A) = 1$ if A is a true proposition, and Truth $(A) = 0$ if A is false. Besides this case, we mention the *n-ary*, *fuzzy* and *modal* logics.

In the present book, all reasons will respect the binary logic.

A. Mathematical logic. First, using a binary function Truth, we sketch the *Propositional Calculus* on the set $\mathbb{P}$ of all propositions, which is a study of how the truth of composite sentences depends on the constituent terms. Most frequently, we construct composite propositions using *negation* ($\rceil$), *conjunction* ($\wedge$) and *disjunction* ($\vee$), defined by the *truth table*

A	$\rceil A$	B	$A \wedge B$	$A \vee B$
1	*0*	*1*	*1*	*1*
0	*1*	*1*	*0*	*1*
1		*0*	*0*	*1*
0		*0*	*0*	*0*

Using the operations $\rceil$, $\wedge$ and $\vee$, we can synthesize arbitrary logic formulas. In the switching circuit's theory, it shows that the parallel and series connections can produce arbitrary functioning. In computer sciences, it is the essential tool in realizing arbitrary logical formulas by $\rceil$, $\wedge$ and $\vee$, and possibly other gates (search *PROLOG*, see [ME], [BC], [BT5]).

Proposition 1. $\mathbb{P}$, endowed with $\rceil$, $\wedge$ and $\vee$, has the properties:

(*i*) $A\vee(B \vee C) = (A\vee B) \vee C$; $A\wedge(B\wedge C) = (A\wedge B) \wedge C$ (*associativity*)

(*ii*) $A\vee B = B\vee A$; $A\wedge B = B\wedge A$ (*commutativity*).

(*iii*) $A\vee(B\wedge C) = (A\vee B) \wedge (A\vee C)$; $A\wedge(B\vee C) = (A\wedge B)\vee(A\wedge C)$ (*distributivity*)

(*iv*) $A\wedge(A\vee B) = A$; $A\vee(A\wedge B) = A$ (*absorption*)

(*v*) $(A\wedge \rceil A) \vee B = B$; $(A\vee \rceil A) \wedge B = B$ (*complementary*).

The proof is direct. Because we may derive any other property of $\mathbb{P}$ from the above relations (*i*) - (*v*), the following generalization makes sense:

Definition 1. Let X be an arbitrary non-void set. If the abstract operations $\rceil$, $\wedge$ and $\vee$ satisfy the conditions (*i*) - (*v*), then we say that $(X, \rceil, \wedge, \vee)$ is a *Boolean algebra*.

Algebra $\mathbb{P}$ of proposition, algebra $\mathscr{P}(X)$ of parts (see Part B) and algebra of logical circuits are the most important examples of Boolean algebras.

Primarily, logic has referred to general knowledge, expressed in a natural language, but later on, it was necessary to solve the same problem of inferring truth in axiomatic theories. Nowadays, practical fields of interest (especially those related to computer sciences) are putting forward problems of artificial languages.

The natural sciences, especially mathematics and physics (particularly, we will refer to Einsteinian Relativity from both mathematical and physical points of view) are systems of true statements, frequently called *theories*. The main task of logic is how to deduce true sentences from the accepted axioms, so it is advisable to recall what a theory is.

A global analysis of the scientific texts shows that they work with the following types of abstract symbols (i.e. mental objects):

- A **vocabulary**, which contains *semantically* accepted (i.e. making sense) words like nouns, verbs, adjectives, etc.
- A set of **connectives**, which are used to put together words or groups of words (e.g. “and”, “or”, “if - then” etc.).
- A set of **punctuation marks**, necessary to precise the meaning of the message (e.g. full stop period, comma, colon, brackets, etc.).
- A set of **quantifiers**, especially “All” and “some”, used to define the “quantity” of the sentence. Of course, we may avoid separation of quantifiers as special signs (e.g. $\forall, \exists$) and include them in vocabulary.

By constructing finite sequences with elements from above, we obtain *propositions* (sentences, judgments, phrases etc.). To solve the problem of their truth we may use two devices:

- A set of **axioms**, which represent sentences *a priori* true, and
- A set of correct **inferences**, which allow deriving new true sentences.

Of course, we properly speak of *axioms* only in a theoretical framework, i.e. in exact sciences, since otherwise no axiom is explicitly formulated, as for example in everyday talking. Because each inference reduces the truth of a proposition to several propositions previously established as true, we finally reduce knowledge to the "truth" that we have gained in our daily experience of life. In addition, when using natural languages we make no visible distinction between syntactic and semantic correctness of our words propositions and phrases; this distinction is necessary in formal theories.

We may interpret *inferences* as relations between groups of *k* premises and one conclusion. Remember that we have $k = 1$ in immediate inferences, and $k = 2$ in syllogisms, but in poly-syllogisms we encounter higher values of *k*. Number *k* is known as *aryty* of the considered inference.

To simplify the following definition of a *formal system,* which extends the classical logic, it is useful to precise some notations and terminology:

- A set *S* is said to be *countable* if there is a 1:1 correspondence between *S* and $\mathbb{N}$. If the elements of *S* can be indexed by several natural numbers *1, 2, … n*, for some $n \in \mathbb{N}$, then we say that *S* is *finite*.
- If Σ is a nonvoid set, then Σ^* (sometimes Σ^f) denotes the set of all *finite ordered sequences* made with elements from Σ.
- Set $R \subseteq \mathcal{F}^k \times \mathcal{F}$, where $k \in \mathbb{N}\backslash\{0\}$, is called *k-ary relation* (or relation of *aryty k*) *on* $\mathcal{F}$. To read membership $((\alpha_1, \dots, \alpha_k), \beta) \in R$, we say that "*conclusion β is inferred from the premises* $\alpha_1,\dots,\alpha_k$ *by rule R*".

Definition 2. Let us consider the following sets of symbols:

V = **vocabulary**, which is *at most countable* set;
C = set of **connectives**;
P = set of **punctuation marks**;
Q = set of **quantifiers** (possibly void);
$\Sigma = V \cup C \cup P \cup Q$ = Total set of **symbols**.

The elements of Σ^* are usually called **formulas** (*propositions*, *phrases*), or simply *words*, especially in formal linguistic. Farther, we specify:

- a subset $\mathcal{F} \subseteq \Sigma^*$, which consists of *well constructed formulas*, i.e. formulas that syntactically make sense;
- a subset $\mathcal{A} \subseteq \mathcal{F}$, which consists of *axioms*;
- a set $\mathcal{R}$ of *relations* (inferences) of various aryties on $\mathcal{F}$.

A ***formal system*** $\mathcal{S}$ is defined as a *quartet* of the form $\mathcal{S} = (\Sigma,\mathcal{F},\mathcal{A},\mathcal{R})$.

It is useful to distinguish three steps / levels in this definition:

1) A ground level, reduced to the description of Σ^*, which represents the raw (i.e. not structured) framework of the theory;
2) A syntactic level, where we specify $\mathcal{F}$, which consists of *correct* forms;
3) A semantic level, containing axioms and rules of inference, which deal with *truth* (and *meaning*) of the syntactically correct forms.

Example 1. Let us consider

$$V = \{a, =\} \cup \mathbb{N},\ C = \{\circ, \wedge, +\},\ Q = \varnothing, \text{ and } P = \{(,),\{,\},[,]\}.$$

Let us define the set of formulas by

$$\mathcal{F} = \{[a\text{^}n \circ a\text{^}m = a\text{^}p]: n, m, p \in \mathbb{N} \setminus \{0\}\},$$

where the *e-mail* style is adopted, and for brevity, a^n stands for the string $aa\ldots a$ of *length n* (with n terms), and the brackets in $[(a\text{^}n)\circ(a\text{^}m) = a\text{^}p]$ are omitted by convention. Further let us accept two rules of inference, i.e. $\mathcal{R} = \{R_1, R_2\}$, where R_1 and R_2 are 1-ary relations defined by:

$$R_1 = \{([a\text{^}n \circ a\text{^}m = a\text{^}p], [a\text{^}(n+1) \circ a\text{^}m = a\text{^}(p+1)]): n, m, p \in \mathbb{N}\setminus\{0\}\},$$
$$R_2 = \{([a\text{^}n \circ a\text{^}m = a\text{^}p], [a\text{^}n \circ a\text{^}(m+1) = a\text{^}(p+1)]): n, m, p \in \mathbb{N}\setminus\{0\}\}.$$

It is easy to see that part $\mathcal{F}$, of the set Σ^* of all possible formulas, contains those formulas that are correctly constructed from a *syntactical* point of view (e.g.]n=a^{ is a formula too, but not in $\mathcal{F}$). In other terms, $\mathcal{F}$ consists of grammatically correct formulas, which make sense, and in this case, a natural sense concerns the *multiplication of powers*.

Even so, we cannot yet qualify all the formulas of $\mathcal{F}$ as being *true* (for example $[a\text{^}2 \circ a\text{^}2 = a\text{^}5]$), at least in connection to the *multiplication of powers*. Actually, we cannot decide about their truth as long as another *unknown meaning* is possible. Consequently, we may discuss about *truth* only if we accept some formulas as *true* at the very beginning, i.e. we choose a set of axioms $\mathcal{A} \neq \varnothing$. In this case, if we take $\mathcal{A} = \{[a \circ a = a\text{^}2]\}$, then we can prove the truth of other formulas, e.g. $[a\text{^}7 \circ a\text{^}2 = a\text{^}9]$.

To specify what a *proof* should be, and what formulas are true, we have to introduce the following notions:

Definition 3. We say that $\mathcal{S}$ is an **axiomatic formal system** if $\mathcal{A} \neq \varnothing$, $\mathcal{R} \neq \varnothing$, and for each $\alpha \in \mathcal{F}$, we can effectively decide about $\alpha \in \mathcal{A}$.