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On some double lacunary strong Zweier convergent sequence spaces

Ayhan Esi and Mehmet Acikgoz

Abstract. In this paper we define three classes of new double sequence spaces. We give some relations related to these sequence spaces. We also introduce the concept of double lacunary statistical Zweier convergence and obtain some inclusion relations related to these new double sequence spaces.

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1. Introduction

Before we enter the motivation for this paper and the presentation of the main results we give some preliminaries.

By the convergence of a double a double sequence we mean the convergence on the Pringsheim sense [1] that is, a double sequence $x=(x_{i,j})$ has Pringsheim limit L (denoted by P-limx=L) provided that given $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $|x_{i,j}-L|<\varepsilon$ whenever $i,j>N$. We shall describe such an $x=(x_{i,j})$ more briefly as "$P-convergent$". We shall denote the space of all $P-convergent$ sequences by c^2. By a bounded double sequence we shall mean there exists a positive integer K such that $|x_{i,j}|<K$ for all (i,j) and denote such bounded by l_∞^2.

Zweier sequence spaces for single sequences defined and studied by Şengönül [2], Esi and Sapsızoğlu [3] Khan et.all $[4-5]$.

Definition 1.1. [7] The double sequence $\theta_{r,s}=\{(k_r,l_s)\}$ is called *double lacunary sequence* if there exist two increasing of integers sequences (k_r) and (l_s) such that

$$k_o=0,\ h_r=k_r-k_{r-1}\to\infty \quad as\ r\to\infty$$

and

$$l_o=0,\ \bar{h}_s=l_s-l_{s-1}\to\infty \quad as\ s\to\infty.$$

Notations: $k_{r,s}=k_rl_s$, $h_{r,s}=h_r\bar{h}_s$, and $\theta_{r,s}$ is determined by

$$I_{r,s}=\{(k,l):\ k_{r-1}<k\le k_r \text{ and } l_{s-1}<l\le l_s\},$$

$$q_r=\frac{k_r}{k_{r-1}},\ \bar{q}_s=\frac{l_s}{l_{s-1}}\ and\ q_{r,s}=q_r\bar{q}_s.$$

The space of double lacunary convergent sequence spaces $\left[N_{\theta_{r,s}}\right]_o$, $\left[N_{\theta_{r,s}}\right]$ and $\left[N_{\theta_{r,s}}\right]_\infty$ were defined by Savas in $[2]$, as follows:

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$$\left[N_{\theta_{r,s}}\right]_o=\left\{x=(x_{i,j}):\ P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}|x_{i,j}|=0\right\},$$

$$\left[N_{\theta_{r,s}}\right]=\left\{x=(x_{i,j}):\ P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}|x_{i,j}-L|=0,\ \text{for some } L\right\}$$

and

$$\left[N_{\theta_{r,s}}\right]_\infty=\left\{x=(x_{i,j}):\ \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}|x_{i,j}|<\infty\right\}.$$

In [8], Savaş and Patterson defined the double sequence spaces $\left[W^2\right]$ as follows:

$$\left[W^2\right]=\left\{x=(x_{i,j}):\ P-\lim_{m,n}\frac{1}{mn}\sum_{i,j=1,1}^{m,n}|x_{i,j}-L|=0,\ \text{for some } L\right\}.$$

The purpose of this paper is to introduce and study the concepts of double Zweier lacunary strongly convergence and double Zweier lacunary statistical convergence.

2. Double Zweier lacunary strongly convergence

Define the double sequence $y=(y_{i,j})$ which will be used throughout the paper, as Z-transform of a sequence $x=(x_{i,j})$, i.e.,

$$y_{i,j}=\frac{1}{2}\left(x_{i,j}+x_{i,j-1}\right);(i,j\in\mathbb{N}). \tag{1.1}$$

We introduce the double Zweier sequence spaces $\left[W^2,Z\right]$, $\left[N_{\theta_{r,s}},Z\right]_o$, $\left[N_{\theta_{r,s}},Z\right]$ and $\left[N_{\theta_{r,s}},Z\right]_\infty$ as the set of all double sequences such that Z-transforms of them are in $\left[W^2\right]$, $\left[N_{\theta_{r,s}}\right]_o$, $\left[N_{\theta_{r,s}}\right]$ and $\left[N_{\theta_{r,s}}\right]_\infty$ respectively, that is

$$\left[W^2,Z\right]=\left\{x=(x_{i,j}):\ P-\lim_{m,n}\frac{1}{mn}\sum_{i,j=1,1}^{m,n}|y_{i,j}-L|=0,\ \text{for some } L\right\}.$$

$$\left[N_{\theta_{r,s}},Z\right]_o=\left\{x=(x_{i,j}):\ P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}|y_{i,j}|=0\right\},$$

$$\left[N_{\theta_{r,s}},Z\right]=\left\{x=(x_{i,j}):\ P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}|y_{i,j}-L|=0,\ \text{for some } L\right\}$$

and

$$\left[N_{\theta_{r,s}},Z\right]_\infty=\left\{x=(x_{i,j}):\ \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}|y_{i,j}|<\infty\right\}.$$

Theorem 2.1. *The double sets* $\left[W^2,Z\right]$, $\left[N_{\theta_{r,s}},Z\right]_o$, $\left[N_{\theta_{r,s}},Z\right]$ *and* $\left[N_{\theta_{r,s}},Z\right]_\infty$ *are linear spaces over the set of complex numbers.*

Proof. The proof of the theorem is standard and so we omitted. □

Theorem 2.2. *The double Zweier sequence spaces* $\left[W^2, Z\right]$, $\left[N_{\theta_{r,s}}, Z\right]_o$, $\left[N_{\theta_{r,s}}, Z\right]$ *and* $\left[N_{\theta_{r,s}}, Z\right]_\infty$ *are linearly isomorphic to the double sequence spaces* $\left[W^2\right]$, $\left[N_{\theta_{r,s}}\right]_o$, $\left[N_{\theta_{r,s}}\right]$ *and* $\left[N_{\theta_{r,s}}\right]_\infty$, *respectively, i.e.,* $\left[W^2, Z\right] \approx \left[W^2\right]$, $\left[N_{\theta_{r,s}}, Z\right]_o \approx \left[N_{\theta_{r,s}}\right]_o$, $\left[N_{\theta_{r,s}}, Z\right] \approx \left[N_{\theta_{r,s}}\right]$ *and* $\left[N_{\theta_{r,s}}, Z\right]_\infty \approx \left[N_{\theta_{r,s}}\right]_\infty$.

Proof. We consider only $\left[N_{\theta_{r,s}}, Z\right]_o$. We should show the existence of a linear bijection between the double sequence spaces $\left[N_{\theta_{r,s}}, Z\right]_o$ and $\left[N_{\theta_{r,s}}\right]_o$. Consider the transformation Z define, with the notation of (1.1), from $\left[N_{\theta_{r,s}}, Z\right]_o$ to $\left[N_{\theta_{r,s}}\right]_o$ by

$$\begin{array}{rcl} Z : \left[N_{\theta_{r,s}}, Z\right]_o & \rightarrow & \left[N_{\theta_{r,s}}\right]_o \\ x & \rightarrow & Zx = y,\ y = (y_{i,j}) \end{array}$$

and $y_{i,j} = \frac{1}{2}\left(x_{i,j} + x_{i,j-1}\right); (i, j \in \mathbb{N})$. The linearity of Z is clear. Further, it is trivial that $x = 0$ whenever $Zx = 0$ and hence Z is injective. Let $y = (y_{i,j}) \in \left[N_{\theta_{r,s}}\right]_o$ and define the sequence $x = (x_{i,j})$ by

$$x_{i,j} = 2\sum_{k=0}^{j} (-1)^{j-k}\, y_{i,k} \quad (\forall i \in \mathbb{N})$$

Then

$$\begin{aligned} \|x\|_{\left[N_{\theta_{r,s}}, Z\right]_o} &= \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} \left| \frac{1}{2}\left(x_{i,j} + x_{i,j-1}\right) \right| \\ &= \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} \left| \frac{1}{2}\left(2\sum_{k=0}^{j} (-1)^{j-k}\, y_{i,k} + 2\sum_{k=0}^{j} (-1)^{(j-1)-k}\, y_{i,k} \right) \right| \\ &= \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} |y_{i,j}| \end{aligned}$$

which says us that $x = (x_{i,j}) \in \left[N_{\theta_{r,s}}, Z\right]_o$. Additionally, we observe that

$$\|x\|_{\left[N_{\theta_{r,s}}, Z\right]_\infty} = \|y\|_{\left[N_{\theta_{r,s}}\right]_\infty}.$$

Thus, we have that the transform Z is surjective. Hence, Z is linear bijection which therefore says us the double sequence spaces $\left[N_{\theta_{r,s}}, Z\right]_o$ and $\left[N_{\theta_{r,s}}\right]_o$ are linearly isomorphic. The others can be proved similarly. This completes the proof. □

Theorem 2.3. *Let* $\theta_{r,s}$ *be a double lacunary sequence. Then*

- **(i)** $\left[W^2, Z\right] \subset \left[N_{\theta_{r,s}}, Z\right]$ if $\liminf q_r > 1$ and $\liminf \overline{q_s} > 1$;
- **(ii)** $\left[N_{\theta_{r,s}}, Z\right] \subset \left[W^2, Z\right]$ if $\limsup q_r < \infty$ and $\limsup \overline{q_s} < \infty$;
- **(iii)** $\left[N_{\theta_{r,s}}, Z\right] = \left[W^2, Z\right]$ if $1 < \liminf q_r < \infty$ and $1 < \limsup \overline{q_s} < \infty$.

Proof. **(i).** Suppose that $\liminf q_r > 1$ and $\liminf \overline{q_s} > 1$.Then there exists $\delta > 0$ such that both $q_r > 1 + \delta$ and $\overline{q_s} > 1 + \delta$. This implies $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{\overline{h_s}}{l_s} \geq \frac{\delta}{1+\delta}$. If $x = (x_{i,j}) \in \left[W^2, Z\right]$ then we obtain the following:

$$\begin{aligned} A_{r,s} &= \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} |y_{i,j} - L| = \frac{1}{h_{r,s}} \sum_{i=1}^{k_r} \sum_{j=1}^{l_s} |y_{i,j} - L| \\ &-\frac{1}{h_{r,s}} \sum_{i=1}^{k_{r-1}} \sum_{j=1}^{l_{s-1}} |y_{i,j} - L| - \frac{1}{h_{r,s}} \sum_{i=k_{r-1}+1}^{k_r} \sum_{j=1}^{l_s} |y_{i,j} - L| - \frac{1}{h_{r,s}} \sum_{j=l_{s-1}+1}^{l_s} \sum_{i=1}^{k_{r-1}} |y_{i,j} - L| \end{aligned}$$

$$= \frac{k_r l_s}{h_{r,s}} \left(\frac{1}{k_r l_s} \sum_{i=1}^{k_r} \sum_{j=1}^{l_s} |y_{i,j} - L| \right) - \frac{k_{r-1} l_{s-1}}{h_{r,s}} \left(\frac{1}{k_{r-1} l_{s-1}} \sum_{i=1}^{k_{r-1}} \sum_{j=1}^{l_{s-1}} |y_{i,j} - L| \right)$$

$$- \frac{1}{h_r} \sum_{i=k_{r-1}+1}^{k_r} \frac{l_{s-1}}{\overline{h_s}} \frac{1}{l_{s-1}} \sum_{j=1}^{l_{s-1}} |y_{i,j} - L| - \frac{1}{\overline{h_s}} \sum_{j=l_{s-1}+1}^{l_s} \frac{k_{r-1}}{h_r} \frac{1}{k_{r-1}} \sum_{i=1}^{k_{r-1}} |y_{i,j} - L| .$$

Since $x = (x_{i,j}) \in \left[W^2, Z\right]$ the last two terms tend to zero in the Pringsheim sense, thus

$$A_{r,s} = \frac{k_r l_s}{h_{r,s}} \left(\frac{1}{k_r l_s} \sum_{i=1}^{k_r} \sum_{j=1}^{l_s} |y_{i,j} - L| \right) - \frac{k_{r-1} l_{s-1}}{h_{r,s}} \left(\frac{1}{k_{r-1} l_{s-1}} \sum_{i=1}^{k_{r-1}} \sum_{j=1}^{l_{s-1}} |y_{i,j} - L| \right) + o(1).$$

Since $h_{r,s} = k_r l_s - k_r l_{s-1} - k_{r-1} l_s + k_{r-1} l_{s-1}$ we are granted the following:

$$\frac{k_r l_s}{h_{r,s}} \le \left(\frac{1+\delta}{\delta} \right)^2 \quad \text{and} \quad \frac{k_{r-1} l_{s-1}}{h_{r,s}} \le \frac{1}{\delta}.$$

The terms

$$\frac{k_r l_s}{h_{r,s}} \left(\frac{1}{k_r l_s} \sum_{i=1}^{k_r} \sum_{j=1}^{l_s} |y_{i,j} - L| \right) \quad \text{and} \quad \frac{k_{r-1} l_{s-1}}{h_{r,s}} \left(\frac{1}{k_{r-1} l_{s-1}} \sum_{i=1}^{k_{r-1}} \sum_{j=1}^{l_{s-1}} |y_{i,j} - L| \right)$$

are both Pringsheim null sequences. Thus $A_{r,s}$ is a Pringsheim null sequence. Therefore $x = (x_{i,j}) \in \left[N_{\theta_{r,s}}, Z\right]$.

(ii) Suppose that $\limsup q_r < \infty$ and $\limsup \overline{q_s} < \infty$, then there exists $K > 0$ such that $q_r \le K$, $\overline{q_s} \le K$ for all r and s. Let $x = (x_{i,j}) \in \left[N_{\theta_{r,s}}, Z\right]$ and $\varepsilon > 0$.Also there exist $r_o > 0$ and $s_o > 0$ such that for every $k \ge r_o$ and $l \ge s_o$

$$A_{k,l} = \frac{1}{h_{k,l}} \sum_{(i,j) \in I_{k,l}} |y_{i,j} - L| < \varepsilon.$$

Let $M = max \left\{A_{k,l} : \ 1 \le k \le r_o \text{ and } 1 \le l \le s_o\right\}$ and p and q be such that

$$k_{r-1} < p \le k_r \quad \text{and} \quad l_{s-1} < q \le l_s.$$

Thus we obtain the following

$$\frac{1}{pq} \sum_{i,j=1,1}^{p,q} |y_{i,j} - L| \le \frac{1}{k_{r-1} l_{s-1}} \sum_{i=1}^{k_r} \sum_{j=1}^{l_s} |y_{i,j} - L|$$

$$\le \frac{1}{k_{r-1} l_{s-1}} \sum_{p,q=1,1}^{r,s} \left(\sum_{(i,j) \in I_{p,q}} |y_{i,j} - L| \right)$$

$$= \frac{1}{k_{r-1} l_{s-1}} \sum_{p,q=1,1}^{r_o,s_o} h_{p,q} A_{p,q} + \frac{1}{k_{r-1} l_{s-1}} \sum_{(r_o < p \le r) \cup (s_o < q \le s)} h_{p,q} A_{p,q}$$

$$\le \frac{M}{k_{r-1} l_{s-1}} \sum_{p,q=1,1}^{r_o,s_o} h_{p,q} + \frac{1}{k_{r-1} l_{s-1}} \sum_{(r_o < p \le r) \cup (s_o < q \le s)} h_{p,q} A_{p,q}$$

$$\le \frac{M k_{r_o} l_{s_o} r_o s_o}{k_{r-1} l_{s-1}} + \left(\sup_{(p \ge r_o) \cup (q \ge s_o)} A_{p,q} \right) \frac{1}{k_{r-1} l_{s-1}} \sum_{(r_o < p \le r) \cup (s_o < q \le s)} h_{p,q}$$

$$\leq \frac{Mk_{r_o}l_{s_o}r_os_o}{k_{r-1}l_{s-1}} + \varepsilon\frac{1}{k_{r-1}l_{s-1}} \sum_{(r_o<p\leq r)\cup(s_o<q\leq s)} h_{p,q} \leq \frac{Mk_{r_o}l_{s_o}r_os_o}{k_{r-1}l_{s-1}} + \varepsilon K^2.$$

Since k_r and l_s both approaches infinity as both r and s approaches infinity, it follows that

$$P - \lim_{p,q} \frac{1}{pq} \sum_{i,j=1,1}^{p,q} |y_{i,j} - L| = 0.$$

Therefore $x = (x_{i,j}) \in \left[W^2, Z\right]$.

(iii) Combining (i) and (ii) we have the proof of (iii). □

3. Double Zweier lacunary statistical convergence

The following definition was presented by Mursaleen and Edely in [9]:

Definition 3.1. [9] A real double sequence $x = (x_{i,j})$ is said to be *statistically convergent to L,* provided that for each $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} |\{(i,j) : \ i \leq m \ and \ j \leq n, |x_{i,j} - L| \geq \varepsilon\}| = 0$$

where the vertical bars indicate the numbers of elements in the enclosed set.

Recently in [6], Savaş defined double lacunary statistical convergence as follows:

Definition 3.2. [6] A real double sequence $x = (x_{i,j})$ is said to be $S_{\theta_{r,s}}$*-convergent to L,* provided that for each $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(i,j) \in I_{r,s} : \ \ |x_{i,j} - L| \geq \varepsilon\}| = 0.$$

Definition 3.3. A real double sequence $x = (x_{i,j})$ is said to be *double lacunary statistical Zweier convergent to L,* provided that for each $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(i,j) \in I_{r,s} : \ \ |y_{i,j} - L| \geq \varepsilon\}| = 0$$

where $y_{i,j}$ is the form in (1.1). We shall denote the set of all double Zweier lacunary statistical convergent double sequences $x = (x_{i,j})$ by $\left[S_{\theta_{r,s}}, Z\right]$ and if $x = (x_{i,j}) \in \left[S_{\theta_{r,s}}, Z\right]$, then we will write $x_{i,j} \to L\left(\left[S_{\theta_{r,s}}, Z\right]\right)$.

Theorem 3.1. *Let $\theta_{r,s}$ be a double lacunary sequence. If $x_{i,j} \to L\left(\left[N_{\theta_{r,s}}, Z\right]\right)$, then $x_{i,j} \to L\left(\left[S_{\theta_{r,s}}, Z\right]\right)$.*

Proof. If $\varepsilon > 0$ and $x_{i,j} \to L\left(\left[N_{\theta_{r,s}}, Z\right]\right)$ then we can write

$$\begin{aligned} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} |y_{i,j} - L| \ &\geq \ \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s} \ \& \ \left|\frac{1}{2}(x_{i,j}+x_{i,j-1})-L\right|\geq\varepsilon} |y_{i,j} - L| \\ &\geq \ \frac{1}{h_{r,s}} |\{(i,j) \in I_{r,s} : \ \ |y_{i,j} - L| \geq \varepsilon\}| \end{aligned}$$

It follows that $x_{i,j} \to L\left(\left[S_{\theta_{r,s}}, Z\right]\right)$, that is $\left[N_{\theta_{r,s}}, Z\right] \subset \left[S_{\theta_{r,s}}, Z\right]$ and the inclusion is strict. To show this, we can establish an example as follows. □

Example 3.1. Let $y_{i,j}$ is the form in (1.1) and $y=(y_{i,j})$ be defined as follows:

$$y_{i,j}=\begin{pmatrix} 1 & 2 & 3 & ... & \left[\sqrt[3]{h_{r,s}}\right] & 0 & 0 & ... \\ 2 & 2 & 3 & ... & [h_{r,s}] & 0 & 0 & ... \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 2 & \left[\sqrt[3]{h_{r,s}}\right] & \left[\sqrt[3]{h_{r,s}}\right] & ... & \left[\sqrt[3]{h_{r,s}}\right] & 0 & 0 & ... \\ 0 & 0 & 0 & ... & 0 & 0 & 0 & ... \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

It is clear that $x=(x_{i,j})$ is an unbounded double sequence and

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\left|\{(i,j)\in I_{r,s}:\ \ |y_{i,j}-L|\geq\varepsilon\}\right|=P-\lim_{r,s}\frac{\left[\sqrt[3]{h_{r,s}}\right]}{h_{r,s}}=0.$$

Therefore $x_{i,j}\to 0\left(\left[S_{\theta_{r,s}},Z\right]\right)$. But

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}|y_{i,j}|=P-\lim_{r,s}\frac{\left[\sqrt[3]{h_{r,s}}\right]\left(\left[\sqrt[3]{h_{r,s}}\right]\left(\left[\sqrt[3]{h_{r,s}}\right]+1\right)\right)}{2h_{r,s}}=\frac{1}{2}.$$

Therefore $x_{i,j}\nrightarrow 0\left(\left[N_{\theta_{r,s}},Z\right]\right)$. This completes the proof.

Theorem 3.2. *Let $\theta_{r,s}$ be a double lacunary sequence. If $x=(x_{i,j})\in l_\infty^2$ and $x_{i,j}\to L\left(\left[S_{\theta_{r,s}},Z\right]\right)$ then $x_{i,j}\to L\left(\left[N_{\theta_{r,s}},Z\right]\right)$.*

Proof. Suppose that $x=(x_{i,j})\in l_\infty^2$, then there exists a positive integer K such that $|y_{i,j}-L|<K$ for all $i,j\in\mathbb{N}$. Therefore we have, for every $\varepsilon>0$

$$\begin{aligned} P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}|y_{i,j}-L| &= \frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}\ \&\ \left|\frac{1}{2}(x_{i,j}+x_{i,j-1})-L\right|\geq\varepsilon}|y_{i,j}-L| \\ &+ \frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}\ \&\ \left|\frac{1}{2}(x_{i,j}+x_{i,j-1})-L\right|<\varepsilon}|y_{i,j}-L| \\ &\leq \frac{K}{h_{r,s}}\left|\{(i,j)\in I_{r,s}:\ \ |y_{i,j}-L|\geq\varepsilon\}\right|+\varepsilon. \end{aligned}$$

Therefore $x=(x_{i,j})\in l_\infty^2$ and $x_{i,j}\to L\left(\left[S_{\theta_{r,s}},Z\right]\right)$ implies $x_{i,j}\to L\left(\left[N_{\theta_{r,s}},Z\right]\right)$. □

Corollary 3.3. *Let $\theta_{r,s}$ be a double lacunary sequence, then*

$$\left[N_{\theta_{r,s}},Z\right]\cap l_\infty^2=\left[S_{\theta_{r,s}},Z\right]\cap l_\infty^2.$$

Proof. It follows directly from Theorem 3.1. and Theorem 3.2. □

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Fuzzy deductive systems in BE-semigroups

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Abstract. In this paper, we introduce the notion of fuzzy deductive systems and investigate some of their properties. Also we give the construction of quotient self-distributive BE-semigroup X/μ induced by a fuzzy deductive system μ and discuss their interesting pro– perties.

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Key words and phrases. BE-semigroup, deductive system, fuzzy deductive systems.

1. Introduction

Imai and Iséki introduced two classes of abstract algebras, namely, BCK-algebras and BCI-algebras [5], [6]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [11], Neggers and Kim introduced the notion of d-algebras which is a generalization of BCK-algebras. Moreover, Jun et al. [7] introduced a new notion, called a BH-algebra, which is a generalization of BCK/BCI-algebras. Recently, as another generalization of BCK-algebras, the notion of a BE-algebra was introduced by Kim and Kim [9]. They provided an equivalent condition of the filters in BE-algebras using the notion of upper sets. In [2], [3], Ahn and So introduced the notion of ideals in BE-algebras and proved several characterizations of such ideals. In [1], Ahn and Kim combined BE-algebras and semigroups and introduced the notion of BE-semigroups. Also, congruences and BE-Relations in BE-Algebras was studied by Yon et al. [15]. Recently, Handam introduced the notion of BE-homomorphisms between BE-semigroups [4].

The theory of fuzzy sets was first developed by Zadeh [16] and has been applied to many branches in mathematics. The fuzzification of algebraic structures was initiated by Rosenfeld [13] and he introduced the notion of fuzzy subgroups. In 1975, Zadeh [17] introduced the concept of interval valued fuzzy subset, where the values of the membership functions are intervals of numbers instead of the numbers. Later on, Song et al. [14] introduced the concept of a fuzzy ideals in BE-algebras. Recently, Rezaei and Saeid [12] introduced the concepts of fuzzy BE-subalgebras and fuzzy topological BE-algebras. In this paper, we introduce the concept of fuzzy deductive systems and investigate some of their properties. We give the construction of quotient self-distributive BE-semigroup X/μ via a fuzzy deductive system μ. In addition, we establish a generalization of fundamental BE-homomorphism theorem in self-distributive BE-semigroups by using fuzzy deductive systems.

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