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New generalized inequalities using arbitrary operator means and their duals

Leila Nasiri and Mojtaba Bakherad

Abstract. In this article, we present some operator inequalities via arbitrary operator means and unital positive linear maps. For instance, we show that if $A, B \in \mathbb{B}(\mathbb{H})$ are two positive invertible operators such that $0 < m \leq A, B \leq M$ and σ is an arbitrary operator mean, then

$$\Phi^p(A\sigma B) \leq K^p(h)\Phi^p(B\sigma^\perp A),$$

where $\sigma^\perp$ is dual σ, $p \geq 0$ and $K(h) = \frac{(M+m)^2}{4Mm}$ is the classical Kantorovich constant. We also generalize the above inequality for two arbitrary means σ_1, σ_2 which lie between σ and $\sigma^\perp$.

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1. Introduction

In this paper, $\mathbb{B}(\mathbb{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathbb{H}, \langle\cdot,\cdot\rangle)$. I stands for the identity operator. A self-adjoint operator $A \in \mathbb{B}(\mathbb{H})$ is said to be positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{H}$, and in this case we write $A \geq 0$. For self-adjoint operators $A, B \in \mathbb{B}(\mathbb{H})$, the order relation $A \leq B$ means that $B - A \geq 0$. A linear map Φ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital provided that it preserves the identity operator, that is, $\Phi(I) = I$.

The axiomatic theory for pairs of positive operators has been developed by Kubo and Ando [10]. If $A, B \in \mathbb{B}(\mathbb{H})$ are two positive invertible operators, then the $\nu-$weighted arithmetic mean, geometric mean and harmonic mean of A and B are denoted by $A\nabla_\nu B$, $A\sharp_\nu B$ $A!_\nu B$, respectively, and are defined as follows

$$A\nabla_\nu B = \nu A + (1-\nu)B, \qquad A\sharp_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\nu A^{\frac{1}{2}},$$

and

$$A!_\nu B = (\nu A^{-1} + (1-\nu)B^{-1})^{-1}.$$

When $\nu = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$ and $A!B$ for the arithmetic mean, geometric mean and harmonic mean, respectively. The $\nu-$weighted arithmetic-geometric (AM-GM) operator inequality, which is proved in [16] says that if $A, B \in \mathbb{B}(\mathbb{H})$ are two positive operators and $0 \leq \nu \leq 1$, then $A\sharp_\nu B \leq A\nabla_\nu B$. For a particular case, when $\nu = \frac{1}{2}$, we obtain the AM-GM operator inequality

$$A\sharp B \leq \frac{A+B}{2}. \tag{1}$$

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For two positive operators $A, B \in \mathbb{B}(\mathbb{H})$, the Löwner–Heinz inequality states that, if $A \leq B$, then

$$A^p \leq B^p, \qquad (0 \leq p \leq 1). \tag{2}$$

In general (2) is not true for $p > 1$.
Lin [13, Theorem 2.1] showed a squaring of a reverse of (1), namely that the inequality

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^2 \Phi^2(A\sharp B) \tag{3}$$

as well as

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^2 (\Phi(A)\sharp\Phi(B))^2, \tag{4}$$

where Φ is a positive unital linear map.
The Löwner–Heinz inequality and two inequalities (3) and (4) follow that for $0 < p \leq 2$,

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^p \Phi^p(A\sharp B) \tag{5}$$

and

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^p (\Phi(A)\sharp\Phi(B))^p. \tag{6}$$

In [6], the authors showed that inequalities (5) and (6) for $p \geq 2$ hold.
For more improvements and refinements on the above inequalities see [7, 14, 15] and references therein.
Let σ be an operator mean with the representing function f. The operator mean with the representing function $\frac{t}{f(t)}$ is called the dual of σ and denoted by $\sigma^{\perp}$. For $A, B \in \mathbb{B}(\mathbb{H})$,

$$A\sigma^{\perp}B = (B^{-1}\sigma A^{-1})^{-1}.$$

It is trivial that for two invertible operators $A, B \in \mathbb{B}(\mathbb{H})$, $A\nabla^{\perp}B = A!B$ and $A!B \leq A\sharp B$.
Let $0 < m \leq A, B \leq M$, Φ be a positive unital linear map and σ, τ be two arbitrary means between the harmonic and arithmetic means. In [8], the authors obtained the following inequality:

$$\Phi^2(A\sigma B) \leq K^2(h)\Phi^2(A\tau B), \tag{7}$$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.
The authors in [5] generalized inequality (7) for the higher powers as follows:

$$\Phi^p(A\sigma B) \leq K^p(h)\Phi^p(A\tau B), \tag{8}$$

where $p > 0$.
Motivated by the above discussion, in this paper we first obtain the following inequality:

$$\Phi^2(A\sigma B) \leq K^2(h)\Phi^2(B\sigma^{\perp}A), \tag{9}$$

where $0 < m \le A, B \le M$, σ is an arbitrary mean and $\sigma^{\perp}$ is its dual and $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant. Then, we generalize inequality (9) for two arbitrary means σ_1 and σ_2 between σ and $\sigma^{\perp}$.

2. Main results

To obtain the main results we need to recall the following Lemmas.

Lemma 2.1. [3](*Choi's inequality*) *Let* $A \in \mathbb{B}(\mathbb{H})$ *be positive and* Φ *be a positive unital linear map. Then*

$$\Phi(A)^{-1} \le \Phi\left(A^{-1}\right). \tag{10}$$

Lemma 2.2. [16] *Suppose that* $0 < m \le A \le M$. *Then*

$$A + MmA^{-1} \le M + m.$$

Lemma 2.3. [4, 1, 2] *Let* $A, B \in \mathbb{B}(\mathbb{H})$ *be positive and* $\lambda > 0$. *Then*
(i) $||\mathrm{AB}|| \le \frac{1}{4}||\mathrm{A}+\mathrm{B}||^2$.
(ii) *If* $\lambda > 1$, *then* $||A^{\lambda} + B^{\lambda}|| \le ||(A+B)^{\lambda}||$.
(iii) $\mathrm{A} \le \lambda \mathrm{B}$ *if and only if* $||A^{\frac{1}{2}}B^{-\frac{1}{2}}|| \le \lambda^{\frac{1}{2}}$.

Lemma 2.4. [9] *Let* $X \in \mathbb{B}(\mathbb{H})$. *Then* $\|X\| \le t$ *if and only if*

$$\begin{pmatrix} tI & X \\ X^* & tI \end{pmatrix} \ge 0.$$

Theorem 2.5. *Let* $0 < m \le A, B \le M$ *such that* $0 < m < M$ *and* σ *be an arbitrary mean. Then*

$$\Phi^2(A\sigma B) \le K^2(h)\Phi^2(B\sigma^{\perp}A), \tag{11}$$

where $\sigma^{\perp}$ *is dual* σ *and* $K(h) = \frac{(M+m)^2}{4Mm}$ *is the Kantorovich constant.*

Proof. It follows from $0 < m \le A, B \le M$ that $(M-A)(m-A)A^{-1} \le 0$ and $(M-B)(m-B)B^{-1} \le 0$. Therefore

$$A + MmA^{-1} \le M + m \quad \text{and} \quad B + MmB^{-1} \le M + m.$$

Now, the subadditivity and monotonicity properties of the operator mean to conclude that

$$\begin{aligned} A\sigma B + Mm(A^{-1}\sigma B^{-1}) &\le (A + MmA^{-1})\sigma(B + MmB^{-1}) \\ &\le (M+m)\sigma(M+m) \\ &= M + m. \end{aligned}$$

Using the linearity and positivity of Φ and the latter inequality, we get

$$\Phi(A\sigma B) + Mm\Phi(A^{-1}\sigma B^{-1}) \le M + m. \tag{12}$$

Applying two inequalities (10) and (12), respectively, we have

$$\begin{aligned} \Phi(A\sigma B) + Mm\Phi^{-1}(B\sigma^{\perp}A) &\le \Phi(A\sigma B) + Mm\Phi(B\sigma^{\perp}A)^{-1} \\ &\le \Phi(A\sigma B) + Mm\Phi(A^{-1}\sigma B^{-1}) \\ &\le M + m. \end{aligned}$$

By Lemma 2.3(i) and the latter inequality, we get

$$\begin{aligned}\left\|\Phi(A\sigma B)Mm\Phi^{-1}(B\sigma^{\perp}A)\right\| &\le \frac{1}{4}\left\|\Phi(A\sigma B)+Mm\Phi(B\sigma^{\perp}A)^{-1}\right\|^2\\ &\le \Phi(A\sigma B)+Mm\Phi(A^{-1}\sigma B^{-1})\\ &\le M+m.\end{aligned}$$

This proves the assertion as desired. □

Remark 2.1. In special case, when $\sigma=\nabla$, since $\sigma^{\perp}=!$ and $!\le\sharp$, inequality (11) becomes inequality (3).

Corollary 2.6. *Let $0<m\le A,B\le M$ such that $0<m<M$, σ be an arbitrary mean and let $p\ge 0$. Then*

$$\Phi^p(A\sigma B)\le K^p(h)\Phi^p(B\sigma^{\perp}A), \tag{13}$$

where $\sigma^{\perp}$ is dual σ and $K(h)=\frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.

Proof. If $0\le p\le 2$, then $0\le\frac{p}{2}\le 1$. Applying inequality (11) we obtain the desired result. If $p>2$, then

$$\begin{aligned}&\left\|\Phi^{\frac{p}{2}}(A\sigma B)M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(B\sigma^{\perp}A)\right\|\\ &\le\frac{1}{4}\left\|\Phi^{\frac{p}{2}}(A\sigma B)+M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(B\sigma^{\perp}A)\right\|^2 \ \text{(by Lemma 2.3 (i))}\\ &\le\frac{1}{4}\left\|\Phi(A\sigma B)+Mm\Phi^{-1}(B\sigma^{\perp}A)\right\|^p \ \text{(by Lemma 2.3 (ii))}\\ &\le\frac{1}{4}\left\|\Phi(A\sigma B)+Mm\Phi((B\sigma^{\perp}A))^{-1}\right\|^p \ \text{(by (10))}\\ &=\frac{1}{4}\left\|\Phi(A\sigma B)+Mm\Phi(A^{-1}\sigma^{\perp}B^{-1})\right\|^p\\ &\le\frac{1}{4}(M+m)^p \ \text{(by inequality (12))}.\end{aligned}$$

Therefore, by Lemma 2.3(iii) we have

$$\Phi^p(A\sigma B)\le K^p(h)\Phi^p(B\sigma^{\perp}A).$$

□

Remark 2.2. Using the same reason as in Remark 2.1 says that inequality (13) is a generalization of inequality (5) which is presented in [6].

In the following theorem, we generalize inequality (7).

Theorem 2.7. *Let $0<m\le A,B\le M$, σ_1 and σ_2 be two arbitrary means which lie between σ and $\sigma^{\perp}$ and let $p\ge 0$. Then for every positive unital linear map Φ,*

$$\Phi^p(A\sigma_2 B)\le K^p(h)\Phi^p(B\sigma_1 A), \tag{14}$$

where $\sigma^{\perp}$ is dual σ and $K(h)=\frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.

Proof. To prove (14), let $\sigma_1 \geq \sigma^{\perp}$ and $\sigma_2 \leq \sigma$. Therefore,

$$\begin{aligned}
\Phi(A\sigma_2 B) + Mm\Phi^{-1}(B\sigma_1 A) &\leq \Phi(A\sigma_2 B) + Mm\Phi(B\sigma_1 A)^{-1} \text{ (by (10))} \\
&\leq \Phi(A\sigma B) + Mm\Phi(B\sigma^{\perp} A)^{-1} \\
&= \Phi(A\sigma B) + Mm\Phi(A^{-1}\sigma B^{-1}) \\
&\leq M + m \text{ (by (12))}.
\end{aligned}$$

Using the same ideas as used in the proof of Theorem 2.5 and Corollary 2.6, one can obtain the desired result. □

To find a better bound than the obtained bound in inequality (13), we need to state the following Lemma.

Lemma 2.8. [13] *Let $0 < m \leq A, B \leq M$ and σ be an arbitrary mean. Then for every positive unital linear map Φ*

$$\|\Phi^2(A\sigma B) + M^2m^2\Phi^n((A\sigma B)^{-1})\| \leq M^2 + m^2.$$

Theorem 2.9. *Let $0 < m \leq A, B \leq M$, σ be an arbitrary mean and $p \geq 4$. Then*

$$\Phi^p(A\sigma B) \leq \left(\frac{K(h)(M^2+m^2)}{2^{\frac{4}{p}}Mm}\right)^p \Phi^p(B\sigma^{\perp}A), \tag{15}$$

where $\sigma^{\perp}$ is dual σ and $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.

Proof. By Theorem 2.5 we have

$$\Phi^{-2}(B\sigma^{\perp}A) \leq K^2(h)\Phi^{-2}(A\sigma B). \tag{16}$$

A simple computation shows that

$$\begin{aligned}
&\left\|\Phi^{\frac{p}{2}}(A\sigma B)M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(B\sigma^{\perp}A)\right\| \\
&\leq \frac{1}{4}\left\|K^{\frac{p}{4}}(h)\Phi^{\frac{p}{2}}(A\sigma B) + \left(\frac{M^2m^2}{K(h)}\right)^{\frac{p}{4}}\Phi^{-\frac{p}{2}}(B\sigma^{\perp}A)\right\|^2 \text{ (by Lemmas 2.3(i))} \\
&\leq \frac{1}{4}\left\|K\Phi^2(A\sigma B) + \frac{M^2m^2}{K(h)}\Phi^{-2}(B\sigma^{\perp}A)\right\|^{\frac{p}{2}} \text{ (by Lemmas 2.3(ii))} \\
&\leq \frac{1}{4}\left\|K(h)\Phi^2(A\sigma B) + M^2m^2K(h)\Phi^{-2}(A\sigma B)\right\|^{\frac{p}{2}} \text{ (by (16))} \\
&\leq \frac{1}{4}K^{\frac{p}{2}}(h)\left\|\Phi^2(A\sigma B) + M^2m^2\Phi^2(A\sigma B)^{-1}\right\|^{\frac{p}{2}} \text{ (by (10))} \\
&\leq \frac{1}{4}\left(K(h)\left(M^2+m^2\right)\right)^{\frac{p}{2}} \text{ (by Lemma 2.8)}.
\end{aligned}$$

Therefore

$$\left\|\Phi^{\frac{p}{2}}(A\sigma B)\Phi^{-\frac{p}{2}}(B\sigma^{\perp}A)\right\| \leq \frac{1}{4}\left(\frac{K(h)\left(M^2+m^2\right)}{Mm}\right)^{\frac{p}{2}}.$$

The latter relation is equivalent to

$$\Phi^p(A\sigma B) \leq \left(\frac{K(h)\left(M^2+m^2\right)}{2^{\frac{4}{p}}Mm}\right)^p \Phi^p(B\sigma^{\perp}A).$$

This proves the desired result. □

Remark 2.3. When $p \geq 4$, the derived result in Theorem 2.9 is tighter than inequality (13).

Moreover, we show that Theorem 2.9 holds for $0 \leq p \leq 4$.

Corollary 2.10. *Let $0 < m \leq A, B \leq M$, σ be an arbitrary mean and let $0 \leq p \leq 4$. Then*

$$\Phi^p(A\sigma B) \leq \left(\frac{K(h)\left(M^2+m^2\right)}{2Mm}\right)^p \Phi^p(B\sigma^{\perp}A),$$

where $\sigma^{\perp}$ is dual σ and $K(h) = \frac{(M+m)^2}{4MM}$.

Proof. By Theorem 2.5 we have

$$\Phi^4(A\sigma B) \leq \left(\frac{K(h)\left(M^2+m^2\right)}{2Mm}\right)^4 \Phi^4(B\sigma^{\perp}A).$$

If $0 \leq p \leq 4$, then $0 \leq \frac{p}{4} \leq 1$. With the aid of the latter inequality and inequality (2), we conclude the desired inequality. □

Theorem 2.11. *Let $0 < m \leq A, B \leq M$, σ_1 and σ_2 be two arbitrary means between σ and $\sigma^{\perp}$, $1 < \alpha \leq 2$ and $p \geq 2\alpha$. Then for every positive unital linear map Φ*

$$\Phi^p(A\sigma_2 B) \leq \frac{(K^{\frac{\alpha}{2}}(h)(M^\alpha+m^\alpha))^{\frac{2p}{\alpha}}}{16M^pm^p}\Phi^p(B\sigma_1 A) \tag{17}$$

where $\sigma^{\perp}$ is dual σ and $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.

Proof. At once from inequality (14) follows that for $1 < \alpha \leq 2$

$$\Phi^{-\alpha}(B\sigma_1 A) \leq K^\alpha(h)\Phi^{-\alpha}(A\sigma_2 B). \tag{18}$$

Using the fact that $0 < m \leq A, B \leq M$, it deduces that $0 < m \leq A\sigma_2 B \leq M$. Now, the linearity property Φ results that $0 < m \leq \Phi(A\sigma_2 B) \leq M$. Since $1 < \alpha \leq 2$, one can easily prove that

$$\Phi^\alpha(A\sigma_2 B) + M^\alpha m^\alpha \Phi^{-\alpha}(A\sigma_2 B) \leq M^\alpha + m^\alpha. \tag{19}$$

Therefore

$$\begin{aligned}
&\left\|M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{\frac{p}{2}}(A\sigma_2 B)\Phi^{-\frac{p}{2}}(B\sigma_1 A)\right\| \\
&\leq \frac{1}{4}\left\|K^{-\frac{p}{4}}(h)M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(B\sigma_1 A) + K^{\frac{p}{4}}(h)\Phi^{\frac{p}{2}}(A\sigma_2 B)\right\|^2 \text{ (by Lemma 2.3(i))} \\
&\leq \frac{1}{4}\left\|\left(K^{-\frac{\alpha}{2}}(h)M^\alpha m^\alpha\Phi^{-\alpha}(B\sigma_1 A) + K^{\frac{\alpha}{2}}(h)\Phi^\alpha(A\sigma_2 B)\right)^{\frac{p}{2\alpha}}\right\|^2 \text{ (by Lemma 2.3(ii))} \\
&= \frac{1}{4}\left\|K^{-\frac{\alpha}{2}}(h)M^\alpha m^\alpha\Phi^{-\alpha}(B\sigma_1 A) + K^{\frac{\alpha}{2}}(h)\Phi^\alpha(A\sigma_2 B)\right\|^{\frac{p}{\alpha}} \\
&\leq \frac{1}{4}\left\|K^{\frac{\alpha}{2}}(h)M^\alpha m^\alpha\Phi^{-\alpha}(A\sigma_2 B) + K^{\frac{\alpha}{2}}(h)\Phi^\alpha(A\sigma_2 B)\right\|^{\frac{p}{\alpha}} \text{ (by (18))} \\
&\leq \frac{1}{4}K^{\frac{p}{2}}(h)(M^\alpha+m^\alpha)^{\frac{p}{\alpha}} \text{(by (19))},
\end{aligned}$$

that is

$$\left\|\Phi^{\frac{p}{2}}(A\sigma_2 B)\Phi^{-\frac{p}{2}}(B\sigma_1 A)\right\| \leq \frac{K^{\frac{p}{2}}(h)(M^\alpha+m^\alpha)^{\frac{p}{\alpha}}}{4M^{\frac{p}{2}}m^{\frac{p}{2}}},$$

or equivalently

$$\Phi^p(A\sigma_2 B) \leq \frac{\left(K^{\frac{\alpha}{2}}(h)(M^\alpha + m^\alpha)\right)^{\frac{2p}{\alpha}}}{16Mm}\Phi^p(B\sigma_1 A).$$

□

Remark 2.4. In special case, for $\alpha = 2$, inequality (17) becomes inequality (15).

Remark 2.5. By taking $\sigma = \nabla$ in inequality (17), we get inequality (8).

Theorem 2.12. *Let $0 < m \leq A, B \leq M$ such that $0 < m < M$ and σ be an arbitrary mean. Then for every positive unital linear map Φ and two arbitrary means σ_1 and σ_2 which lie between σ and $\sigma^\perp$ and $p \geq 0$, the following inequality holds*

$$\Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A) + \Phi^{-p}(B\sigma_1 A)\Phi^p(A\sigma_2 B) \leq 2K^p(h)\Phi^p(B\sigma_1 A) \tag{20}$$

where $\sigma^\perp$ is dual σ and $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.

Proof. It follows from (14) that

$$\|\Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A)\| \leq K^p(h). \tag{21}$$

Applying Lemma 2.4 we have

$$\begin{pmatrix} K(h)^p I & \Phi^{-p}(B\sigma_1 A)\Phi^p(A\sigma_2 B) \\ \Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A) & K(h)^p I \end{pmatrix} \geq 0$$

and

$$\begin{pmatrix} K(h)^p I & \Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A) \\ \Phi^{-p}(B\sigma_1 A)\Phi^p(A\sigma_2 B) & K(h)^p I \end{pmatrix} \geq 0.$$

Summing up two above inequalities, we obtain the following inequality

$$\begin{pmatrix} 2K(h)^p I & \beta_1 \\ \beta_2 & 2K(h)^p I \end{pmatrix} \geq 0,$$

where

$$\beta_1 = \Phi^{-p}(B\sigma_1 A)\Phi^p(A\sigma_2 B) + \Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A)$$

and

$$\beta_2 = \Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A) + \Phi^{-p}(B\sigma_1 A)\Phi^p(A\sigma_2 B).$$

Again using Lemma 2.4 we get the desired result. □

Remark 2.6. Put $\sigma = \nabla$, inequality (20) reduces to some results in [2]

3. A refined inequality for the arithmetic-geometric mean

Let $A, B \in \mathbb{B}(\mathbb{H})$ be two invertible positive operators, $0 \leq \nu \leq 1$ and $-1 \leq q \leq 1$. We use from the notation $A\sharp_{q,\nu}B$ to define the power mean

$$A\sharp_{q,\nu}B = A^{\frac{1}{2}}\left((1-\nu)I + \nu\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)^q\right)^{\frac{1}{q}}A^{\frac{1}{2}}.$$

For more information see [11]. The authors in [12] proved that if $0 < m \leq A, B \leq M$ such that $0 < m < M$ and $0 < \nu \leq \mu < 1$, $-1 \leq q \leq 1$. Then for every positive unital linear map Φ and $p \geq 0$, the following inequality holds

$$\Phi^p\left(A\nabla_\nu B + \frac{\nu}{\mu}Mm\left(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_{q,\mu}B^{-1}\right)\right) \leq K^p(h)\Phi^p(A\sharp_{q,\nu}B), \tag{22}$$

where $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.
Using the following theorem, we obtain a generalization from inequality (22).

Theorem 3.1. *Suppose that $0 < m \leq A, B \leq M$ such that $0 < m < M$ and $0 < \nu \leq \mu < 1$, $-1 \leq q \leq 1$ and $1 < \alpha \leq 2$. Then for every positive unital linear map Φ and $p \geq 0$, the following inequality holds*

$$\Phi^p\left(A\nabla_\nu B + \frac{\nu}{\mu}Mm(A^{-1}\nabla_\nu B^{-1} - A^{-1}\sharp_{q,\mu}B^{-1})\right) \leq \frac{\left(K^{\frac{\alpha}{4}}(h)(M^\alpha + m^\alpha)\right)^{\frac{2p}{\alpha}}}{16M^pm^p}\Phi^p(A\sharp_{q,\nu}B), \tag{23}$$

where $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.

Proof. For $1 < \alpha \leq 2$, by inequality (22), we have

$$\Phi^\alpha\left(A\nabla_\nu B + \frac{\nu}{\mu}Mm\left(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_{q,\mu}B^{-1}\right)\right) \leq K^\alpha(h)\Phi^\alpha(A\sharp_{q,\nu}B) \tag{24}$$

The last inequality deduces using a process similar to inequality (19). This shows that

$$\left\|\Phi^{\frac{p}{2}}\left(A\nabla_\nu B + \frac{\nu}{\mu}Mm\left(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_{q,\mu}B^{-1}\right)\right)\Phi^{-\frac{p}{2}}(A\sharp_{q,\nu}B)\right\| \leq \frac{K^{\frac{p}{2}}(h)(M^\alpha + m^\alpha)^{\frac{p}{\alpha}}}{4M^{\frac{p}{2}}m^{\frac{p}{2}}}.$$

Then

$$\Phi^p\left(A\nabla_\nu B + \frac{\nu}{\mu}Mm\left(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_{q,\mu}B^{-1}\right)\right) \leq \frac{\left(K^{\frac{\alpha}{4}}(h)(M^\alpha + m^\alpha)\right)^{\frac{2p}{\alpha}}}{16M^pm^p}\Phi^p(A\sharp_{q,\nu}B).$$

□

Remark 3.1. Taking $\alpha = 2$, inequality (23) becomes inequality (22).

Remark 3.2. By putting $\alpha = 2, \mu = \frac{1}{2}$ and taking $q \to 0$, inequality (23) collapse to the derived result in [2].

References

[1] T. Ando and X. Zhan, Norm inequalities related to operator monoton functions, *Math. Ann.* **315** (1999), 771–780.

[2] M. Bakherad, Refinements of a reversed AM-GM operator inequality, *Linear Multilinear Algebra* **64** (2016), no. 9, 1687–1695.

[3] R. Bhatia, *Positive Definite Matrices*, Princeton University Press, Princeton, 2007.
[4] R. Bhatia and F. Kittaneh, Notes on matrix arithmetic-geometric mean inequalities, *Linear Algebra Appl.* **308** (2000), no. 1-3, 203–211.
[5] X. Fu and D.T. Hoa, On some inequalities with matrix means, *Linear Multilinear Algebra* **63** (2015), no. 12, 2373–2378.
[6] X. Fu and C. He, Some operator inequalities for positive linear maps, *Linear Multilinear Algebra* **63** (2015), no. 3, 571–577.
[7] M. Hajmohamadi, R. Lashkaripour, and M. Bakherad, Some matrix power and Karcher means inequalities involving positive linear maps, *Filomat* **32** (2018), no. 7, 2625–2634.
[8] D.T. Hoa, D.T.H. Binh, and H.M. Toan, On some inequalities with matrix means, *RIMS Kokyukoku* **1893** (2014), 67–71, Kyoto University.
[9] R.A. Horn and C.R. Johson, *Topics in matrix analysis*, Cambridge, Cambridge University Press, 1991.
[10] F. Kubo, T. Ando, Means of positive linear operators, *Math. Ann.* **246** (1980), 205–224.
[11] M. Khosravi, Some matrix inequalities for weighted power mean, *Ann. Funct. Anal.* **7** (2016), no. 2, 348–357.
[12] M. Khosravi, M.S. Moslehian, and A. Sheikhhosseini, Some operator inequalities involving operator means and positive linear maps, *Linear Multilinear Algebra* **66** (2018), no. 6, 1186–1198.
[13] M. Lin, Squaring a reverse AM-GM inequality, *Studia Math.* **215** (2013), no. 2, 187–194.
[14] L. Nasiri and M. Bakherad, *Improvements of some operator inequalities involving positive linear maps via the Kantorovich constant*, *Houston J. Math.* **76** (2018), no. 1, 1–16.
[15] L. Nasiri and W. Liao, The new reverses of Young type inequalities for numbers, matrices and operators, *Oper. Matrices* **12** (2018), no. 4, 1063–1071.
[16] T. Furuta, J. Mićić Hot, J. Pečarić, and Y. Seo, *Mond Pečarić method in operator inequalities*, Zagreb, 2005.

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Matrix map between complex uncertain sequences

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Abstract. In this article we define matrix maps between complex uncertain sequences. We introduce the notion of bounded sequences of complex uncertain sequence for almost sure, mean, measure and distribution. We introduce the limitation method for different notion of boundedness of sequence of complex uncertain variables and establish relation between the different notions. The necessary condition for a matrix map to be a limitation method is established.

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1. Introduction

When uncertainty acts neither fuzziness nor randomness, we are unable to deal with this kind of uncertainty by fuzzy set theory or probability theory. In order to handle this, an uncertainty theory was established by Liu [1]. At the same time, Liu proposed convergence concepts in different notion and established relationship between them. Up to now, uncertainty theory has successfully been applied to uncertain programming, uncertain risk analysis and uncertain reliability analysis, uncertain logic, uncertain differential equation, uncertain graphs etc. In our daily life, uncertainty became noticeable in real quantities as well as in complex quantities. The concept of complex uncertain variable was introduced by Peng [9] and after that it has been applied by Chen et al. [8], Tripathy and Dowari [4], Tripathy and Nath [7], Nath and Tripathy [3] for studying sequences of complex uncertain variables. The limitation method was founded in the book of Petersen [6], limitation method is a linear transformation which turns bounded sequences to a bounded sequences. One may refer to Petersen [6], for further details about the limitation method.

2. Preliminaries

In this section, we introduced some basic concepts on uncertainty theory which will be used throughout the paper.

Definition 2.1. ([1]) Let $\mathcal{L}$ be a $\sigma-$algebra on a nonempty set Γ. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following axioms:
Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$;

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Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
Axiom 3. (Subadditivity Axiom) For every countable sequence of $\{\Lambda_j\} \in \mathcal{L}$, we have

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \bigwedge_{j}\right\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space, and each element Λ in $\mathcal{L}$ is called an event.In order to obtain uncertainty measure of compound event, a product uncertain measure is defined by Liu [1] as follows:
Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$, be uncertainty spaces for $k = 1, 2, ...$. The product uncertainty measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, ...$, respectively.

Definition 2.2. ([1]) An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers i.e., for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ is an event.

In this paper, we introduce the notion of bounded sequences of complex uncertain sequence for almost sure, mean, measure and distribution. We introduce the limitation method for different notion of boundedness of sequence of complex uncertain variables.

Definition 2.3. Let (ζ_n) be a sequence of complex uncertain variables in the uncertainty space $(\Gamma, L, \mathcal{M})$. Then, the sequence (ζ_n) is almost surely bounded if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ and there exists $H > 0$ such that $\sup_n \|\zeta_n(\gamma)\| < H$ for every $\gamma \in \Lambda$.

Definition 2.4. A complex uncertain sequence (ζ_n) is bounded in measure, if there exists $H > 0$ such that $\sup_n \mathcal{M}\{\|\zeta_n\| > H\} = 0$.

Definition 2.5. A complex uncertain sequence (ζ_n) is bounded in mean, if there exists $H > 0$ such that $\sup_n E[\|\zeta_n\|] < H$.

Definition 2.6. Let, $\phi, \phi_1, \phi_2, ...$ be the complex uncertain distribution of complex uncertain variables $\xi_1, \xi_2, ...$, respectively. We say the complex uncertain sequence (ζ_n) is bounded in distribution if $\sup_n \phi_n(c) \leq \phi(c)$ for all c at which $\phi(c)$ is continuous.

Definition 2.7. A linear transformation T from a complex uncertain sequence into complex uncertain sequence is said to be almost sure limitation method if it transforms bounded complex uncertain sequence into a bounded complex uncertain sequence i.e there exists constants $G, H > 0$ such that $\sup_n \|T(\zeta_n(\gamma))\| < G$ for every $\sup_n \|\zeta_n(\gamma)\| < H$ in $(\Gamma, L, \mathcal{M})$.

Definition 2.8. A linear transformation is said to be limitation method in measure if

$\sup_n \mathcal{M}\{\|T(\zeta_n)\| > G\} = 0$ for every $\sup_n \mathcal{M}\{\|\zeta_n\| > H\} = 0$.

Definition 2.9. A linear transformation is said to be limitation method in mean, if $\sup_n E[\|T(\zeta_n)\|] < G$ for every $\sup_n E[\|\zeta_n\|] < H$.

Definition 2.10. Let, $\phi, \phi_1, \phi_2, ...$ be the complex uncertain distribution of complex uncertain variables $\zeta_1, \zeta_2, ...$ respectively. We say the linear transformation T is limitation method in distribution, if $\sup_n \phi_n(T(c)) \leq \phi(T(c))$ for every $\sup_n \phi_n(c) \leq \phi(c)$ for all c at which $\phi(c)$ and $\phi(T(c))$ is continuous.

3. Main results

In this section, some relationships among the Limitation methods will be studied and derive some new results for a matrix map to be a limitation methods.

Theorem 3.1. *If the linear transformation T is limitation method in mean, then T is a limitation method in measure.*

Proof. From the definition of limitation method in mean it follows that, $\sup_n E\left(\|T(\zeta_n)\|\right) < G$ whenever $\sup_n E\left(\|\zeta_n\|\right) < H$.

For any $\frac{H}{\varepsilon} > 0$, we have from the Markov inequality,

$$\mathcal{M}\{\|\zeta_n\| \geq \frac{H}{\varepsilon}\} \leq \frac{E[\|\zeta_n\|]}{\frac{H}{\varepsilon}} < \varepsilon,$$

since ε is arbitrary.

Therefore, $\sup_n \mathcal{M}\left\{\|\zeta_n\| > \frac{H}{\varepsilon}\right\} = 0.$

For any $\frac{G}{\varepsilon} > 0$, we have from the Markov inequality,

$$\mathcal{M}\{\|T(\zeta_n)\| \geq \frac{G}{\varepsilon}\} \leq \frac{E[\|T(\zeta_n)\|]}{\frac{G}{\varepsilon}} < \varepsilon,$$

since ε is arbitrary.

Therefore, $\sup_n \mathcal{M}\left\{\|T(\zeta_n)\| > \frac{G}{\varepsilon}\right\} = 0.$ □

Remark 3.1. Bounded in measure does not imply bounded in mean in general.

Example 3.1. Consider an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\gamma_1, \gamma_2, ...$ with

$$M\{\Lambda\} = \begin{cases} \sup \dfrac{1-n}{n^2}, & \text{if } \sup\limits_{\gamma_n \in \Lambda} \dfrac{1-n}{n^2} < \dfrac{1}{2}; \\ 1 - \sup \dfrac{1-n}{n^2}, & \text{if } \sup\limits_{\gamma_n \in \Lambda^c} \dfrac{1-n}{n^2} < \dfrac{1}{2}; \\ \dfrac{1}{2}, & \text{otherwise.} \end{cases}$$

Then, we define complex uncertain variables by,

$$\zeta_n(\gamma) = \begin{cases} in^2(n^2+n-1), & \text{if } \gamma = \gamma_n; \\ 0, & \text{otherwise,} \end{cases}$$

for $n = 1, 2....$ and $\zeta_n = 0$. For some small number $\varepsilon > 0$, we have,

$$\begin{aligned} \mathcal{M}\{\|\zeta_n\| > \varepsilon\} &= \mathcal{M}\{\gamma | \|\zeta_n(\gamma)\| > \varepsilon\} \\ &= \mathcal{M}\{\gamma_n\} = \frac{1-n}{n^2}. \end{aligned}$$

Therefore, $\sup_n \mathcal{M}\{\|\zeta_n\| > \varepsilon\} = 0$. However, for each $n \geq 2$, the complex uncertain distribution of ζ_n,

$$\phi_n(c) = \phi_n(a+ib) = \begin{cases} 0, & \text{if } a < 0, b < +\infty; \\ 0, & \text{if } a \geq 0, b < 0; \\ \frac{1-n}{n^2}, & \text{if } a \geq 0, 0 \leq b < (n^2+n-1); \\ 1, & \text{if } a \geq 0, b \geq (n^2+n-1). \end{cases}$$

So for each $n \geq 2$, we have

$$E[\|\zeta_n\|] = \int_0^{(n^2+n-1)n^2} (1 - \frac{1-n}{n^2})dx \; = \; (n^2+n-1)^2.$$

Therefore, $\sup_n E[\|\zeta_n\|]$ is infinite.

Theorem 3.2. *Let (ζ_n) be a sequence of complex uncertain variables. Then, (ζ_n) is bounded in measure if its real and imaginary parts are bounded in measure.*

Proof. Let,(ξ_n) and (η_n) be real and imaginary parts of (ζ_n) respectively which are bounded in measure. From the definition of bounded in measure, there exists $H > 0$ such that

$$\sup_n \mathcal{M}\left\{\|\xi_n\| > \frac{H}{\sqrt{2}}\right\} = 0,$$

and

$$\sup_n \mathcal{M}\left\{\|\eta_n\| > \frac{H}{\sqrt{2}}\right\} = 0.$$

Also,

$$\|\zeta_n\| = \sqrt{|\xi_n|^2 + |\eta_n|^2}.$$

Thus, we have

$$\{\|\zeta_n\| > H\} \subset \left\{|\xi_n| > \frac{H}{\sqrt{2}}\right\} \bigcup \left\{|\eta_n| > \frac{H}{\sqrt{2}}\right\}.$$

Using sub additivity axiom of uncertain measure, we obtain

$$\mathcal{M}\{\|\zeta_n\| > H\} \leq \mathcal{M}\left\{|\xi_n| > \frac{H}{\sqrt{2}}\right\} + \mathcal{M}\left\{|\eta_n| > \frac{H}{\sqrt{2}}\right\}.$$

Hence, we have

$$0 \leq \sup \mathcal{M}\{\|\zeta_n\| > H\} \leq \sup \mathcal{M}\left\{|\xi_n| > \frac{H}{\sqrt{2}}\right\} + \sup \mathcal{M}\left\{|\eta_n| > \frac{H}{\sqrt{2}}\right\} = 0.$$

Therefore,

$$\sup \mathcal{M}\{\|\zeta_n\| > H\} = 0.$$

Hence, (ζ_n) is bounded in measure. □

Theorem 3.3. *Let, T be a linear transformation. Then, T is limitation method in measure if its real and imaginary parts are bounded in measure.*

Proof. The proof is similar to Theorem 3.2, so it is omitted. □

Theorem 3.4. *Let (ζ_n) be a complex uncertain sequence with real and imaginary parts (ξ_n) and (η_n), respectively, for $n = 1, 2.....$ If uncertain sequences (ξ_n) and (η_n) bounded in measure, then the complex uncertain sequence (ζ_n) bounded in distribution.*

Proof. Let, $c = a + ib$ be a given continuity point of the complex uncertainty distribution ϕ.

For any $\alpha > a, \beta > b$, we have,

$$\begin{aligned}\{\xi_n \leqslant a, \eta_n \leqslant b\} = &\{\xi_n \leqslant a, \eta_n \leqslant b, \xi_k \leqslant \alpha, \eta_k \leqslant \beta\} \bigcup \{\xi_n \leqslant a, \eta_n \leqslant b, \xi_k > \alpha, \eta_k > \beta\} \\ &\bigcup \{\xi_n \leqslant a, \eta_n \leqslant b, \xi_k \le \alpha, \eta_k > \beta\} \bigcup \{\xi_n \leqslant a, \eta_n \leqslant b, \xi_k > \alpha, \eta_k \le \beta\} \\ &\subset \{\xi_k \le \alpha, \eta_k \le \beta\} \bigcup \{|\xi_n - \xi_k| \ge \alpha - a\} \bigcup \{|\xi_n - \zeta_k| \ge \beta - b\}.\end{aligned}$$

$$\begin{aligned}\phi_n(c) &= \phi_n(a+ib) \\ &\le \phi(\alpha + i\beta) + \mathcal{M}\{|\xi_n - \xi_k| \ge \alpha - a\} + \mathcal{M}\{|\xi_n - \eta_k| \ge \beta - b\} \\ &\le \phi(\alpha + i\beta) + \mathcal{M}\{|\xi_n| \ge \frac{\alpha - a}{2}\} + \mathcal{M}\{|\xi_k| \ge \frac{\alpha - a}{2}\} + \mathcal{M}\{|\eta_n| \ge \frac{\beta - b}{2}\} \\ &\quad + \mathcal{M}\{|\eta_k| \ge \frac{\beta - b}{2}\}.\end{aligned}$$

Since (ξ_n) and (η_n) are bounded in measure, we have

$$\sup_n \mathcal{M}\{|\xi_n| \ge \frac{\alpha - a}{2}\} = 0, \ \sup_n \mathcal{M}\{|\xi_k| \ge \frac{\alpha - a}{2}\} = 0,$$

and

$$\sup_n \mathcal{M}\{|\eta_n| \ge \frac{\beta - b}{2}\} = 0, \ \sup_n \mathcal{M}\{|\eta_k| \ge \frac{\beta - b}{2}\} = 0.$$

Thus, we obtain

$$\sup_n \phi_n(c) \leqslant \phi(\alpha + i\beta),$$

for any $\alpha > a, \ \beta > b$. Taking $\alpha + i\beta \longrightarrow a + ib$, we have

$$\sup_n \phi_n(c) \leqslant \phi(c).$$

□

Remark 3.2. Bounded in distribution does not imply bounded in measure in general.

Example 3.2. Consider an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\gamma_1, \gamma_2, ...$ with

$$M\{\Lambda\} = \begin{cases} \sup \dfrac{n}{2n+1}, & \text{if } \sup\limits_{\gamma_n \in \Lambda} \dfrac{n}{2n+1} < \dfrac{1}{2}; \\ 1 - \sup \dfrac{n}{2n+1}, & \text{if } \sup\limits_{\gamma_n \in \Lambda^c} \dfrac{n}{2n+1} < \dfrac{1}{2}; \\ \dfrac{1}{2}, & \text{otherwise.} \end{cases}$$

Then, we define complex uncertain variables by,

$$\zeta_n(\gamma) = \begin{cases} in, & \text{if } \gamma = \gamma_n, \\ 0, & \text{otherwise,} \end{cases}$$

for $n = 1, 2....$ and $\zeta_n = 0$. For some small number $\varepsilon > 0$, we have,

$$\mathcal{M}\{\|\zeta_n\| \ge \varepsilon\} = \mathcal{M}\{\gamma | \|\zeta_n(\gamma)\| \ge \varepsilon\} = \mathcal{M}\{\gamma_n\} = \frac{n}{2n+1}.$$