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Cohomological properties of the massless tensor field with the mixed symmetry $(k, 1)$. I. Results on the cohomology of the exterior longitudinal differential

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Abstract

In this former part of a paper dedicated to the computation of local BRST cohomology for a free massless tensor field with the mixed symmetry $(k, 1)$ $(k \geq 4)$ we focus on the main cohomological properties of the exterior longitudinal differential.

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1 Introduction

Real tensor fields transforming according to exotic representations of $GL(D, \mathbb{R})$ corresponding to two-column Young diagrams with $(k+1)$ cells and k lines ("hook" diagrams) or, briefly, tensor fields with the mixed symmetry $(k, 1)$ have been studied starting more than two decades ago in [1]–[5] and more recently (inclusively within the BRST setting) in [6]–[9]. Such fields are present for instance in the bosonic sector of Chern–Simons gravities in odd dimensions [10]–[12] due to the fact that the free limit of their massless version describes one of the dual formulations of linearized gravity in $k + 3$ spacetime dimensions. The limit $k = 1$ provides the linearized Einstein–Hilbert action without cosmological terms, known as the Pauli–Fierz model [13, 14].

The aim of this paper is to analyze the main properties of the local BRST cohomology for the free theory describing a massless tensor field with the mixed symmetry $(k, 1)$ for $k \geq 4$. The case $k = 2$ is covered in [15] and $k = 3$ respectively in [16]. To this end we rely on the general BRST cohomological results for gauge field theories with a well-defined Cauchy order [17]–[20] completed by specific techniques and results from [15, 16] and [21]–[26]. In this context the findings on some BRST cohomological aspects related to a massless tensor field corresponding to a two-column non-rectangular Young tableau [27] are also interesting. More precisely, in this former part we will evaluate the cohomology of the exterior longitudinal differential and its local version. The latter part [28] will be dedicated to the computation of the local cohomology of the Koszul–Tate differential and of its invariant version and finally to the study of the core properties of the local cohomology of the BRST differential in maximum form degree. We use the conventions, notations, and results from [29] on the Lagrangian formulation and BRST symmetry of a single massless tensor field $(k, 1)$.

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2 Lagrangian formulation. BRST symmetry

Let $t_{\mu_1\ldots\mu_k|\alpha}$ be a real tensor field with the mixed symmetry $(k,1)$ on a D-dimensional Minkowski space $\mathcal{M}$, meaning it is antisymmetric with respect to its first k indices and satisfies the identity $t_{[\mu_1\ldots\mu_k|\alpha]} \equiv 0$, where $[\mu\ldots\nu]$ stands for full antisymmetry. We assume that $\mathcal{M}$ is endowed with the metric $\sigma_{\mu\nu} = \sigma^{\mu\nu} = (-+\ldots+)$. The trace of this field, $t_{\mu_1\ldots\mu_{k-1}} = t_{\mu_1\ldots\mu_k|\alpha}\sigma^{\mu_k\alpha}$, defines an antisymmetric tensor of order $(k-1)$.

The Lagrangian formulation of a free, massless tensor field $(k,1)$ $(k \geq 4)$ relies on the general principle of gauge invariance in terms of a generating set of gauge symmetries

$$\delta_{\overset{(1)}{\theta},\overset{(1)}{\epsilon}} t_{\mu_1\ldots\mu_k|\alpha} = \partial_{[\mu_1}\overset{(1)}{\theta}_{\mu_2\ldots\mu_k]|\alpha} + \partial_{[\mu_1}\overset{(1)}{\epsilon}_{\mu_2\ldots\mu_k\alpha]} + (-)^{k+1}(k+1)\partial_\alpha \overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k}, \tag{1}$$

which renders in the limits $k=2$ and $k=3$ the well-known results [15, 16]. The gauge parameters $\overset{(1)}{\theta}$ display the mixed symmetry $(k-1,1)$, so they are antisymmetric in their first $(k-1)$ indices and fulfill the identity $\overset{(1)}{\theta}_{[\mu_1\ldots\mu_{k-1}|\alpha]} \equiv 0$, while $\overset{(1)}{\epsilon}$ are fully antisymmetric. It has been shown in [29] that the corresponding Lagrangian reads as

$$S_0^{\mathrm{t}}[t_{\mu_1\ldots\mu_k|\alpha}] = -\tfrac{1}{2\cdot(k+1)!}\int \left[F_{\mu_1\ldots\mu_{k+1}|\alpha}F^{\mu_1\ldots\mu_{k+1}|\alpha} - (k+1)F_{\mu_1\ldots\mu_k}F^{\mu_1\ldots\mu_k}\right]d^Dx, \tag{2}$$

where $D \geq k+2$ in order to ensure a non-negative number of physical degrees of freedom. The tensor $F_{\mu_1\ldots\mu_{k+1}|\alpha}$ is linear in the first-order derivatives of field components

$$F_{\mu_1\ldots\mu_{k+1}|\alpha} = \partial_{[\mu_1}t_{\mu_2\ldots\mu_{k+1}]|\alpha}, \tag{3}$$

exhibits the mixed symmetry $(k+1,1)$, and possesses the gauge transformation

$$\delta_{\overset{(1)}{\theta},\overset{(1)}{\epsilon}} F_{\mu_1\ldots\mu_{k+1}|\alpha} = (-)^{k+1}k\partial_\alpha\partial_{[\mu_1}\overset{(1)}{\epsilon}_{\mu_2\ldots\mu_{k+1}]}. \tag{4}$$

Its trace, $F_{\mu_1\ldots\mu_k} = F_{\mu_1\ldots\mu_{k+1}|\alpha}\sigma^{\mu_{k+1}\alpha}$, is completely antisymmetric

$$F_{\mu_1\ldots\mu_k} = \partial_{[\mu_1}t_{\mu_2\ldots\mu_k]} + (-)^k\partial^\alpha t_{\mu_1\ldots\mu_k|\alpha}, \tag{5}$$

and presents the gauge variation $\delta_{\overset{(1)}{\theta},\overset{(1)}{\epsilon}} F_{\mu_1\ldots\mu_k} = -k\partial^\alpha\partial_{[\alpha}\overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k]}$. The generating set of gauge transformations of action (2), given by (1), has been shown in [29] to be Abelian and off-shell reducible of order $(k-1)$.

The field equations

$$\frac{\delta S_0^{\mathrm{t}}}{\delta t_{\nu_1\ldots\nu_k|\alpha}} \equiv \tfrac{1}{k!}T^{\nu_1\ldots\nu_k|\alpha} \approx 0, \tag{6}$$

are expressed in terms of the tensor $T^{\nu_1\ldots\nu_k|\alpha}$, linear in the field components $t_{\mu_1\ldots\mu_k|\beta}$, first-order in its derivatives, and with the mixed symmetry $(k,1)$

$$\begin{aligned} T^{\nu_1\ldots\nu_k|\alpha} \;=\; & \Box t^{\nu_1\ldots\nu_k|\alpha} + \partial_\mu\big((-)^k\partial^{[\nu_1}t^{\nu_2\ldots\nu_k]\mu|\alpha} - \partial^\alpha t^{\nu_1\ldots\nu_k|\mu}\big) + (-)^{k+1}\partial^\alpha\partial^{[\nu_1}t^{\nu_2\ldots\nu_k]} \\ & +\sigma^{\alpha[\nu_1}\big[(-)^k\Box t^{\nu_2\ldots\nu_k]} + \partial_\mu\big((-)^{k+1}\partial_\beta t^{\nu_2\ldots\nu_k]\mu|\beta} - \partial^{\nu_2}t^{\nu_3\ldots\nu_k]\mu}\big)\big]. \end{aligned} \tag{7}$$

It is useful to write $T^{\nu_1\ldots\nu_k|\alpha}$ in terms of the tensor $F_{\mu_1\ldots\mu_{k+1}|\beta}$ introduced in (3)

$$T^{\nu_1\ldots\nu_k|\alpha} = \partial_\mu F^{\mu\nu_1\ldots\nu_k|\alpha} - \sigma^{\alpha[\nu_1}\partial_\mu F^{\nu_2\ldots\nu_k\mu]}. \tag{8}$$

The most general gauge-invariant quantities constructed out of $t_{\mu_1\ldots\mu_k|\alpha}$ and its space-time derivatives are given by the components of the "curvature tensor"

$$K_{\mu_1\ldots\mu_{k+1}|\alpha\beta} = \partial_\alpha F_{\mu_1\ldots\mu_{k+1}|\beta} - F_{\mu_1\ldots\mu_{k+1}|\alpha} \equiv \partial_{[\mu_1} t_{\mu_2\ldots\mu_{k+1}]|[\beta,\alpha]}, \tag{9}$$

together with their derivatives. The tensor $K_{\mu_1\ldots\mu_{k+1}|\alpha\beta}$ is linear in the original field, second-order in its derivatives, and displays the mixed symmetry $(k+1,2)$, so it is separately antisymmetric in its first $(k+1)$ indices and in the last two ones and satisfies the first Bianchi identity $K_{[\mu_1\ldots\mu_{k+1}|\alpha]\beta} \equiv 0$. Moreover, it satisfies the second Bianchi identity

$$\partial_{[\mu_1} K_{\mu_2\ldots\mu_{k+2}]|\alpha\beta} \equiv 0, \quad K_{\mu_1\ldots\mu_{k+1}|[\alpha\beta,\gamma]} \equiv 0. \tag{10}$$

The invariance of action (2) with respect to the gauge transformations (1) is equivalent to the Noether identities $\partial_{\nu_1} T^{\nu_1\ldots\nu_k|\alpha} \equiv 0$, $\partial_\alpha T^{\nu_1\ldots\nu_k|\alpha} \equiv 0$, while the reducibility of this generating set of gauge symmetries shows that not all Noether identities are independent. The free theory of a massless $(k,1)$ tensor field satisfies the general regularity conditions [30] and generates a linear gauge theory with the Cauchy order equal to $(k+1)$.

Next, we briefly review the antibracket-antifield BRST symmetry of this free theory, exposed in [29]. The first step of this procedure requires the identification of the algebra on which the BRST differential s acts. The BRST generators are of two kinds: fields/ghosts and antifields. The ghost spectrum is composed of the tensor fields

$$\Big\{ \big\{ \overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}|\alpha}, \overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m+1}} \big\}_{m=\overline{1,k-1}}, \overset{(k)}{\eta}_\mu \Big\}, \tag{11}$$

where $\overset{(1)}{C}$ and $\overset{(1)}{\eta}$ are respectively associated with the gauge parameters $\overset{(1)}{\theta}$ and $\overset{(1)}{\epsilon}$ from (1), while the other ghost fields correspond to the reducibility parameters detailed in [29]. We ask that $\overset{(m)}{C}$ with $m = \overline{1,k-1}$ possess the mixed symmetry $(k-m,1)$, and therefore are antisymmetric in their first $(k-m)$ (where applicable) and fulfill the identities $\overset{(m)}{C}_{[\mu_1\ldots\mu_{k-m}|\alpha]} \equiv 0$, $m = \overline{1,k-1}$, while $\overset{(m)}{\eta}$ with $m = \overline{1,k-1}$ remain antisymmetric. For further purposes we make the compact notation

$$\Phi^A \equiv \Big\{ t_{\mu_1\ldots\mu_k|\alpha}, \big\{ \overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}|\alpha}, \overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m+1}} \big\}_{m=\overline{1,k-1}}, \overset{(k)}{\eta}_\mu \Big\}. \tag{12}$$

The antifield spectrum corresponds to the original fields and to the newly added ghosts, being structured into

$$\Phi^*_A \equiv \Big\{ t^{*\mu_1\ldots\mu_k|\alpha}, \big\{ \overset{(m)}{C}{}^{*\mu_1\ldots\mu_{k-m}|\alpha}, \overset{(m)}{\eta}{}^{*\mu_1\ldots\mu_{k-m+1}} \big\}_{m=\overline{1,k-1}}, \overset{(k)}{\eta}{}^{*\mu} \Big\}. \tag{13}$$

The mixed symmetry/antisymmetry properties of the antifields are the same with those of the corresponding fields/ghosts, so in particular $t^{*[\mu_1\ldots\mu_k|\alpha]} \equiv 0$, $\overset{(m)}{C}{}^{*[\mu_1\ldots\mu_{k-m}|\alpha]} \equiv 0$, $m = \overline{1,k-1}$.

The BRST differential of this free model splits into

$$s = \delta + \gamma, \quad s^2 = 0 \Leftrightarrow (\delta^2 = 0,\, \gamma^2 = 0,\, \delta\gamma + \gamma\delta = 0), \tag{14}$$

with δ the Koszul–Tate differential, $\mathbb{N}$-graded in terms of the antighost number agh ($\mathrm{agh}(\delta) = -1$) and γ the exterior longitudinal derivative, which is a true differential here, anticommuting with δ and $\mathbb{N}$-graded according to the pure ghost number pgh (pgh$(\gamma) = 1$). These two degrees are independent (agh$(\gamma) = 0$, pgh$(\delta) = 0$). The overall degree of the BRST differential is the ghost number (gh), defined as the difference between pgh and agh, such that gh(s) = gh(δ) = gh$(\gamma) = 1$. Consequently, the BRST differential is $\mathbb{Z}$-graded in terms of gh. The standard rules of the antibracket-antifield formalism endow the generators of the BRST complex with the gradings collected in Table 1, where ε denotes the Grassmann parity.

BRST generator	pgh	agh	gh	ε
$t_{\mu_1\ldots\mu_k\vert\alpha}$	0	0	0	0
$\left\{\overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}\vert\alpha}\right\}_{m=\overline{1,k-1}}$	m	0	m	m mod 2
$\left\{\overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m+1}}\right\}_{m=\overline{1,k}}$	m	0	m	m mod 2
$t^{*\mu_1\ldots\mu_k\vert\alpha}$	0	1	-1	1
$\left\{\overset{(m)}{C}{}^{*\mu_1\ldots\mu_{k-m}\vert\alpha}\right\}_{m=\overline{1,k-1}}$	0	$m+1$	$-(m+1)$	$(m+1)$ mod 2
$\left\{\overset{(m)}{\eta}{}^{*\mu_1\ldots\mu_{k-m+1}}\right\}_{m=\overline{1,k}}$	0	$m+1$	$-(m+1)$	$(m+1)$ mod 2

Table 1: Gradings of BRST generators.

The actions of the operators δ and γ on the BRST generators (assuming they act like right derivations) that comply with all the BRST requirements are expressed by

$$\gamma t_{\mu_1\ldots\mu_k\vert\alpha} = \partial_{[\mu_1}\overset{(1)}{C}_{\mu_2\ldots\mu_k]\vert\alpha} + \partial_{[\mu_1}\overset{(1)}{\eta}_{\mu_2\ldots\mu_k\alpha]} + (-)^{k+1}(k+1)\partial_\alpha\overset{(1)}{\eta}_{\mu_1\ldots\mu_k}, \tag{15}$$

$$\begin{aligned}\gamma\overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}\vert\alpha} &= \partial_{[\mu_1}\overset{(m+1)}{C}_{\mu_2\ldots\mu_{k-m}]\vert\alpha} + \partial_{[\mu_1}\overset{(m+1)}{\eta}_{\mu_2\ldots\mu_{k-m}\alpha]}\\ &+(-)^{k-m+1}(k-m+1)\partial_\alpha\overset{(m+1)}{\eta}_{\mu_1\ldots\mu_{k-m}},\quad m=\overline{1,k-2},\end{aligned} \tag{16}$$

$$\gamma\overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m+1}} = \tfrac{k-m}{k-m+2}\partial_{[\mu_1}\overset{(m+1)}{\eta}_{\mu_2\ldots\mu_{k-m+1}]},\quad m=\overline{1,k-1}, \tag{17}$$

$$\gamma\overset{(k-1)}{C}_{\mu_1\vert\alpha} = \partial_{(\mu_1}\overset{(k)}{\eta}_{\alpha)},\quad \gamma\overset{(k)}{\eta}_\mu = 0,\quad \gamma\Phi^*_A = 0, \tag{18}$$

$$\delta\Phi^A = 0,\quad \delta t^{*\mu_1\ldots\mu_k\vert\alpha} = -\tfrac{1}{k!}T^{\mu_1\ldots\mu_k\vert\alpha}, \tag{19}$$

$$\delta\overset{(1)}{C}{}^{*\mu_1\ldots\mu_{k-1}\vert\alpha} = -\partial_\mu\left(kt^{*\mu\mu_1\ldots\mu_{k-1}\vert\alpha} + (-)^k t^{*\mu_1\ldots\mu_{k-1}\alpha\vert\mu}\right), \tag{20}$$

$$\begin{aligned}\delta\overset{(m)}{C}{}^{*\mu_1\ldots\mu_{k-m}\vert\alpha} &= (-)^m\partial_\mu\Bigg((k-m+1)\overset{(m-1)}{C}{}^{*\mu\mu_1\ldots\mu_{k-m}\vert\alpha}\\ &+(-)^{k-m+1}\overset{(m-1)}{C}{}^{*\mu_1\ldots\mu_{k-m}\alpha\vert\mu}\Bigg),\quad m=\overline{2,k-2},\end{aligned} \tag{21}$$

$$\delta\overset{(k-1)}{C}{}^{*\mu_1\vert\alpha} = (-)^{k-1}\partial_\mu\overset{(k-2)}{C}{}^{*\mu(\mu_1\vert\alpha)},\quad \delta\overset{(1)}{\eta}{}^{*\mu_1\ldots\mu_k} = (-)^k(k+1)\partial_\alpha t^{*\mu_1\ldots\mu_k\vert\alpha}, \tag{22}$$

$$\begin{aligned}\delta\overset{(m)}{\eta}{}^{*\mu_1\ldots\mu_{k-m+1}} &= (-)^k(k-m+2)\partial_\alpha\overset{(m-1)}{C}{}^{*\mu_1\ldots\mu_{k-m+1}\vert\alpha}\\ &+\tfrac{(-)^m(k-m+2)(k-m+1)}{k-m+3}\partial_\mu\overset{(m-1)}{\eta}{}^{*\mu\mu_1\ldots\mu_{k-m+1}},\quad m=\overline{2,k},\end{aligned} \tag{23}$$

with $T^{\mu_1\cdots\mu_k|\alpha}$ like in (7). These definitions may be written more compactly if we perform some linear transformations on the ghosts/antifields without affecting their homogeneity with respect to the various gradings

$$\overset{(m)}{C'}_{\mu_1\ldots\mu_{k-m}||\alpha} \equiv \overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}|\alpha} + (k-m+2)\overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m}\alpha}, \tag{24}$$

$$\overset{(m)}{C'}{}^{*\mu_1\ldots\mu_{k-m}||\alpha} \equiv \overset{(m)}{C}{}^{*\mu_1\ldots\mu_{k-m}|\alpha} + \tfrac{1}{k-m+2}\overset{(m)}{\eta}{}^{*\mu_1\ldots\mu_{k-m}\alpha}, \tag{25}$$

with $m=\overline{1,k-1}$. The double bar "$||$" means full antisymmetry with respect to the indices placed before (if applicable) without further identities. The redefined variables are useful at various computations since for every $m=\overline{1,k-1}$ the independent components of the ghost tensor $\overset{(m)}{C'}$ are given by the union between the independent components of all ghosts of pure ghost number m, namely $\overset{(m)}{C}$ and $\overset{(m)}{\eta}$. Similarly, $\overset{(m)}{C'}{}^{*}$ gather all the independent components of the antifields with the antighost number equal to $(m+1)$. Now, some of formulas (15)–(23) take a simpler form

$$\gamma t_{\mu_1\ldots\mu_k|\alpha} = \partial_{[\mu_1}\overset{(1)}{C'}_{\mu_2\ldots\mu_k]||\alpha} - \tfrac{1}{k+1}\partial_{[\mu_1}\overset{(1)}{C'}_{\mu_2\ldots\mu_k||\alpha]}, \tag{26}$$

$$\gamma\overset{(m)}{C'}_{\mu_1\ldots\mu_{k-m}||\alpha} = \partial_{[\mu_1}\overset{(m+1)}{C'}_{\mu_2\ldots\mu_{k-m}]||\alpha},\ \ m=\overline{1,k-2}, \tag{27}$$

$$\gamma\overset{(k-1)}{C'}_{\mu_1||\alpha} = 2\partial_{\mu_1}\overset{(k)}{\eta}_{\alpha}, \delta\overset{(1)}{C'}{}^{*\mu_1\ldots\mu_{k-1}||\alpha} = -k\partial_\mu t^{*\mu\mu_1\ldots\mu_{k-1}|\alpha}, \delta\overset{(k)}{\eta}{}^{*\alpha} = (-)^k 2\partial_{\mu_1}\overset{(k-1)}{C'}{}^{*\mu_1||\alpha}, \tag{28}$$

$$\delta\overset{(m)}{C'}{}^{*\mu_1\ldots\mu_{k-m}||\alpha} = (-)^m(k-m+1)\partial_\mu\overset{(m-1)}{C'}{}^{*\mu\mu_1\ldots\mu_{k-m}||\alpha},\ \ m=\overline{2,k-1}. \tag{29}$$

3 Local BRST cohomology. Generalities

All cohomological computations will be carried out on the algebra of local differential forms with coefficients from the BRST algebra without explicit dependence on the spacetime coordinates x^μ, to be denoted by Λ. In other words, the form coefficients are elements of the BRST algebra $\mathcal{A}$ of local "functions" that do not explicitly depend on the global coordinates of the Minkowski spacetime $\mathcal{M}$, and therefore polynomials in ghosts, antifields, and their spacetime derivatives up to a finite order, 'smooth' in the original tensor field with the mixed symmetry $(k,1)$ and also polynomials in its derivatives up to a finite order. Consequently, the algebra Λ will inherit the four gradings of the BRST algebra [the $\mathbb{Z}_2$-grading in terms of the Grassmann parity ε, the $\mathbb{Z}$-grading according to gh as well as the two $\mathbb{N}$-gradings involving agh and pgh] introduced via Table 1, accompanied by

$$\varepsilon(dx^\mu)=1, \quad \varepsilon(dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}) = p \bmod 2, \tag{30}$$

$$\mathrm{agh}(dx^\mu)=0, \quad \mathrm{pgh}(dx^\mu)=0, \quad \mathrm{gh}(dx^\mu)=0, \tag{31}$$

where $\wedge$ is the symbol for wedge product. In addition, Λ is endowed with a supplementary $\mathbb{N}$-grading in terms of the form degree deg

$$\Lambda = \bigoplus_{p\in\mathbb{N}}\overset{[p]}{\Lambda}, \quad \deg(\overset{[p]}{\omega}) = p \Leftrightarrow \overset{[p]}{\omega}\in\overset{[p]}{\Lambda}, \tag{32}$$

$$\overset{[p]}{\omega} = \tfrac{1}{p!}a_{\mu_1\ldots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p},\ \ a_{\mu_1\ldots\mu_p}\in\mathcal{A}. \tag{33}$$

Since the dimension of $\mathcal{M}$ is by hypothesis finite and denoted by D, the decomposition (32) stops at D; all forms that are homogeneous with respect to deg like in (33) and with the form degree $p > D$ vanish. The operators δ, γ, and s are extended to the algebra Λ via relations (15)–(23) together with

$$\delta(dx^\mu) = 0, \quad \gamma(dx^\mu) = 0, \quad s(dx^\mu) = 0 \tag{34}$$

and by assuming their actions as right derivatives on Λ with respect to the wedge product. In this context we recall that for any element of the form (33) with $a_{\mu_1 \ldots \mu_p} \in \mathcal{A}$ of well-defined Grassmann parity, $\varepsilon(a)$, we have that

$$\varepsilon(\overset{[p]}{\omega}) = [\varepsilon(a) + p] \bmod 2. \tag{35}$$

In this way all the properties of the operators δ, γ, and $s = \delta + \gamma$ are transferred from the BRST algebra $\mathcal{A}$ to the algebra of local forms Λ. In particular, these operators remain differentials and δ still anticommutes with γ. Also, δ continues to be acyclic on Λ in strictly positive values of the antighost number agh, $H(\delta) \equiv H_0(\delta)$, and it makes sense to compute the cohomology algebras $H(\gamma|H_0(\delta))$ and $H(s)$. Moreover, the isomorphisms $H^j(s) \simeq H^0_{-j}(\delta)$ for $j < 0$ and $H^l(s) \simeq H^l(\gamma|H_0(\delta))$ for $l \geq 0$ still hold. Regarding the last relations, j from $H^j(s)$ stands for the ghost number, $(-j)$ from $H^0_{-j}(\delta)$ represents the antighost number, and the superscript 0 refers to the value equal to zero of pgh; l from $H^l(s)$ means the ghost number, while l from $H^l(\gamma|H_0(\delta))$ signifies the pure ghost number. From (34) we notice that the form degree of δ, γ, and s is equal to zero

$$\deg(\delta) = \deg(\gamma) = \deg(s) = 0. \tag{36}$$

We define a linear operator $d : \Lambda \to \Lambda$ as an odd, right derivation

$$da = \partial_\mu a \, dx^\mu, \;\; a \in \mathcal{A}, \quad d(dx^\mu) = 0, \tag{37}$$

$$d(\omega_1 \wedge \omega_2) = \omega_1 \wedge d\omega_2 + (-)^{\varepsilon(\omega_2)} (d\omega_1) \wedge \omega_2, \;\; \omega_{1,2} \in \Lambda, \tag{38}$$

where it was assumed that $\omega_{1,2}$ possess well-defined Grassmann parities. The operator d becomes a differential on Λ with respect to deg, traditionally known as the exterior spacetime differential: $\varepsilon(d) = 1$, $\deg(d) = +1$, $d^2 = 0$. From (31) and (37) it follows that

$$\mathrm{agh}(d) = 0, \quad \mathrm{pgh}(d) = 0, \quad \mathrm{gh}(d) = 0. \tag{39}$$

The operators δ, γ, and s are also differentials that anticommute with d on Λ

$$O^2 = 0 = d^2, \quad Od + dO = 0, \quad O = \delta, \gamma, s. \tag{40}$$

Their gradings to not interfere

$$\mathrm{grad}(d) = 0, \quad \deg(O) = 0, \quad \mathrm{grad} = \mathrm{agh}, \mathrm{pgh}, \mathrm{gh}, \tag{41}$$

so it makes sense to compute the local cohomologies $H(O|d)$ in Λ. These are standardly defined like the set of equivalence classes of local forms O-closed modulo d, $O\omega + dj = 0$, modulo the local forms that are O-exact modulo d, $\omega' = Ow + dm$. We highlight that there is a strict correspondence between O and grad, namely: $O = \delta \leftrightarrow \mathrm{grad} = \mathrm{agh}$, $O = \gamma \leftrightarrow \mathrm{grad} = \mathrm{pgh}$, $O = s \leftrightarrow \mathrm{grad} = \mathrm{gh}$. These means that whenever $O = s$ the local BRST cohomology $H(s|d)$ is a vector space simultaneously $\mathbb{Z}_2$-graded (according to the

Grassmann parity) and $\mathbb{Z}$-graded in terms of gh, $H(s|d) = \bigoplus_{g\in\mathbb{Z}} H^g(s|d)$, where for every $g \in \mathbb{Z}$ the space $H^g(s|d)$ is in turn $\mathbb{N}$-graded according to the form degree, $H^g(s|d) = \bigoplus_{p=0}^{D} H^{g,p}(s|d)$. The subspace $H^{g,p}(s|d)$ is called local BRST cohomology in ghost number g and form degree p. If $O = \delta$, then the local cohomology of the Koszul–Tate differential $H(\delta|d)$ is a vector space $\mathbb{Z}_2$-graded and meanwhile $\mathbb{N}$-graded in terms of agh, $H(\delta|d) = \bigoplus_{j\in\mathbb{N}} H_j(\delta|d)$, where for every $j \in \mathbb{N}$ the space $H_j(\delta|d)$ is again $\mathbb{N}$-graded according to deg, $H_j(\delta|d) = \bigoplus_{p=0}^{D} H_j^p(\delta|d)$. The subspace $H_j^p(\delta|d)$ is known as the local cohomology of the Koszul–Tate differential in antighost number j and form degree p. Finally, if $O = \gamma$, then the local cohomology of the exterior longitudinal differential $H(\gamma|d)$ is a vector space $\mathbb{Z}_2$-graded, but also $\mathbb{N}$-graded in terms of pgh, $H(\gamma|d) = \bigoplus_{l\in\mathbb{N}} H^l(\gamma|d)$, where for every $l \in \mathbb{N}$ the space $H^l(\gamma|d)$ is $\mathbb{N}$-graded according to deg, $H^l(\gamma|d) = \bigoplus_{p=0}^{D} H^{l,p}(\gamma|d)$. The subspace $H^{l,p}(\gamma|d)$ means the local cohomology of the exterior longitudinal differential in pure ghost number l and form degree p.

The study of the local BRST cohomology is an essential step in view of constructing consistent interactions involving a massless tensor field with the mixed symmetry $(k, 1)$ by means of the deformation of the solution to the master equation [31]–[34]. This deformation method requires the computation of the local BRST cohomology in ghost number 0 and in maximum form degree. From this perspective in what follows we approach the main cohomological ingredients related to the spaces $H(\gamma)$ and $H(\gamma|d)$.

4 $H(\gamma)$ and $H(\gamma|d)$

In the sequel we evaluate the cohomology algebra $H(\gamma)$ in the algebra of local forms Λ, defined like the set of equivalence classes of γ-closed local forms modulo γ-exact ones. Due to the second relation in (34) it is enough to compute $H(\gamma)$ in the BRST algebra of local "functions" $\mathcal{A}$, defined as the set of equivalence classes of γ-closed elements from $\mathcal{A}$ modulo γ-exact ones. The computation of the cohomology $H(\gamma)$ in $\mathcal{A}$ or in Λ makes sense since the operator γ is a true differential on both algebras in this case, with $\mathrm{pgh}(\gamma) = +1$, $\gamma^2 = 0$. We recall that $H(\gamma)$ defines a supercommutative algebra ($\mathbb{Z}_2$-graded), $\mathbb{N}$-graded in terms of pgh, $H(\gamma) = \bigoplus_{l\in\mathbb{N}} H^l(\gamma)$. Moreover, if we work on Λ, then for every $l \in \mathbb{N}$ the space $H^l(\gamma)$ is also $\mathbb{N}$-graded with respect to the form degree

$$H^l(\gamma) = \bigoplus_{p=0}^{D} H^{l,p}(\gamma), \quad l \in \mathbb{N}. \tag{42}$$

We rely on definitions (15)–(18) and approach the construction gradually, according to the increasing values of pgh.

From Table 1 we observe that there are no BRST generators with negative pure ghost numbers, such that in pgh $= 0$ the cohomology $H^0(\gamma)$ coincides with the kernel of γ, $H^0(\gamma) = (\mathrm{Ker}(\gamma))^0$ and, due to the additive behavior of pgh with respect to the multiplication operation on $\mathcal{A}$, it will actually be an algebra. Table 1 helps us to identify the BRST generators of pure ghost number equal to 0 being given by the antifields Φ_A^* introduced in (13) and their spacetime derivatives up to a finite order together with the field $t_{\mu_1\ldots\mu_k|\alpha}$ and its derivatives up to a finite order. The last definition from (18) implies that the (polynomial) dependence on $[\Phi_A^*]$ produces elements belonging to $(\mathrm{Ker}(\gamma))^0$, and thus implicitly to $H^0(\gamma)$, where the generic notation $f[\varphi]$ means that f depends on φ and its derivatives up to a finite order. Relation (15) compared with (1) shows that the action of γ on the field with the mixed symmetry $(k, 1)$ follows from its gauge transformation by replacing the gauge parameters $\left\{\overset{(1)}{\theta}, \overset{(1)}{\epsilon}\right\}$ respectively with the ghosts $\left\{\overset{(1)}{C}, \overset{(1)}{\eta}\right\}$. Since the most general gauge-invariant quantities constructed out of the field with the mixed symmetry $(k,1)$ and its derivatives are given by the curvature tensor (9) together with its derivatives, we obtain that the entire dependence on $t_{\mu_1\ldots\mu_k|\alpha}$ of the elements from $H^0(\gamma)$ is represented by polynomials (in order to ensure the spacetime locality) in $[K_{\mu_1\ldots\mu_{k+1}|\alpha\beta}]$. In conclusion, $H^0(\gamma)$ computed in the BRST algebra of local "functions" $\mathcal{A}$ is precisely the algebra of invariant polynomials (local "functions" with pgh $= 0$ that are γ-invariant and therefore true polynomials in $[\Phi^*_A]$ and $[K_{\mu_1\ldots\mu_{k+1}|\alpha\beta}]$ since they are not allowed to depend on the undifferentiated components of the field t)

$$H^0(\gamma) \text{ in } \mathcal{A} = \{\text{algebra of invariant polynomials}\} \equiv \{\alpha([\Phi^*_A],[K])\}. \tag{43}$$

Consequently, $H^0(\gamma)$ computed in the algebra of local differential forms Λ will also be an algebra (where the function multiplication must be replaced with the wedge product among the forms) allowing for a decomposition of the form (42) with $l = 0$, where the elements of each space $H^{0,p}(\gamma)$ are p-forms whose coefficients are invariant polynomials

$$H^0(\gamma) = \bigoplus_{p=0}^{D} H^{0,p}(\gamma), \quad H^{0,p}(\gamma) \ni \overset{[p]}{\alpha} = \tfrac{1}{p!}\alpha_{\mu_1\ldots\mu_p}([\Phi^*_A],[K])dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}. \tag{44}$$

In the next step, from Table 1 we identify the BRST generators of pure ghost number 1 being expressed by the ghosts $\overset{(1)}{C}$ and $\overset{(1)}{\eta}$ together with their derivatives up to a finite order and further use definitions (16) and (17) for $m = 1$. Equivalently, from (24) for $m = 1$ we get that the BRST generators of pure ghost number 1 are given by linear combinations of the ghosts $\overset{(1)}{C}{}'$ and of their derivatives up to a finite order. From the action of γ on the latter generators, given by (27) for $m = 1$, we deduce the most general γ-closed quantities (so from Ker(γ) at pgh $= 1$) linear in $\overset{(1)}{C}{}'$ and their derivatives up to a finite order under the form

$$\left\{\partial_{[\mu_1}\overset{(1)}{C}{}'_{\mu_2\ldots\mu_k]||\alpha}, \partial_{\rho_1}\partial_{[\mu_1}\overset{(1)}{C}{}'_{\mu_2\ldots\mu_k]||\alpha}, \cdots, \partial_{\rho_1\ldots\rho_n}\partial_{[\mu_1}\overset{(1)}{C}{}'_{\mu_2\ldots\mu_k]||\alpha}\right\} \in (\mathrm{Ker}(\gamma))^1.$$

It is more convenient to introduce the notations

$$\partial_{[\mu_1}\overset{(1)}{C}{}'_{\mu_2\ldots\mu_k]||\alpha} \equiv \overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha}, \tag{45}$$

in terms of which the previous relation becomes

$$\left\{\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha}, \partial_{\rho_1}\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha}, \partial_{\rho_1\rho_2}\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha}, \cdots, \partial_{\rho_1\ldots\rho_n}\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha}\right\} \in (\mathrm{Ker}(\gamma))^1. \tag{46}$$

With the help of formula (26) it can be shown that

$$\partial_{\rho_1}\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha} = \gamma(\partial_{\rho_1}t_{\mu_1\ldots\mu_k|\alpha} + (-)^{k+1}\partial_{[\mu_1}t_{\mu_2\ldots\mu_k\alpha]|\rho_1}), \tag{47}$$

such that all the elements from (46) excepting the first one are γ-exact (or, in other words, trivial in $H(\gamma)$)

$$\left\{\partial_{\rho_1}\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha}, \partial_{\rho_1\rho_2}\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha}, \cdots, \partial_{\rho_1\ldots\rho_n}\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha}\right\} \in (\mathrm{Im}(\gamma))^1. \tag{48}$$