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Quasi-invariant convergence in a normed space

Fatih Nuray

Abstract. In this study, notions of quasi-invariant convergence and quasi-invariant statistical convergence, which are related to invariant limits, are defined and discussed.

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Key words and phrases. Invariant limits, statistical convergence.

1. Introduction

Let σ be a one-to-one mapping of the set of positive integer into itself such that $\sigma^m(n) \neq n$ for all positive integers m and n, where $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$; $m = 1, 2, 3, ...$

Let X be a real normed space. A continuous linear functional φ on the space of bounded sequences is an *invariant mean* or *σ-limit* if
(1) $\varphi(x) \geq 0$ when the sequence $x = (x_n) \in X$ has $x_n \geq 0$ for all n,
(2) $\varphi(1, 1, 1, ...) = 1$ and
(3) $\varphi(x_{\sigma(n)}) = \varphi(x)$
for all bounded sequences x. We denote by V_σ the set of bounded sequences all of whose invariant means are equal. In case $\sigma(n) = n + 1$, a σ-limit is often called a *Banach limit* and V_σ is the set of almost convergent sequences. It is known that a bounded sequence $x = (x_n) \in X$ is invariant convergent to $s \in X$ if and only if

$$\lim_{p\to\infty} \|\frac{1}{p}\sum_{i=0}^{p-1} x_{\sigma^i(k)} - s\| = 0 \tag{1}$$

uniformly in $k(= 1, 2, 3, ...)$. It is known that $c \subset V_\sigma \subset l_\infty$ where c is the space of all convergent sequences and l_∞ is the space of all bounded sequences in a real normed space X. Over the years invariant convergence has been examined in summability theory.

A sequence $(x_i) \in X$ is said to be *statistically convergent* to $s \in X$ if for each $\epsilon > 0$

$$\lim_{p\to\infty} \frac{1}{p}|\{i \leq p : \|x_i - s\| \geq \epsilon\}| = 0$$

where $|A|$ denotes the number of members of a set A. The concept of statistical convergence was first introduced by Fast [2] and also independently by Buck [1] and Schoenberg [8] for real and complex sequences. Further this concept was studied by Salat [6], Fridy [3] and many others. Recently Savas and Nuray [5] introduced σ-statistical convergence for real and complex sequences as follows: A sequence (x_i) is

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said to be *invariant* or σ- *statistically convergent* to real or complex number s if for each $\epsilon > 0$

$$\lim_{p\to\infty}\frac{1}{p}|\{i \le p : |x_{\sigma^i(k)} - s| \ge \epsilon\}| = 0$$

uniformly in k. We can generalize this definition to the sequences in a real normed space X as follows: A sequence $(x_i) \in X$ is said to be invariant or σ- statistically convergent to $s \in X$ if for each $\epsilon > 0$

$$\lim_{p\to\infty}\frac{1}{p}|\{i \le p : \|x_{\sigma^i(k)} - s\| \ge \epsilon\}| = 0$$

uniformly in k.

The plan of this paper is as follows. First we will show the existence of an another family of functionals defined on the space l_∞. Then we define a new method of summability of sequences $(x_i) \in l_\infty$ which will be called quasi invariant convergence and we will give a theorem which contains a necessary and sufficient condition for a bounded sequence to be quasi invariant convergent. Next, we shall prove a theorem which shows that if a bounded sequence is invariant convergent to s, then it is quasi invariant convergent to s. Finally we will introduce quasi invariant statistical convergence for sequences in a real normed space and show that if a sequence is invariant statistical convergent to s, then it is quasi invariant statistical convergent to s.

2. Quasi-invariant convergence

Let us define on the space l_∞ the function q by

$$q(x) \equiv q(x_i) = \overline{\lim_{p\to\infty}}\{sup_n\frac{1}{p}\|\sum_{i=0}^{p-1} x_{\sigma^i(np)}\|\} \tag{2}$$

The functional q clearly is real-valued and it satisfies following properties:
(i) $q(x) \ge 0$,
(ii) $q(\alpha x) = |\alpha| q(x)$,
(iii) $q(x+y) \le q(x) + q(y)$ $(\alpha \in \mathbb{R}; x, y \in l_\infty)$
that is, q is a symmetric convex functional on the space l_∞. According to a corollary of Hahn-Banach theorem there must exist a nontrivial linear functional L on the space l_∞ such that $|L(x_i)| \le q(x_i)$.

The following lemma is well known in the literature.

Lemma 2.1. *Let X be a real linear space and $q : X \to \mathbb{R}$ be a functional such that the following assertions are valid: $q(x) \ge 0, q(\alpha x) = |\alpha| q(x)$, $q(x+y) \le q(x)+q(y)$ $(\alpha \in \mathbb{R}; x, y \in l_\infty)$. Then for each $x_0 \in X$, there exists a linear functional L on X such that*

$$(\forall x \in X) \quad |L(x)| \le q(x), \quad L(x_0) = q(x_0).$$

Denoting now by $\sum$ the family of functionals satisfying the above conditions then for each $s \in X$ we have

$$(\forall L \in \sum) \quad L(x_i - s) = 0 \text{ iff } q(x_i - s) = 0 \quad ((x_i) \in l_\infty). \tag{3}$$

Now we can state following theorem.

Theorem 2.2. *There exists the family of non trivial functionals L defined on the space l_∞ such that for all $\alpha, \beta \in \mathbb{R}$, each $s \in X$ and all $(x_i), (y_i) \in l_\infty$, the following assertions are valid:*

(a) $L(\alpha x_i + \beta y_i) = \alpha L(x_i) + \beta L(y_i)$,

(b) $L(x_{\sigma(i)}) = L(x_i)$,

(c) $|L(x_i)| \leq q(x_i)$,

(d) $L(x_i - s) = 0$ *iff* $q(x_i - s) = 0$.

Having obtained the functionals $L \in \sum$ we can proceed to the investigation of the sequences $(x_i) \in l_\infty$ which all the functionals $L \in \sum$ assigned the same value.

Definition 2.1. A sequence $(x_i) \in l_\infty$ is *quasi invariant convergent* to $s \in X$ or *quasi σ-summable* to s if

$$(\forall L \in \sum) \quad L(x_i - s) = 0. \tag{4}$$

in this case we will write $(Q - \sigma) - lim_{i\to\infty} x_i = s$.

It is easy to see that quasi invariant limit of a sequence defined in such way is unique.

Theorem 2.3. *A bounded sequence (x_i) quasi invariant convergent to $s \in X$ iff*

$$\|\frac{1}{p}\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\| \to 0 \text{ as } p \to +\infty \tag{5}$$

uniformly in n $(= 1, 2, ...)$.

Proof. Suppose for a bounded sequence (x_i), we have $(Q-\sigma)-lim_{i\to\infty} x_i = s$. Then, by (4) and (3), we have $q(x_i - s) = 0$ or, by (2), we have

$$\overline{lim}_{p\to\infty}\{sup_n \frac{1}{p}\|\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\|\} = 0.$$

Therefore for any $\epsilon > 0$, there exists an integer $p_0 > 0$ such that for all $p > p_0$ and $n = 1, 2, 3, ...$, we have

$$\frac{1}{p}\|\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\| < \epsilon$$

Since $\epsilon > 0$, arbitrary, we have

$$\frac{1}{p}\|\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\| \to 0 \text{ as } p \to \infty$$

uniformly in n, so the condition (5) is necessary. Conversely, let the condition (5) be true. This means that

$$sup_n \frac{1}{p}\|\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\| \to 0 \text{ as } p \to \infty$$

or

$$q(x_i - s) = \overline{\lim}_{p\to\infty}\{sup_n \frac{1}{p}\|\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\|\} = 0$$

hence by (3), we have

$$(\forall L \in \sum) \quad L(x_i - s) = 0,$$

which by (4), means that $(Q-\sigma)-lim_{i\to\infty} x_i = s$, so the condition (5) is sufficient. ☐

Theorem 2.4. *If a bounded sequence $x = (x_i)$ invariant convergent to $s \in X$, then it is quasi invariant convergent to s.*

Proof. Let bounded sequence $x = (x_i)$ be invariant convergent to $s \in X$. Then by (1) for any $\epsilon > 0$ there exists an integer $p_0 > 0$ such that

$$\|\frac{1}{p}\sum_{i=0}^{p-1} x_{\sigma^i(k)} - s\| < \epsilon \quad (p > p_0, k = 1, 2, 3, ...).$$

hence for $k = np$ $(p > p_0,\ n = 1, 2, 3, ...)$ we have

$$\|\frac{1}{p}\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\|\frac{1}{p}\sum_{i=0}^{p-1} x_{\sigma^i(np)} - s\| \to 0 \text{ as } p \to \infty$$

uniformly in n which, by (5), means that (x_i) quasi invariant convergent. □

When $\sigma(i) = i + 1$ we have quasi almost convergence which was defined and discussed in [4].

3. Quasi-invariant statistical convergence

Definition 3.1. A sequence (x_i) is said to be *quasi invariant statistically convergent* to $s \in X$ if for each $\epsilon > 0$

$$\lim_{p\to\infty} \frac{1}{p}|\{i \le p : \|x_{\sigma^i(np)} - s\| \ge \epsilon\}| = 0$$

uniformly in n.

When $\sigma(i) = i + 1$ we have the following definition of quasi almost statistical convergence which have not appeared anywhere by this time.

Definition 3.2. A sequence (x_i) is said to be *quasi almost statistically convergent* to $s \in X$ if for each $\epsilon > 0$

$$\lim_{p\to\infty} \frac{1}{p}|\{i \le p : \|x_{np+i} - s\| \ge \epsilon\}| = 0$$

uniformly in n.

Theorem 3.1. *If a sequence $x = (x_i) \in X$ invariant statistically convergent to $s \in X$, then it is quasi invariant statistically convergent to s.*

Proof. Let $x = (x_i)$ be invariant statistically convergent to $s \in X$. Then for any $\epsilon > 0$ there exists an integer $p_0 > 0$ such that

$$\frac{1}{p}|\{i \le p : \|x_{\sigma^i(k)} - s\| \ge \epsilon\}| < \epsilon \ (p > p_0, k = 1, 2, ...).$$

Hence for $k = np$ $(p > p_0, n = 1, 2, ...)$ we have

$$\frac{1}{p}|\{i \le p : \|x_{\sigma^i(np)} - s\| \ge \epsilon\}| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\lim_{p\to\infty} \frac{1}{p} |\{i \leq p : \|x_{\sigma^i(np)} - s\| \geq \epsilon\}| = 0$$

uniformly in n which means that (x_i) is quasi invariant statistically convergent to s convergent. □

We remark that from the comparison of the definitions of invariant statistical convergence and quasi invariant statistical convergence, follows that there is a big possibility that there exist sequences that are quasi invariant statistical convergent, but not invariant statistically convergent. Proof of that is still an open problem. The similar remark also stands for relationship between the quasi invariant convergence and invariant convergence.

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Some further results on belonging of trigonometric series to Orlicz space

Xhevat Z. Krasniqi

Abstract. Here in this paper we have introduced a new condition which is not worse than the condition that satisfy numerical sequences of Rest Bounded Variation Mean Sequences. This condition is used to obtain some integrability conditions of the functions $g(x)$ and $f(x)$ (which denote formal sine and cosine trigonometric series respectively) such that these functions are going to belong to the Orlicz space. This study may be considered as a continuation of the investigations previously done by L. Leindler [5] and S. Tikhonov [14].

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Key words and phrases. Trigonometric series, Integrability, Orlicz space.

1. Introduction

Many authors have studied the integrability of the formal series

$$g(x) := \sum_{n=1}^{\infty} \lambda_n \sin nx \tag{1}$$

and

$$f(x) := \sum_{n=1}^{\infty} \lambda_n \cos nx \tag{2}$$

imposing certain conditions on the coefficients λ_n (see for example [2], [3], [8], [9], and [11]–[13]).

As initial example, R. P. Boas in [1] proved the following result for (1):

Theorem 1.1. *If $\lambda_n \downarrow 0$ then for $0 \le \gamma \le 1$, $x^{-\gamma} g(x) \in L[0,\pi]$ if and only if $\sum_{n=1}^{\infty} n^{\gamma-1}\lambda_n$ converges.*

This result had previously been proved for $\gamma = 0$ by W.H. Young [15] and it was later extended by P. Heywood [4] for $1 < \gamma < 2$.

Later on the monotonicity condition on the coefficients λ_n was replaced to more general ones by S.M. Shah [12] and L. Leindler [7].

Recently, S. Tikhonov [14] has proved two theorems giving sufficient conditions of belonging of $g(x)$ and $f(x)$ to Orlicz space. Before we state his theorems we shall recall some notions and notations.

L. Leindler [7] introduced a class of numerical sequences which has an interesting property and useful in many applications. A sequence $c := \{c_n\}$ of positive numbers

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tending to zero is of rest bounded variation, or briefly R_0^+BVS, if it possesses the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(c)c_m \tag{3}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called almost increasing (decreasing) if there exists constant $C := C(\gamma) \geq 1$ such that

$$C\gamma_n \geq \gamma_m \quad (\gamma_n \leq C\gamma_m)$$

holds for any $n \geq m$.

Here and further C, C_i denote positive constants that are not necessarily the same at each occurrence, and also we use the notion $u \ll w$ ($u \gg w$) at inequalities if there exists a positive constant C such that $u \leq Cw$ ($u \geq Cw$) holds.

We will denote (see [10]) by $\triangle(p,q)$, $(0 \leq q \leq p)$ the set of all nonnegative functions $\Phi(x)$ defined on $[0,1)$ such that $\Phi(0) = 0$ and $\Phi(x)/x^p$ is nonincreasing and $\Phi(x)/x^q$ is nondecreasing. It is clear that $\triangle(p,q) \subset \triangle(p,0)$, $(0 < q \leq p)$. As an example, $\triangle(p,0)$ contains the function $\Phi(x) = \log(1+x)$.

Here and in the sequel, a function $\gamma(x)$ is defined by the sequence γ in the following way: $\gamma\left(\frac{\pi}{n}\right) := \gamma_n$, $n \in \mathbb{N}$ and there exist positive constants C_1 and C_2 such that $C_1\gamma_{n+1} \leq \gamma(x) \leq C_2\gamma_n$ for $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right)$.

A locally integrable almost everywhere positive function $\gamma(x) : [0,\pi] \to [0,\infty)$ is said to be a weight function. Let $\Phi(t)$ be a nondecreasing continuous function defined on $[0,\infty)$ such that $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = +\infty$. For a weight $\gamma(x)$ the weighted Orlicz space $L(\Phi,\gamma)$ is defined by

$$L(\Phi,\gamma) = \left\{h : \int_0^{\pi} \gamma(x)\Phi(\varepsilon|h(x)|)dx < \infty \quad \text{for some} \quad \varepsilon > 0\right\}. \tag{4}$$

Tikhonov's results now can be read as follows:

Theorem 1.2. *Let $\Phi(x) \in \triangle(p,0)$, $0 \leq p$. If $\lambda_n \in R_0^+BVS$, and the sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-1+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^2}\Phi(n\lambda_n) < \infty \Rightarrow \psi(x) \in L(\Phi,\gamma), \tag{5}$$

where a function $\psi(x)$ is either a sine or cosine series.

Theorem 1.3. *Let $\Phi(x) \in \triangle(p,q)$, $0 \leq q \leq p$. If $\lambda_n \in R_0^+BVS$, and the sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-(1+q)+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2+q}}\Phi(n^2\lambda_n) < \infty \Rightarrow g(x) \in L(\Phi,\gamma). \tag{6}$$

A null-sequence c of nonnegative numbers possessing the property

$$\sum_{n=2m}^{\infty} |c_n - c_{n+1}| \leq \frac{K(c)}{m}\sum_{\nu=m}^{2m-1} c_\nu \tag{7}$$

is called a sequence of mean rest bounded variation, in symbols, $c \in MRBVS$.

In [5] L. Leindler extended Theorem 1.2 and Theorem 1.3 so that the sequence $\{\lambda_n\}$ belongs the class $MRBVS$ instead of the class R_0^+BVS. His results are formulated as follows:

Theorem 1.4. *Theorems 1.2 and 1.3 can be improved when the condition $\lambda_n \in R_0^+BVS$ is replaced by the assumption $\lambda_n \in MRBVS$. Furthermore the conditions of (5) and (6) may be modified as follows:*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^2} \Phi\left(\sum_{\nu=n}^{2n-1} \lambda_\nu\right) < \infty \Rightarrow \psi(x) \in L(\Phi, \gamma), \tag{8}$$

and

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2+q}} \Phi\left(n \sum_{\nu=n}^{2n-1} \lambda_\nu\right) < \infty \Rightarrow g(x) \in L(\Phi, \gamma), \tag{9}$$

respectively.

Let $C := C_n := \frac{1}{n+1} \sum_{i=0}^{n} c_k$, where c_k is a sequence of nonnegative numbers. Very recently, R. N. Mohapatra and B. Szal [16] introduced the following class of sequences of nonnegative numbers:

If $C \in RBVS$, i.e.

$$\sum_{k=m}^{\infty} |C_k - C_{k+1}| \leq K(c) C_m, \tag{10}$$

then it is said that C is of rest bounded variation means sequence, briefly denoted by $C \in RBVMS$.

Aiming to prove the counterparts of Theorem 1.2 and Theorem 1.3 so that the sequence $\{\lambda_n\}$ belongs the class $RBVMS$ instead of the classes $MRBVS$ or R_0^+BVS, we were not in able. However, we have proved two theorems, when not a worse condition than (10) will be fulfilled. Indeed, we have required that the sequence $\{\lambda_n\}$ satisfies condition (obviously not worse than condition (10))

$$\sum_{k=n}^{\infty} k|V_k - V_{k+1}| \leq K V_n, \quad (n = 1, 2, \dots), \tag{11}$$

where $V_k := \frac{1}{k} \sum_{j=1}^{k} \lambda_j$.

To prove our main results we need some helpful statements given in next section.

2. Auxiliary lemmas

We shall use the following lemmas for the proof of the main results.

Lemma 2.1 ([6]). *If $a_n \geq 0$, $b_n > 0$, and if $p \geq 1$, then*

$$\sum_{n=1}^{\infty} b_n \left(\sum_{v=1}^{n} a_v\right)^p \leq C \sum_{n=1}^{\infty} b_n^{1-p} a_n^p \left(\sum_{v=n}^{\infty} a_v\right)^p.$$

Lemma 2.2 ([10]). *Let $\Phi \in \triangle(p, q)$, $0 \leq q \leq p$, and $t_j \geq 0$, $j = 1, 2, \dots, n$, $n \in \mathbb{N}$. Then*

(1) $\theta^p \Phi(t) \leq \Phi(\theta t) \leq \theta^q \Phi(t)$, $0 \leq \theta \leq 1$, $t \geq 0$,

(2) $\Phi\left(\sum_{j=1}^{n} t_j\right) \leq \left(\sum_{j=1}^{n} \Phi^{1/p*}(t_j)\right)^{p*}$, $p* := \max(1, p)$.

Lemma 2.3. *Let $\Phi \in \triangle(p, q)$, $0 \leq q \leq p$. If $\rho_n > 0$, $\lambda_n \geq 0$, and if*

$$V_{\nu+j} \ll V_\nu, \quad V_\nu := \frac{1}{\nu} \sum_{j=1}^{\nu} \lambda_j \tag{12}$$