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Existence and stability results for fixed points of multivalued F contractions and application to Volterra type non homogeneous integral equation of second kind

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Abstract. In this paper we introduce multivalued modified F-contraction on a metric space. This is a multivalued mapping obtained by incorporating the idea of the recently introduced F-contraction which has attracted much attention in contemporary research. We explore the fixed point problem associated with the above contractive mapping. We also investigate the data dependence and stability properties of the fixed point sets associated with these multivalued contractions. We discuss an illustration of the main result and present an application of the single valued version of the main theorem to a problem of an integral equation of Volterra type. The domain of the study is fixed point theory and set valued analysis.

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Key words and phrases. Metric space; Hausdorff distance; fixed point; data dependency; stability.

1. Mathematical background and preliminaries

Our consideration in this paper is a study related to fixed points of some multivalued operators on metric spaces. We use rational terms in our inequality. The use of rational terms in contraction inequalities in the domain of metric fixed point theory was initiated by Dass et al. in their work [18] in which they extended the Banach's contraction principle [7] by using a contractive rational inequality. After that the rational inequalities have been used in fixed point and related problems in several works as for instances in [6, 10, 11, 28].

Fixed points of multivalued mappings have been treated extensively in its various aspects. An early reference in this direction is due to Nadler [31] in which the Banach contraction principle [7] is extended to the domain of setvalued analysis. Some references of fixed point results of multivalued mappings are noted in [2, 8, 36].

In the present paper we consider a multivalued mapping on a complete metric space. The multivalued mapping is assumed to satisfy a contraction inequality where there are rational terms on pairs of points which are connected by a specific way. We show that the fixed point set of this mapping is nonempty. As special cases of the fixed point result we obtain several corollaries which are infact multivalued extension of certain results in metric fixed point theory of ordinary functions. We consider the problem of data-dependence and stability associated with the fixed point sets of these mappings.

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Our result is for a α- dominated mappings. These concepts are defined here and are conceptual extension of the admissibility condition. Various types of admissibility conditions have been used in fixed point theory in works like [17, 20, 26].

Recently, Wardowski [37] introduced a new family of mappings so called F or $\mathfrak{F}$ family. Using the mapping from this family, he introduced a new contraction condition namely F- contraction. Many researchers have generalized his concept, see for example [3, 4, 5, 19]. This $F-$ contraction nicely generalizes famous contraction conditions.

In this paper, we combine the ideas mentioned above to introduce some new contraction conditions for multivalued mappings and corresponding fixed point theorem. We also show that many new results in different settings can be obtained from our result. We also discuss Data dependence and stability of fixed point sets of affrosaid contractive mapping.

It may be mentioned here that the problem of data dependence and stability have been discussed in works like [33, 34] and [9, 16, 29, 30] respectively in recent times. We use the data dependence to deduce stability result for our mappings. None often than not, such problems are discussed for multivalued mappings. This is due to the fact that the fixed point sets of setvalued mappings are generally wider than their singlevalued counterparts and also have more complicated structures. Several research papers on data dependence have been published in recent literatures of which we mention a few in references [12, 13].

Let $T_1,\ T_2 : X \to N(X)$ be two multivalued mappings, where $(X,\ d)$ is a metric space and $N(X)$ is the set of nonempty subsets of X such that $H(T_1x,\ T_2x) \leq \eta$, for all $x \in X$, where η is some positive number. Then a data dependence problem is to estimate the distance between the fixed point sets of these two mappings. The above is meaningful only if we have an assurance of nonempty fixed point sets of these two operators. There are also some variants of the problem.

In continuation of the data dependence result in section four, by particularly considering a special case in which both the mappings are assumed to satisfy the conditions of the main theorem in section three, we establish a stability result for fixed point sets of these mappings.

Stability is related to the limiting behavior of a system which, in this case, is the relation of the fixed point sets associated with a sequence of multivalued mappings with the limit function to which the sequence converges. There are several studies related to stabilities of fixed point sets, some of which are noted in [14, 16, 17].

2. Introduction and mathematical preliminaries

Let $(X,\ d)$ be a metric space and $N(X) :=$ the collection of all nonempty subsets of X, $CB(X) :=$ the collection of all nonempty closed and bounded subsets of X and $K(X) :=$ the collection of all nonempty compact subsets of X. We use the following notations and definitions

$$D(x, B) = \inf\ \{d(x, y) : y \in B\},\ \text{where } x \in X \text{ and } B \in CB(X),$$

$$D(A, B) = \inf\ \{d(a, b) : a \in A,\ b \in B\},\ \text{where } A, B \in CB(X),$$

$$H(A, B) = \max\ \{\underset{x\in A}{sup}\ D(x, B),\ \ \underset{y\in B}{sup}\ D(y, A)\},\ \text{where } A, B \in CB(X).$$

H is known as the Hausdorff metric on $CB(X)$ [31].

Lemma 2.1 ([16])**.** Let $B \in K(X)$, where $(X,\ d)$ is a metric space. Then for every $x \in X$ there exists $y \in B$ such that $d(x,\ y) = D(x,\ B)$.

Definition 2.1. Let $(X,\ d)$ be a metric space, $T : X \to N(X)$ be a setvalued mapping. Then a point $x \in X$ is called a fixed point of T if $x \in Tx$.

$Fix(T)$ denotes the set of all fixed points of T.

Definition 2.2. Let $T : X \to CB(Y)$ be a multivalued mapping, where $(X,\ \rho)$, $(Y,\ d)$ are two metric spaces and H is the Hausdorff metric on $CB(Y)$. The mapping T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X, $H(Tx,\ Tx_n) \to 0$, as $n \to \infty$ whenever $\rho(x,\ x_n) \to 0$, as $n \to \infty$.

Recently, Kutbi and Sintunavarat [27] introduced the concept of the α-continuity for multivalued mappings in metric spaces.

Definition 2.3 ([27])**.** Let $T : X \to CB(Y)$ be a multivalued mapping and $\alpha : X \times X \to [0,\ \infty)$, where $(X,\ \rho)$, $(Y,\ d)$ are two metric spaces and H is the Hausdorff metric on $CB(Y)$. The mapping T is said to have α-continuity at $x \in X$ if for any sequence $\{x_n\}$ in X, $H(Tx,\ Tx_n) \to 0$ whenever $\rho(x,\ x_n) \to 0$ as $n \to \infty$ and $\alpha(x_n,\ x_{n+1}) \geq 1$, for all n.

Remark 2.1 ([27])**.** The continuity of a mapping implies its α-continuity for any $\alpha : X \times X \to [0,\ \infty)$. In general, the converse is not true.

For our study, we introduce the α-dominated mapping which is a varied version of admissibility conditions which are quite extensively used in fixed point theory now-a-days. It was introduced in the work of Samet et al. [35] and was further elaborated through works like [2, 15, 20].

Definition 2.4. Let $(X,\ d)$ be a metric space, $T : X \to X$ and $\alpha : X \times X \to [0,\ \infty)$ be two mappings. Then T is called α- dominated if $\alpha(x,\ Tx) \geq 1$, for all $x \in X$.

Definition 2.5. Let $(X,\ d)$ be a metric space, $T : X \to N(X)$ and $\alpha : X \times X \to [0,\ \infty)$ be two mappings. Then T is called α- dominated if $\alpha(x,\ u) \geq 1$, for all $x \in X$ and $u \in Tx$.

Definition 2.6 ([35])**.** Let $(X,\ d)$ be a metric space and $\alpha : X \times X \to [0,\ \infty)$. Then X is said to have α- regular property if for every convergent sequence $\{x_n\}$ with limit x in X and $\alpha(x_n,\ x_{n+1}) \geq 1$, for all n implies that $\alpha(x_n,\ x) \geq 1$, for all n.

For the purpose of our works we define the following classes of functions.

Let Θ denote the class of all functions $\theta : [0,\ \infty)^5 \to [0,\ \infty)$ having the properties:
(i) : θ is continuous and nondecreasing in each coordinate,
(ii): $\phi(t) \leq t$, for all $t > 0$, where $\phi(t) = \theta(t,\ t,\ t,\ t,\ t)$.

Let $\mathfrak{G}$ be the collections of all mappings $G : [0,\ \infty)^5 \to [0,\ \infty)$ having the following properties
(i) : G is continuous in each coordinate,
(ii) : for all $t_1,\ t_2,\ t_3,\ t_4,\ t_5 \in [0,\ \infty)$, with $t_1.t_2.t_3.t_4.t_5 = 0$, there exists $\tau > 0$ such that

$$G(t_1,\ t_2,\ t_3,\ t_4,\ t_5) = \tau.$$

We introduce a generalized weak contraction. To fulfill our purpose we use the recent technique, which was given by Wardowski [37]. For the sake of completeness, we will discuss the basic lines [23, 24, 25].

Definition 2.7 ([32]). Let Ω be the collections of all mappings $F : (0,\ \infty) \to \mathbb{R}$ satisfying the following properties:
(i) : F is strictly increasing,
(ii) : for any sequence $\{a_n\}$ in $\mathbb{R}^+$, $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} F(a_n) = -\infty$ are equivalent, (iii): there exists $k \in (0,\ 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Now we define generalized F- weak contraction in case of single and multivalued mappings.

Definition 2.8. $(X,\ d)$ be a metric space, $\alpha : X \times X \to [0,\ \infty)$, $T\colon X \to X$ be two mappings. Let $\theta \in \Theta$, $F \in \Omega$, $G \in \mathfrak{G}$ and $\tau > 0$. Then T is said to be a generalized F- weak contraction if for all $x,\ y \in X$ with $\alpha(x,\ y) \geq 1$ and $d(x,\ Tx) > 0$,

$$G\Big(d(x,y),\ d(x,Tx),\ d(y,Ty),\ d(x,Ty),\ d(y,Tx)\Big) + F(d(Tx,\ Ty)) \leq F(L(x,\ y)), \tag{2.1}$$

$$\text{where}\ \ L(x,\ y) = \theta\Big(d(x,y),\ \frac{d(x,Tx)+d(y,Ty)}{2},\ \frac{d(x,Ty)+d(y,Tx)}{2},\ \frac{d(x,Tx)\ d(y,Ty)}{1+d(x,\ y)},\ \frac{d(x,Ty)\ d(y,Tx)}{1+d(x,\ y)}\Big).$$

Definition 2.9. $(X,\ d)$ be a metric space, $\alpha : X \times X \to [0,\ \infty)$, $T\colon X \to K(X)$ be two mappings. Let $\theta \in \Theta$, $F \in \Omega$, $G \in \mathfrak{G}$ and $\tau > 0$. Then T is said to be a generalized multivalued F- weak contraction if for all $x,\ y \in X$ with $\alpha(x, y) \geq 1$ and $D(x,\ Tx) > 0$,

$$G\Big(d(x,y), D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)\Big) + F(H(Tx,Ty)) \leq F(M(x,y)), \tag{2.2}$$

$$\text{where}\ \ M(x,\ y) = \theta\Big(d(x,y),\ \frac{D(x,Tx)+D(y,Ty)}{2},\ \frac{D(x,Ty)+D(y,Tx)}{2},\ \frac{D(x,Tx)\ D(y,Ty)}{1+d(x,y)},\ \frac{D(x,Ty)\ D(y,Tx)}{1+d(x,y)}\Big).$$

Definition 2.10. Let $(X,\ d)$ be a metric space and $\{T_n : X \to CB(X) : n \in \mathbb{N}\}$ be a sequence of mappings. Then the fixed point sets $F(T_n)$ of a sequence $\{T_n\}$ are called stable if $H(F(T_n),\ F(T)) \to 0$, as $n \to \infty$, where $T = \lim\limits_{n\to\infty} T_n$.

3. Main results

Theorem 3.1. Let $(X,\ d)$ be a complete metric space, $T : X \to K(X)$ be a multivalued mapping and $\alpha : X \times X \to [0,\ \infty)$. Suppose that (i) (a) T is α- continuous or (b) X has α- regular property ; (ii) T is α- dominated; (iii) there exist $\theta \in \Theta$, $F \in \Omega$, $G \in \mathfrak{G}$ and $\tau > 0$ such that T is a generalized F- weak contraction. Then T has a fixed point in X.

Proof. Let $x_0 \in X$. Since $Tx_0 \in K(X)$, by Lemma 2.1, there exists $x_1 \in Tx_0$ such that $D(x_0,\ Tx_0) = d(x_0,\ x_1)$. By the condition (ii) of the theorem, we have $\alpha(x_0,\ x_1) \geq 1$. Again as $Tx_1 \in K(X)$, by Lemma 2.1, there exists $x_2 \in Tx_1$ such that $D(x_1,\ Tx_1) = d(x_1,\ x_2)$. By the condition (ii) of the theorem, we have $\alpha(x_1,\ x_2) \geq 1$. Continuing in this process we obtain a sequence $\{x_n\}$ in X such that

$$x_{n+1} \in Tx_n \text{ with } \alpha(x_n, x_{n+1}) \geq 1 \text{ and } D(x_n, Tx_n) = d(x_n, x_{n+1}), \text{ for all } n \geq 0. \tag{3.1}$$

If $D(x_{n_0},\ Tx_{n_0}) = 0$, for some n_0 then $x_{n_0} \in \overline{Tx_{n_0}} = Tx_{n_0}$, where $\overline{Tx_{n_0}}$ is the closure of Tx_{n_0} and in such case x_{n_0} is a fixed point of T. Therefore, we assume that

$$d(x_n, x_{n+1}) = D(x_n, Tx_n) > 0, \quad \text{for all } n \geq 0. \tag{3.2}$$

By condition (iii) and using (3.1) and (3.2), we have

$$\begin{aligned}
&G\Big(d(x_n, x_{n+1}),\ D(x_n, Tx_n),\ D(x_{n+1}, Tx_{n+1}),\ D(x_n, Tx_{n+1}),\ D(x_{n+1}, Tx_n)\Big) \\
&\qquad\qquad + F(d(x_{n+1}, x_{n+2})) \\
&= G\Big(d(x_n, x_{n+1}),\ D(x_n, Tx_n),\ D(x_{n+1}, Tx_{n+1}),\ D(x_n, Tx_{n+1}),\ D(x_{n+1}, Tx_n)\Big) \\
&\qquad\qquad + F(D(x_{n+1},\ Tx_{n+1})) \\
&\leq G\Big(d(x_n, x_{n+1}),\ D(x_n, Tx_n),\ D(x_{n+1}, Tx_{n+1}),\ D(x_n, Tx_{n+1}),\ D(x_{n+1}, Tx_n)\Big) \\
&\qquad\qquad + F(H(Tx_n,\ Tx_{n+1})) \leq F(M(x_n,\ x_{n+1})).
\end{aligned} \tag{3.3}$$

By the properties of G and θ, we have

$$\begin{aligned}
&G\Big(d(x_n, x_{n+1}),\ D(x_n, Tx_n),\ D(x_{n+1}, Tx_{n+1}),\ D(x_n, Tx_{n+1}),\ D(x_{n+1}, Tx_n)\Big) \\
&\quad = G\Big(d(x_n, x_{n+1}),\ D(x_n, Tx_n),\ D(x_{n+1}, Tx_{n+1}),\ D(x_n, Tx_{n+1}),\ 0\Big) = \tau
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
&M(x_n,\ x_{n+1}) \\
&= \theta\Big(d(x_n, x_{n+1}),\ \frac{D(x_n, Tx_n) + D(x_{n+1}, Tx_{n+1})}{2},\ \frac{D(x_n, Tx_{n+1}) + D(x_{n+1}, Tx_n)}{2}, \\
&\qquad \frac{D(x_n, Tx_n)\ D(x_{n+1}, Tx_{n+1})}{1 + d(x_n,\ x_{n+1})},\ \frac{D(x_n, Tx_{n+1})\ D(x_{n+1}, Tx_n)}{1 + d(x_n,\ x_{n+1})}\Big) \\
&\leq \theta\Big(d(x_n, x_{n+1}),\ \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2},\ \frac{d(x_n, x_{n+2})}{2} \\
&\qquad \frac{d(x_n, x_{n+1})\ d(x_{n+1}, x_{n+2})}{1 + d(x_n,\ x_{n+1})},\ \frac{d(x_n, x_{n+2})\ d(x_{n+1}, x_{n+1})}{1 + d(x_n,\ x_{n+1})}\Big) \\
&\leq \theta\Big(d(x_n, x_{n+1}),\ \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2},\ \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2} \\
&\qquad \frac{d(x_n, x_{n+1})\ d(x_{n+1}, x_{n+2})}{1 + d(x_n,\ x_{n+1})},\ 0\Big)
\end{aligned}$$

$$\leq \theta(d(x_n, x_{n+1}),\ \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2},\ \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}$$
$$d(x_{n+1}, x_{n+2}),\ 0)$$
$$\leq \theta(\max\{d(x_{n+1}, x_{n+2}),\ d(x_n, x_{n+1})\},\ \max\{d(x_{n+1}, x_{n+2}),\ d(x_n, x_{n+1})\},$$
$$\max\{d(x_{n+1}, x_{n+2}),\ d(x_n, x_{n+1})\},\ \max\{d(x_{n+1}, x_{n+2}),\ d(x_n, x_{n+1})\},$$
$$\max\{d(x_{n+1}, x_{n+2}),\ d(x_n, x_{n+1})\})$$
$$= \phi(\max\{d(x_{n+1}, x_{n+2}),\ d(x_n, x_{n+1})\}) \tag{3.5}$$

From (3.3), (3.4) and (3.5), we have

$$F(d(x_{n+1}, x_{n+2})) \leq F(M(x_n, x_{n+1})) - \tau$$
$$\leq F(\phi(\max\{d(x_{n+1}, x_{n+2}),\ d(x_n, x_{n+1})\})) - \tau. \tag{3.6}$$

Suppose that $d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2})$. Then $d(x_{n+1}, x_{n+2}) > 0$ and as $\tau > 0$ from (3.6), we have

$$F(d(x_{n+1}, x_{n+2})) \leq F(\phi(d(x_{n+1}, x_{n+2}))) - \tau$$
$$\leq F(d(x_{n+1}, x_{n+2})) - \tau < F(d(x_{n+1}, x_{n+2})),$$

which is a contradiction. Therefore, we have $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$ and also $d(x_n, x_{n+1}) = D(x_n,\ Tx_n) > 0$. From (3.6), we have

$$F(d(x_{n+1}, x_{n+2})) \leq F(\phi(d(x_n, x_{n+1}))) - \tau \leq F(d(x_n, x_{n+1})) - \tau. \tag{3.7}$$

By repeated application of (3.7), we have

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})) - \tau \leq F(d(x_{n-1}, x_n)) - 2\tau$$
$$\leq F(d(x_{n-2}, x_{n-1})) - 3\tau \leq \ldots \leq F(d(x_0, x_1)) - (n+1)\tau. \tag{3.8}$$

Also from (3.6), we have $F(d(x_{n+1}, x_{n+2})) < F(\phi(d(x_n, x_{n+1})))$, the property of F implies that

$$d(x_{n+1}, x_{n+2}) < \phi(d(x_n, x_{n+1}), \quad \text{for all} \ \ n \geq 0. \tag{3.9}$$

Let $a_n = d(x_n,\ x_{n+1})$, for all $n \geq 0$. Then $a_n > 0$, for all $n \geq 0$ and hence taking limit as $n \to \infty$ in (3.8) and using the property of F, we have $\lim_{n\to\infty} F(a_n) = -\infty$, which by the property of F implies that $\lim_{n\to\infty} a_n = 0$. Again from the property of F, there exists $k \in (0,\ 1)$ such that $\lim_{n\to\infty} a_n^k F(a_n) = 0$. Therefore, from (3.8), we have

$$a_n^k F(a_n) - a_n^k F(a_0) \leq -a_n^k n\tau \leq 0, \ \text{for all} \ \ n > 0.$$

Taking limit as $n \to \infty$ in above inequality, we have $\lim_{n\to\infty} n\ a_n^k = 0$. Therefore, there exists n_1 such that $n\ a_n^k \leq 1$, for all $n \geq n_1$. Therefore, we have $a_n \leq \frac{1}{n^{\frac{1}{k}}}$, for all $n \geq n_1$. Let $m > n \geq n_1$. Now we have

$$\sum_{n=1}^{\infty} d(x_n,\ x_{n+1}) = \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} < \infty,$$

which implies that $d(x_n,\ x_m) \to 0$, as $m,\ n \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence in X. As (X, d) is complete, there exists $x \in X$ such that

$$x_n \to x, \ \text{as} \ n \to \infty. \tag{3.10}$$

Suppose $(i)(a)$ **holds:**
As $\alpha(x_n,\ x_{n+1}) \geq 1$, for all n and $x_n \to x$, as $n \to \infty$, the $\alpha-$ continuity of T implies that $H(Tx_n, Tx) \to 0$ as $n \to \infty$. It follows that $D(x_{n+1},\ Tx) \to 0$ as $n \to \infty$, that is, $D(x,\ Tx) = 0$, that is, $x \in \overline{Tx} = Tx$, where $\overline{Tx}$ is the closure of Tx. Therefore, x is a fixed point of T.

Suppose $(i)(b)$ **holds:**
By (3.1), we have that $D(x_n, Tx_n) > 0$ and $\alpha(x_n,\ x_{n+1}) \geq 1$, for all $n \geq 0$. By (3.10) and the assumption (i)(b), we have $\alpha(x_n,\ x) \geq 1$, for all $n > 0$. Using the assumption (iii), (3.1) and the property of F, we have for all $n > 0$

$$\begin{aligned} &G(d(x_n, x),\ D(x_n, Tx_n),\ D(x, Tx),\ D(x_n, Tx),\ D(x, Tx_n)) + F(D(x_{n+1},\ Tx)) \\ &\qquad \leq G(d(x_n, x),\ D(x_n, Tx_n),\ D(x, Tx),\ D(x_n, Tx),\ D(x, Tx_n)) \\ &\qquad\quad + F(H(Tx_n,\ Tx)) \leq F(M(x_n,\ x)), \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} M(x_n,\ x) &= \theta\Big(d(x_n,\ x),\ \frac{D(x_n,\ Tx_n) + D(x,\ Tx)}{2},\ \frac{D(x_n,\ Tx) + D(x,\ Tx_n)}{2}, \\ &\qquad \frac{D(x_n, Tx_n)\ D(x, Tx)}{1 + d(x_n,\ x)},\ \frac{D(x_n, Tx)\ D(x, Tx_n)}{1 + d(x_n,\ x)}\Big) \\ &\leq \theta\Big(d(x_n,\ x),\ \frac{d(x_n,\ x_{n+1}) + D(x,\ Tx)}{2},\ \frac{D(x_n,\ Tx) + d(x,\ x_{n+1})}{2}, \\ &\qquad \frac{d(x_n, x_{n+1})\ D(x, Tx)}{1 + d(x_n,\ x)},\ \frac{D(x_n, Tx)\ d(x, x_{n+1})}{1 + d(x_n,\ x)}\Big). \end{aligned} \tag{3.12}$$

From (3.11) using property of G, we have

$$F(D(x_{n+1},\ Tx)) \leq F(M(x_n,\ x)).$$

Increasing property of F implies that

$$D(x_{n+1},\ Tx) \leq M(x_n,\ x). \tag{3.13}$$

Taking limsup as $n \to \infty$ in (3.12), we have

$$\begin{aligned} \limsup_{n\to\infty} M(x_n,\ x) &= \theta(0,\ \frac{D(x,\ Tx)}{2},\ \frac{D(x,\ Tx)}{2},\ 0,\ 0) \\ &\leq \theta(\frac{D(x, Tx)}{2},\ \frac{D(x, Tx)}{2},\ \frac{D(x, Tx)}{2},\ \frac{D(x, Tx)}{2},\ \frac{D(x, Tx)}{2}) = \phi(\frac{D(x, Tx)}{2}). \end{aligned} \tag{3.14}$$

We claim that $D(x,\ Tx) = 0$. If possible, suppose that $D(x,\ Tx) > 0$, then $\frac{D(x,\ Tx)}{2} > 0$. Taking limsup as $n \to \infty$ in (3.13) and using (3.14) and using a property of θ, we have $D(x,\ Tx) \leq \phi(\frac{D(x,\ Tx)}{2}) \leq \frac{D(x,\ Tx)}{2}$, which is a contradiction. Therefore, we have $D(x,\ Tx) = 0$, which implies that $x \in \overline{Tx} = Tx$, where $\overline{Tx}$ is the closure of Tx. Therefore, x is a fixed point of T.

Now we present a few special cases illustrating the applicability of Theorem 3.1.

Remark 3.1. In Theorem 3.1, taking $\alpha(x,\ y) = 1$, for all $x,\ y \in X$ and $G(t_1,\ t_2,\ t_3,\ t_4,\ t_5) = \tau$, for all $t_1,\ t_2,\ t_3,\ t_4,\ t_5 \in [0,\ \infty)$ and choosing any one of

(i) $\theta(t_1, t_2, t_3, t_4, t_5) = t_1$,
(ii) $\theta(t_1, t_2, t_3, t_4, t_5) = t_2$,
(iii) $\theta(t_1, t_2, t_3, t_4, t_5) = t_3$,
(iv) $\theta(t_1, t_2, t_3, t_4, t_5) = \max\ \{t_1,\ t_2,\ t_3\}$,
respectively, where $\tau > 0$, we have the following corollaries.

Corollary 3.2. A multivalued mapping $T : X \to K(X)$, where (X, d) is a complete metric space, has a fixed point if for all $x, y \in X$ with $D(x,\ Tx) > 0$, T satisfies the following inequality:

$$\tau + F(H(Tx,\ Ty)) \leq F(d(x,\ y)),\ \text{where } \tau > 0.$$

Corollary 3.3. A multivalued mapping $T : X \to K(X)$, where (X, d) is a complete metric space, has a fixed point if for all $x, y \in X$ with $D(x,\ Tx) > 0$, T satisfies the following inequality:

$$\tau + F(H(Tx,\ Ty)) \leq F\Big(\frac{1}{2}[D(x, Tx) + D(y, Ty)]\Big),\ \text{where } \tau > 0.$$

Corollary 3.4. A multivalued mapping $T : X \to K(X)$, where (X, d) is a complete metric space, has a fixed point if for all $x, y \in X$ with $D(x,\ Tx) > 0$, T satisfies the following inequality:

$$\tau + F(H(Tx,\ Ty)) \leq F\Big(\frac{1}{2}[D(x, Ty) + D(y, Tx)]\Big),\ \text{where } \tau > 0.$$

Corollary 3.5. A multivalued mapping $T : X \to K(X)$, where (X, d) is a complete metric space, has a fixed point if for all $x, y \in X$ with $D(x,\ Tx) > 0$, T satisfies the following inequality:

$$\tau + F(H(Tx, Ty)) \leq F\Big(\max\ \{d(x, y),\ \frac{D(x, Tx) + D(y, Ty)}{2},\ \frac{D(x, Ty) + D(y, Tx)}{2}\}\Big),$$

where $\tau > 0$.

Example 3.1. Let $X = [0,\ \infty)$ and d be the usual metric on X. Let $T : X \to K(X)$, $\alpha : X \times X \to [0,\ \infty)$, $F : (0,\ \infty) \to \mathbb{R}$, $G : [0,\ \infty)^5 \to [0,\ \infty)$ and $\theta : [0,\ \infty)^5 \to [0,\ \infty)$ be defined respectively as follows:

$$Tx = \Big[0, \frac{x}{8}\Big],\ \text{for } x \in X, \qquad \alpha(x,\ y) = \begin{cases} e^{x+y}, \text{ if } & x \in X \text{ and } y \leq \frac{x}{8}, \\ 0, & \text{otherwise}, \end{cases}$$

$$F(t) = \ln t,\ \text{for } \ t \in (0,\ \infty),$$

$$G(t_1, t_2, t_3, t_4, t_5) = \frac{\ln 8}{1 + \min\ \{t_1, t_2, t_3, t_4, t_5\}},\ \text{for } t_1, t_2, t_3, t_4, t_5 \in [0,\ \infty)$$

and

$$\theta(t_1, t_2, t_3, t_4, t_5) = \max\ \{t_1, t_2, t_3\},\ \text{for } t_1, t_2, t_3, t_4, t_5 \in [0,\ \infty).$$

(i) **(a)** As T is continuous, it is α- continuous or **(b)** Let $\{x_n\}$ be a sequence converging to x and $\alpha(x_n,\ x_{n+1}) \geq 1$, for all n. Then $x_{n+1} \leq \frac{x_n}{8}$, which implies that $x \leq \frac{x_n}{8}$, for all n, which implies that $\alpha(x_n,\ x) \geq 1$. Hence, X has α- regular property.

(ii) For all $x \in X$ and $y \in Tx$, we have $y \leq \frac{x}{8}$ and hence $\alpha(x,\ y) \geq 1$. Therefore, T is α- dominated.

(iii) Let $x,\ y \in X$ with $\alpha(x,\ y) \geq 1$ and $D(x,\ Tx) > 0$. Then $x \in (0,\ \infty)$ and $y \leq \frac{x}{8}$ and for those value of $x,\ y \in X$, we have $H(Tx,\ Ty) = \frac{\mid x - y \mid}{8} = \frac{d(x,\ y)}{8} \leq \frac{1}{8}M(x,\ y)$, that is,

$$\begin{aligned}\ln(H(Tx,\ Ty)) &\leq \ln(M(x,\ y)) - \ln 8\\ &\leq \ln(M(x,\ y)) - \frac{\ln 8}{1 + \min\{d(x,y),\ D(x,Tx),\ D(y,Ty),\ D(x,Ty),\ D(y,Tx)\}},\end{aligned}$$

which implies that

$$\begin{aligned}&\frac{\ln 8}{1 + \min\{d(x,y),\ D(x,Tx),\ D(y,Ty),\ D(x,Ty),\ D(y,Tx)\}} + \ln(H(Tx,\ Ty))\\ &\quad\leq \ln(M(x,\ y)),\end{aligned}$$

that is, $G(d(x,y),\ D(x,Tx),\ D(y,Ty),\ D(x,Ty),\ D(y,Tx)) + F(H(Tx,\ Ty)) \leq F(M(x,\ y))$.
Then T is generalized F- weak contraction. Hence all the conditions of Theorem 3.1 are satisfied and T has a fixed point $0 \in X$.

4. Data dependence of fixed point sets

In this section, we investigate the data dependence result of fixed point sets of the set valued contractions mentioned in section 3. For this section we consider the following assumption:

(A): $\Psi(t) = \sum_{n=0}^{\infty} \phi^n(t) < \infty$, where $\phi(t) = \theta(t,\ t,\ t,\ t,\ t,\ t)$.

Theorem 4.1. Let $(X,\ d)$ be a complete metric space, $T_j : X \to K(X),\ j = 1,\ 2$ be two multivalued mappings and $\alpha : X \times X \to [0,\ \infty)$. Suppose that the assumptions of Theorem 3.1 holds for the function T_2 and the space X. Then $Fix(T_2)$ is nonempty. Also suppose that $Fix(T_1)$ is nonempty, the assumption (A) holds and there exists a $\eta > 0$ such that $H(T_1x,\ T_2x) \leq \eta$, for all $x \in X$. Then $\sup\limits_{z \in Fix(T_1)} D(z, Fix(T_2)) \leq \Psi(\eta)$.

Proof. From theorem 3.1, the set of fixed points of T_2 are nonempty, that is, $Fix(T_2) \neq \emptyset$. By the assumption of the theorem, $Fix(T_1)$ is nonempty. Let x_0 be a fixed point of T_1, that is, $x_0 \in Fix(T_1)$, that is, $x_0 \in T_1x_0$. Then by Lemma 2.1, there exists $x_1 \in T_2x_0$ such that $D(x_0,\ T_2x_0) = d(x_0,\ x_1)$. By the assumption (ii), we have $\alpha(x_0,\ x_1) \geq 1$. Again as $T_2x_1 \in K(X)$, by Lemma 2.1, there exists $x_2 \in T_2x_1$ such that $D(x_1,\ T_2x_1) = d(x_1,\ x_2)$. By the assumption (ii), we have $\alpha(x_1,\ x_2) \geq 1$. Continuing in this process, we obtain a sequence $\{x_n\} \subseteq X$ such that

$$x_{n+1} \in T_2x_n \text{ with } \alpha(x_n, x_{n+1}) \geq 1 \text{ and } D(x_n, T_2x_n) = d(x_n, x_{n+1}), \text{ for all } n > 0.$$

Arguing similarly as in the proof of Theorem 3.1, we can show that
- the inequality (3.9) is satisfied,
- $\{x_n\}$ is a Cauchy sequence in X,
- there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

• u is a fixed point of T_2, that is, $u \in T_2u$.
By repeated application of (3.9), we have

$$d(x_{n+1}, x_{n+2}) < \phi(d(x_n, x_{n+1}) < \phi^2(d(x_{n-1}, x_n) < ... < \phi^{n+1}(d(x_0, x_1). \tag{4.1}$$

From the definition of η, we have

$$d(x_0,\ x_1) = D(x_0,\ T_2x_0) \leq H(T_1x_0,\ T_2x_0) \leq \eta. \tag{4.2}$$

By (4.1) and (4.2), we have

$$d(x_0,\ u) \leq \sum_{i=0}^{n} d(x_i,\ x_{i+1}) + d(x_{n+1},\ u) \leq \sum_{i=0}^{n} \phi^i(d(x_0,\ x_1)) + d(x_{n+1},\ u).$$

Taking limit as $n \to \infty$ in the above inequality and using (4.2) and the property of ϕ, we have

$$d(x_0,\ u) \leq \sum_{i=0}^{\infty} \phi^i(d(x_0,\ x_1)) \leq \sum_{i=0}^{\infty} \phi^i(\eta) = \Psi(\eta).$$

Thus given arbitrary $x_0 \in Fix(T_1)$, we can find $u \in Fix(T_2)$ for which $d(x_0,\ u) \leq \Psi(\eta)$. Hence, it follows that $D(x_0, Fix(T_2)) \leq \Psi(\eta)$. As $x_0 \in Fix(T_1)$ is arbitrary, we have $\sup\limits_{z \in Fix(T_1)} D(z, Fix(T_2)) \leq \Psi(\eta)$. This completes the proof of the theorem.

5. Stability of fixed point sets

In this section, we investigate the stability result of fixed point sets of the set valued contractions mentioned in Section 3 with the help of data dependence result, discussed in previous section. For this section we take the following assumption

(B): Suppose that for $a \in X$ and any convergent sequence $\{y_n\}$ in X with limit $y \in X$,

$$\alpha(a,\ y_n) \geq 1, \text{ for all } n \in \mathbb{N} \text{ implies that } \alpha(a,\ y) \geq 1.$$

Lemma 5.1. Let $(X,\ d)$ be a metric space and $\alpha : X \times X \to [0,\ \infty)$. Let $\{T_n : X \to K(X) : n \in \mathbb{N}\}$ be a sequence of α- dominated mappings converging to a mapping $T : X \to K(X)$. Suppose that the assumption (B) holds and there exist $\theta \in \Theta$, a continuous function $F \in \Omega$, $G \in \mathfrak{G}$ and $\tau > 0$ such that each T_n, $(n \in \mathbb{N})$ is α- dominated and generalized F- weak contraction. Then T is α-dominated and generalized F- weak contraction.

Proof. First we prove T is α-dominated. Let $x \in X$ and $y \in Tx$. Since $T_n \to T$, as $n \to \infty$, there exists a sequence $\{y_n\}$ in X such that $y_n \in T_nx$ and $y_n \to y$, as $n \to \infty$. Since T_n is α-dominated, for every $n \in \mathbb{N}$, it follows that $\alpha(x,\ y_n) \geq 1$, for every $n \in \mathbb{N}$. By assumption (B), it follows that $\alpha(x,\ y) \geq 1$. Hence T is α-dominated.

Next we prove that T is a generalized F- weak contraction. Let $x, y \in X$ such that $\alpha(x,y) \geq 1$ and $D(x,\ Tx) > 0$. As the sequence $\{T_n\}$ is converging to T, $D(x,\ Tx) > 0$ implies that there exists n_0 such that $D(x, T_nx) > 0$, for all $n \geq n_0$. As each T_n is generalized F- weak contraction, for $x,\ y \in X$ with $\alpha(x,y) \geq 1$, we have for all $n \geq n_0$ that

$$\begin{aligned}
&G(d(x,\ y),\ D(x,T_nx),\ D(y,T_ny),\ D(x,T_ny),\ D(y,T_nx)) + F(H(T_nx,\ T_ny)) \\
&\leq F\Big(\theta\Big(d(x,y),\ \frac{D(x,T_nx)+D(y,T_ny)}{2},\ \frac{D(x,T_ny)+D(y,T_nx)}{2}, \\
&\qquad \frac{D(x,T_nx)\ D(y,T_ny)}{1+d(x,\ y)},\ \frac{D(x,T_ny)\ D(y,T_nx)}{1+d(x,\ y)}\Big)\Big).
\end{aligned}$$

Since the sequence $\{T_n\}$ converges to T and G, θ and F are continuous, taking limit as $n \to \infty$ in the above inequality, we have

$$\begin{aligned}
&G(d(x,\ y),\ D(x,Tx),\ D(y,Ty),\ D(x,Ty),\ D(y,Tx)) + F(H(Tx,\ Ty)) \\
&\leq F\Big(\theta\Big(d(x,y),\ \frac{D(x,Tx)+D(y,Ty)}{2},\ \frac{D(x,Ty)+D(y,Tx)}{2}, \\
&\qquad \frac{D(x,Tx)\ D(y,Ty)}{1+d(x,\ y)},\ \frac{D(x,Ty)\ D(y,Tx)}{1+d(x,\ y)}\Big)\Big).
\end{aligned}$$

Therefore, T is a generalized F- weak contraction.

Theorem 5.2. Let $(X,\ d)$ be a complete metric space, $\{T_n : X \to K(X) : n \in \mathbb{N}\}$ be a sequence of mappings converging uniformly to $T : X \to K(X)$ and $\alpha : X \times X \to [0,\ \infty)$. Suppose that all the assumptions of Theorem 3.1 hold for each T_n, $(n \in \mathbb{N})$ and the space X. Then $Fix(T_n)$, $(n \in \mathbb{N})$ are nonempty. Let $F \in \Omega$ be continuous and the assumption (B) holds. Then $Fix(T)$ is nonempty. Also suppose that the assumption (A) holds and $\Psi(t) \to 0$, as $t \to 0$. Then $\lim_{n\to\infty} H(Fix(T_n),\ Fix(T)) = 0$, that is, the fixed point sets of T_n are stable.

Proof. By Theorem 3.1, we have $Fix(T_n) \neq \emptyset$, for every $n \in \mathbb{N}$. Again, by Lemma 5.1 and Theorem 3.1, it follows that $Fix(T) \neq \emptyset$. Let $\eta_n = \sup_{x\in X} H(T_nx,\ Tx)$, for $n \in \mathbb{N}$. By Theorem 4.1, we have $\sup\limits_{z_n \in Fix(T_n)} D(z_n,\ Fix(T)) \leq \Psi(\eta_n)$ and $\sup\limits_{z \in Fix(T)} D(z,\ Fix(T_n)) \leq \Psi(\eta_n)$, for all $n \in \mathbb{N}$. Then it follows that

$$H(Fix(T_n),\ Fix(T)) \leq \Psi(\eta_n),\ \text{for every } n \in \mathbb{N}. \tag{5.1}$$

Since $T_n \to T$ uniformly, we have $\eta_n \to 0$, as $n \to \infty$. Taking limit as $n \to \infty$ in (5.1) and using the property of Ψ, we have

$$\lim_{n\to\infty} H(Fix(T_n),\ Fix(T)) \leq \lim_{n\to\infty} \Psi(\eta_n) = 0,$$

that is, $\lim_{n\to\infty} H(Fix(T_n),\ Fix(T)) = 0$, that is, the fixed point sets of T_n are stable.

6. Application to the solution of nonlinear integral equation

Every singleton subset of (X, d) is a member of $K(X)$, that is, $\{x\} \in K(X)$, for every $x \in X$. We can treat a mapping $T : X \to X$ as a multivalued mapping in which case Tx is a singleton set for every $x \in X$. Hence the following result is a special case of Theorem 3.1 when T is a single valued mapping.

Theorem 6.1. Let $(X,\ d)$ be a complete metric space, $T : X \to X$ be a single valued mapping and $\alpha : X \times X \to [0,\ \infty)$ be a mapping. Suppose that (i) (a) T is α continuous or (b) X has α- regular property; (ii) T is α- dominated ; (iii) there exist $\theta \in \Theta, F \in \Omega, G \in \mathfrak{G}$ and $\tau > 0$ such that T is generalized $F-$ weak contraction. Then T has a fixed point in X.

Fixed point theorems in metric spaces have several applications in integral equation (see [1, 21, 22] and references there in). In this section, we present an application of Theorem 6.1 to establish the existence of integral equation. Here we prove the existence of a solution of an integral equation using Theorem 6.1.

We consider nonlinear Volterra type non homogeneous integral equation of second kind as follows

$$x(t) = f(t) + \int_{g(t)}^{h(t)} K(t,\ s,\ x(s))\ ds, \tag{6.1}$$

where the unknown function $x(t)$ takes real values, for $t,\ s \in [a,\ b]$.

Let $X = C([a,\ b])$, where $a < b$ be the space of all real valued continuous functions defined on $[a,\ b]$. It is well known that $C([a,\ b])$ endowed with the metric

$$d_\tau(x,\ y) = \max_{t\in[a,\ b]} \mid x(t) - y(t) \mid .e^{-|\tau\ t|}, \quad \text{for} \quad \tau \geq 0 \tag{6.2}$$

is a complete metric space.

Define a mapping $T : X \to X$ by

$$T(x)(t) = f(t) + \int_{g(t)}^{h(t)} K(t,\ s,\ x(s))\ ds, \quad \text{for all} \quad t \in [a,\ b]. \tag{6.3}$$

We designate the following assumptions by $A_1,\ A_2$ and A_3:

A_1 : $f,\ g,\ h \in C([a,\ b])$ and $K : [a,\ b] \times [a,\ b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\tau > 0$.

A_2 : $\mid K(t,s,x) - K(t,s,y) \mid \leq e^{-\tau} \ \mid x - y \mid$, for all $x, y \in X$ whenever $\mid x - Tx \mid > 0$ and for all $t, s \in [a, b]$.

A_3 : $\displaystyle\int_{g(t)}^{h(t)} e^{|\tau\ s|} ds \leq e^{|\tau\ t|}$.

Theorem 6.2. Let $(X,\ d_\tau) = C([a,\ b],\ d_\tau),\ T,\ f,\ g,\ h,\ K(t,s,x)$ satisfy the assumptions $A_1,\ A_2$ and A_3. Then nonlinear Volterra type integral equation (6.1) has a solution $x \in C([a,\ b])$.

Proof. Let us define a mapping $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = 1$, for $x, y \in X$. It is clear that the mapping $T : X \to X$ defined by (6.3) is a α- dominated mapping and X has α- regular property. By assumptions $A_1,\ A_2$ and A_3, for all $x,\ y \in C([a,\ b])$

with $\alpha(x,\ y) = 1$ and $d_\tau(x, Tx) > 0$, that is for $\mid x - Tx \mid > 0$, we have for all $t \in [a,\ b]$

$$\begin{aligned}
&\mid Tx(t) - Ty(t) \mid \\
&= \left| \int_{g(t)}^{h(t)} K(t,s,x(s)) - K(t,s,y(s))\ ds \right| = \int_{g(t)}^{h(t)} \mid K(t,s,x(s)) - K(t,s,y(s)) \mid\ ds \\
&\leq \int_{g(t)}^{h(t)} e^{-\tau} \ \mid x(s) - y(s) \mid\ ds = e^{-\tau} \int_{g(t)}^{h(t)} \mid x(s) - y(s) \mid\ ds \qquad [\text{by } A_2] \\
&= e^{-\tau} \int_{g(t)}^{h(t)} e^{|\tau\ s|} \times \mid x(s) - y(s) \mid\ e^{-|\tau\ s|}\ ds \leq e^{-\tau} \int_{g(t)}^{h(t)} e^{|\tau\ s|} \times d_\tau(x,y)\ ds \\
&= e^{-\tau}\ d_\tau(x,y)\ \times \int_{g(t)}^{h(t)} e^{|\tau\ s|}\ ds \leq e^{-\tau}\ d_\tau(x,y)\ \times e^{|\tau\ t|} \qquad [\text{by } A_3].
\end{aligned}$$

Thus, we have

$$\mid Tx(t) - Ty(t) \mid\ e^{-|\tau\ t|} \quad \leq \quad e^{-\tau}\ d_\tau(x,y),$$

which implies that

$$\begin{aligned}
d_\tau(Tx,Ty)\ e^{\tau} &\leq d_\tau(x,y) \\
&\leq \max \Big\{ d_\tau(x,y),\ \frac{d_\tau(x,Tx) + d_\tau(y,Ty)}{2},\ \frac{d_\tau(x,Ty) + d_\tau(y,Tx)}{2}, \\
&\qquad \frac{d_\tau(x,Tx)\ d_\tau(y,Ty)}{1 + d_\tau(x,\ y)},\ \frac{d_\tau(x,Ty)\ d_\tau(y,Tx)}{1 + d_\tau(x,\ y)} \Big\} = N_\tau(x,\ y).
\end{aligned}$$

Taking $\tau > 0$ and $\theta \in \Theta,\ F \in \Omega,\ G \in \mathfrak{G}$ as

$F(t) = \ln t$, for all $t \in (0,\ \infty)$, $\quad G(t_1,t_2,t_3,t_4,t_5) = \ln \tau$, for all $t_1,t_2,t_3,t_4,t_5 \in [0,\ \infty)$,

and $\theta(t_1,t_2,t_3,t_4,t_5) = \max\{t_1,\ t_2,\ t_3,\ t_4,\ t_5\}$, for all $t_1,\ t_2,\ t_3,\ t_4,\ t_5 \in [0,\ \infty)$ respectively, we obtain

$$\begin{aligned}
&G\Big(d_\tau(x,y),\ d_\tau(x,Tx),\ d_\tau(y,Ty),\ d_\tau(x,Ty),\ d_\tau(y,Tx)\Big) + F(d_\tau(Tx,\ Ty)) \\
&\quad \leq F(N_\tau(x,\ y)),
\end{aligned}$$

which implies that T is generalized F- weak contraction. Therefore, all conditions of Theorem 6.1 are satisfied. Therefore, Theorem 6.1 applies to T, which guarantee the existence of a fixed point x in X. That is, x is a solution of nonlinear Volterra type non homogeneous integral equation (6.1) in $C([a,\ b])$.

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