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A study on K- paracontact and (κ, μ)- paracontact manifold admitting vanishing Cotton tensor and Bach tensor

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Abstract. The object of the present paper is to study K-paracontact manifold admitting parallel Cotton tensor, vanishing Cotton tensor and to study Bach flatness on K-paracontact manifold. In that we prove for a K-paracontact metric manifold M^{2n+1} has parallel Cotton tensor if and only if M^{2n+1} is an η-Einstein manifold and $r = -2n(2n+1)$. Further we show that if g is an η-Einstein K-paracontact metric and if g is Bach flat then g is an Einstein. Also we study vanishing Cotton tensor on (κ, μ)-paracontact manifold for both $\kappa > -1$ and $\kappa < -1$. Finally, we prove that if M^{2n+1} is a (κ, μ)-paracontact manifold for $\kappa \neq -1$ and if M^{2n+1} has vanishing Cotton tensor for $\mu \neq \kappa$, then M^{2n+1} is an η-Einstein manifold.

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1. Introduction

In 1921, the notion of Bach tensor was introduced by R. Bach [1] to study conformal relativity. This is a symmetric traceless $(0, 2)$-type tensor B on an n-dimensional Riemannian manifold (M, g), defined by

$$\begin{aligned} B(X,Y) =& \frac{1}{n-1} \sum_{i,j=1}^{n} \big((\nabla_{e_i} \nabla_{e_j} W)(X, e_i, e_j, Y) \big) \\ &+ \frac{1}{n-2} \sum_{i,j=1}^{n} Ric(e_i, e_j) W(X, e_i, e_j, Y), \end{aligned} \tag{1}$$

where $(e_i), i = 1, ..., n$, is a local orthonormal frame on $(M; g)$, Ric is the Ricci tensor of type $(0, 2)$ and C is the $(0, 3)$-type Cotton tensor defined by[9]

$$\begin{aligned} C(X,Y)Z =& (\nabla_X Ric)(Y, Z) - (\nabla_Y Ric)(X, Z) \\ &- \frac{1}{2(n-1)} [g(Y, Z)(X_r) - g(X, Z)(Y_r)] \end{aligned} \tag{2}$$

and W denotes the Weyl tensor of type $(0, 3)$ defined by[9]

$$\begin{aligned} W(X,Y)Z =& R(X,Y)Z - g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X \\ &- g(QX,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y). \end{aligned} \tag{3}$$

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After Bach[1], many people worked on Bach tensor; In 1993 Pedersen and Swann[13] studied Einstein-Weyl geometry, the Bach tensor and conformal scalar curvature. In 2013-14 H.D. Cao and others ([6] and [7]) studied Bach tensor on gradient shrinking and steady Ricci soliton. In 2017 Ghosh and Sharma [10] studied Sasakian manifolds with purely transversal Bach tensor. In that article they shows a $(2n+1)$-dimensional Sasakian manifold (M, g) with a purely transversal Bach tensor has constant scalar curvature $\geq 2n(2n+1)$, equality holding if and only if (M, g) is Einstein. For dimension 3, M is locally isometric to the unit sphere S^3. For dimension 5, if in addition (M, g) is complete, then it has positive Ricci curvature and is compact with finite fundamental group $\pi_1(M)$. Recently in 2019 Ghosh and Sharma [9] studied classification of (κ, μ)-contact manifold with divergence free Cotton tensor and vanishing Bach tensor.

The study of paracontact geometry was introduced by Kaneyuki and Williams in [11]. A systematic study of paracontact metric manifolds started with the paper [16], were the Levi-Civita connection, the curvature and a canonical connection (analogue to the Tanaka Webster connection of the contact metric case) of a paracontact metric manifold have been described.

There are differences between a contact metric (κ, μ)- space $(M^{2n+1}, \phi, \xi, \eta, g)$ and a paracontact metric (κ, μ)-space $(M^{2n+1}, \phi, \xi, \eta, g)$. Namely, unlike in the contact Riemannian case, a paracontact (κ, μ)-manifold such that $\kappa = -1$ in general is not para-Sasakian. In fact, there are paracontact (κ, μ)-manifolds such that $h^2 = 0$ (which is equivalent to take $\kappa = -1$) but with $h \neq 0$. For 5-dimensional, Cappelletti Montano and Di Terlizzi gave the first example of paracontact metric $(-1, 2)$-space $(M^{2n+1}, \phi, \xi, \eta, g)$ with $h^2 = 0$ but $h \neq 0$ in [5] and then Cappelletti Montano et. al., gave the first paracontact metric structures defined on the tangent sphere bundle and constructed an example with arbitrary n in [2]. Later, for 3-dimensional, the first numerical example was given in [8]. Another important difference with the contact Riemannian case, due to the non-positive definiteness of the metric, is that while for contact metric (κ, μ)-spaces the constant κ can not be greater than 1, paracontact metric (κ, μ)-space has no restriction for the constants κ and μ.

These papers leads interest and gives motivation to us to study Bach and Cotton tensor on K-paracontact and (κ, μ)-paracontact manifold.

After the introduction, we discuss preliminary part, it includes some basic definitions and some important properties of K-paracontact and (κ, μ)-paracontact manifold which are related to our paper and in the third section we study vanishing Cotton tensor on K-paracontact manifold, in the next section we study parallel Cotton tensor on K-paracontact manifold. In section five, we study Bach tensor on η-Einstein K-paracontact manifold $(n > 1)$. Finally in the last two sections, we discuss vanishing Cotton tensor on (κ, μ)-paracontact manifold for both $\kappa > -1$ and $\kappa < -1$.

2. Preliminaries

In this section, we recall some basic definitions, which are helpful for our future studies. For more information we refer [3],[12],[15]. A $(2n+1)$-dimensional smooth manifold M^{2n+1} has a almost paracontact structure (φ, ξ, η) if it admits a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η such that

$$\varphi^2 = I - \eta \cdot \xi, \quad \varphi(\xi) = 0, \quad \eta \cdot \varphi = 0, \quad \eta(\xi) = 1, \tag{4}$$

for all $X, Y \in TM^{2n+1}$ and the eigen distributions D^+ and D^- of φ corresponding to the respective eigenvalues 1 and -1 have equal dimension n. If an almost paracontact manifold is endowed with a semi-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{5}$$

where signature of g is necessarily $(n+1, n)$ for all $X, Y \in TM^{2n+1}$, then $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called an almost paracontact metric manifold. The curvature tensor R is taken with the sign convention $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ (note that an opposite convention is used in [[3],[4],[14]]. By Q and r, we shall denote the Ricci operator determined by $S(X,Y) = g(QX,Y)$ and the scalar curvature of the metric g, respectively. The fundamental 2-form of an almost paracontact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is defined by $\Phi(X,Y) = g(X, \varphi Y)$. If $d\eta = \Phi$, then the manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be paracontact metric manifold and g the associated metric. In such case η is a contact form (that is, $\eta \wedge (d\eta)^n \neq 0$), ξ is its Reeb vector field and M^{2n+1} is a contact manifold. If, in addition, ξ is a Killing vector field (equivalently, $h = \frac{1}{2}\pounds_\xi \varphi = 0$, where $\pounds$ is the usual Lie derivative), then M^{2n+1} is said to be a paracontact metric manifold. In a K-paracontact manifold, we can easily get the following formulas

$$\begin{aligned} \nabla_X \xi &= -\varphi X + \varphi h X, & (6)\\ \nabla_\xi h &= -\varphi + \varphi h^2 - \varphi l & (7)\\ Ric(\xi, \xi) &= g(Q\xi, \xi) = Trl = Tr(h^2) - 2n, & (8) \end{aligned}$$

for all vector fields X, Y on M, where ∇ is the operator of covariant differentiation of g and Q denotes the Ricci operator associated with the Ricci tensor given by $Ric(X,Y) = g(QX,Y)$ for all vector fields X, Y on M. If the vector field ξ is Killing (equivalently, $h = 0$) then M is said to be a K-paracontact manifold. On K-paracontact manifold, the following formulas hold:

$$\begin{aligned} \nabla_X \xi &= -\varphi X & (9)\\ R(X, \xi)\xi &= -X + \eta(X)\xi & (10)\\ Q\xi &= -2n\xi & (11) \end{aligned}$$

Proposition 2.1. *On a K-paracontact manifold $M^{2n+1}(\varphi, \xi, \eta, g)$, we have (from [12])*

$$\begin{aligned} (i) \quad & (\nabla_X Q)\xi = Q\varphi X + 2n\varphi X, & (12)\\ (ii) \quad & (\nabla_\xi Q)X = Q\varphi X - \varphi Q X, & (13) \end{aligned}$$

for any vector field X on M^{2n+1}.

Definition 2.1. (See [2]) A paracontact metric (κ,μ)-manifold M^{2n+1} is a paracontact metric manifold for which the curvature tensor field satisfies

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{14}$$

for all vector fields X, Y on M^{2n+1} and for some real constants κ and μ .

Further, a paracontact metric manifold M satisfies the following properties

$$\begin{aligned} h^2 &= (1+\kappa)\varphi^2, & (15)\\ Q\xi &= 2n\kappa\xi, & (16)\\ (\nabla_X \varphi)Y &= -g(X - hX, Y)\xi + \eta(Y)(X - hX), \ \text{for } \kappa = -1, & (17) \end{aligned}$$

$$\begin{aligned}(\nabla_X h)Y - (\nabla_Y h)X =& -(1+\kappa)2g(X,\varphi Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X \\ & + (1-\mu)\eta(X)\varphi hY - \eta(Y)\varphi hX, \end{aligned} \tag{18}$$

$$\begin{aligned} Q =& (2(1-n)+n\mu)I + (2(n-1)+\mu)h \\ & + (2(n-1)+n(2\kappa-\mu))\eta\otimes\xi, \quad for\ \ \kappa > -1, \end{aligned} \tag{19}$$

$$\begin{aligned} Q =& (2(1-n)+n\mu)I + (2(n+1)+\mu)h \\ & + (2(n-1)+n(2\kappa-\mu))\eta\otimes\xi, \quad for\ \ \kappa < -1, \end{aligned} \tag{20}$$

for any vector fields X, Y on M, where Q denotes the Ricci operator of (M^{2n+1}, g).

Definition 2.2. A Riemannian manifold is called an η-Einstein manifold, if it has *Ricci* tensor Q such that

$$QY = aY + b\eta(Y)\xi \tag{21}$$

where $a, b \in C^\infty(M^{2n+1})$ and if the function $b = 0$ then it is called Einstein.

3. Vanishing Cotton tensor on K-paracontact manifold

Proposition 3.1. *Let M^{2n+1} be a K-paracontact manifold. Then M^{2n+1} has constant scalar curvature if and only if $C(X,\xi)\xi = 0$*

Proof. Setting $Z = \xi$ in (2) we get.

$$C(X,Y)\xi = g((\nabla_X Q)\xi, Y) - g((\nabla_Y Q)\xi, X) - \frac{1}{4n}[(Xr)\eta(Y) - (Yr)\eta(X)] \tag{22}$$

Using equation (12) from Proposition [2.1] in the above equation, we get

$$C(X,Y)\xi = -4ng(\varphi X, Y) + g(Q\varphi X, Y) - g(Q\varphi Y, X) - \frac{1}{4n}[(Xr)\eta(Y) - (Yr)\eta(X)]. \tag{23}$$

Replacing X by φX and Y by φY in (23) we obtain,

$$C(\varphi X, \varphi Y)\xi = 4ng(\varphi X, Y) + g(Q\varphi^2 X, \varphi Y) - g(Q\varphi^2 Y, \varphi X) = 0, \tag{24}$$

which gives

$$-4ng(\varphi X, Y) - g(X, Q\varphi Y) + g(Q\varphi X, Y) = 0. \tag{25}$$

Admitting (25) in (23), we get,

$$(Xr)\eta(Y) - (Yr)\eta(X) = 0. \tag{26}$$

Putting $Y = \xi$ and taking X orthogonal to ξ in the above equation gives

$$Xr = 0. \tag{27}$$

As M is paracontact manifold and $X \in ker\eta$ which implies $Xr = 0$, $\forall X \in TM^{2n+1}$. So r is constant.

Conversely, if r is constant then substituting $Y = \xi$ in the equation (23) gives $C(X,\xi)\xi = 0$.
Hence the proof. ∎

4. Parallel Cotton tensor on K-paracontact manifold M^{2n+1}

Definition 4.1. In a Riemannian manifold M^{2n+1}, if there is a Cotton tensor C such that its covariant differentiation i.e., $(\nabla_W C) = 0$ then the manifold is said to have parallel Cotton tensor.

Theorem 4.1. *Let M^{2n+1} be a K-paracontact metric manifold. Then M has parallel Cotton tensor if and only if M^{2n+1} is an η-Einstein manifold and $r = -2n(2n+1)$.*

Proof. For a K-paracontact manifold M^{2n+1}, the equation (2) for $Y = \xi$ and $Z = Y$ is gives

$$C(X,\xi)Y = 2ng(\varphi X, Y) + g(Q\varphi X, Y) - \frac{1}{4n}\{(Xr)\eta(Y)\}. \tag{28}$$

Taking $Y = \xi$ in the above equation, we get

$$C(X,\xi)\xi = -\frac{1}{4n}\{(Xr)\}. \tag{29}$$

Using (29) in (22) we calculate the following relations

$$\nabla_W C(X,\xi)\xi = -\tfrac{1}{4n}\{g(\nabla_W X, Dr) + g(X, \nabla_W Dr)\}, \tag{30}$$

$$C(\nabla_W X,\xi)\xi = -\tfrac{1}{4n}\{g(\nabla_W X, Dr)\}, \tag{31}$$

$$C(X,\varphi W)\xi = 4ng(\varphi X, \varphi W) + g(Q\varphi X, \varphi W) - g(Q\varphi^2 W, X) - \tfrac{1}{4n}\{-(\varphi W r)\eta(X)\}, \tag{32}$$

$$C(X,\xi)\varphi W = 2ng(\varphi X, \varphi W) + g(Q\varphi X, \varphi W). \tag{33}$$

Making use of above group of equations we obtain

$$\begin{aligned}(\nabla_W C)(X,\xi)\xi = &-\frac{1}{4n}\{g(X, \nabla_W Dr)\} + 4ng(\varphi X, \varphi W) + g(Q\varphi X, \varphi W)\\ &-g(Q\varphi^2 W, X) - \frac{1}{4n}\{(\varphi W r)\eta(X)\} + 2ng(\varphi X, \varphi W) + g(\varphi Q X, \varphi W).\end{aligned} \tag{34}$$

Putting $W = \xi$ in the above equation, the parallel Cotton tensor becomes

$$(\nabla_\xi C)(X,\xi)\xi = -\frac{1}{4n}\{g(X, \nabla_\xi Dr)\} = 0. \tag{35}$$

As $\pounds_\xi r = 0, \nabla_\xi Dr = \nabla_{Dr}\xi = -\varphi Dr$, which implies $g(X, \varphi Dr) = 0$, which gives $Dr = 0$ and so r is constant. Then the relation (34) becomes

$$\begin{aligned}6ng(\varphi X, \varphi W) + g(Q\varphi X, \varphi W) - g(X, QW) - 2n\eta(X)\eta(W)\\ -g(X, QW) - 2n\eta(X)\eta(W) = 0.\end{aligned} \tag{36}$$

Replacing X by φX and W by φW in (36) and simplifying we get

$$g(Q\varphi X, \varphi W) = -3ng(\varphi X, \varphi W) + \frac{1}{2}g(QX, W) + n\eta(X)\eta(W). \tag{37}$$

Feeding (37) in (36) we obtain

$$\begin{aligned}6ng(X, W) + 6n\eta(X)\eta(W) - 3ng(X, W) + 3n\eta(X)\eta(W) + \frac{1}{2}g(X, \varphi W)\\ +n\eta(X)\eta(W) - 4n\eta(X)\eta(W) - 2g(X, QW) = 0.\end{aligned} \tag{38}$$

Contracting the equation (38) over X and W we have $r = -2n(2n+1)$ and M is an η-Einstein manifold.

Conversely, suppose M is an η-Einstein manifold and $r = -2n(2n+1)$, which implies

$QY = -2nY$. And so this gives $C(X,Y)Z = 0$.
Hence the proof. □

Lemma 4.2. *Let $M^{2n+1}(n > 1)$ be a K-paracontact manifold. If M^{2n+1} satisfies (21), then a and b are constant functions*

Proof. From the condition (21) we have,

$$(\nabla_X Q)Y = (Xa)Y + (Xb)\eta(Y)\xi + b\left\{g(X,\varphi Y)\xi + \eta(Y)\nabla_X\xi\right\}. \tag{39}$$

From η-Einstein condition, $-2n = a+b$, so $(Xa) = -(Xb)$.
Therefore

$$(\nabla_X Q)Y = (Xa)Y - (Xa)\eta(Y)\xi + \{-2n-a\}\left\{g(X,\varphi Y)\xi - \eta(Y)\varphi X\right\}. \tag{40}$$

Contracting the above equation over X with respect to the orthonormal frame field we get

$$\sum_{i=1}^{2n+1}\epsilon_i\left\langle(\nabla_{e_i}Q)Y, e_i\right\rangle = \sum_{i=1}^{2n+1}\epsilon_i(e_i a)g(Y,e_i) + (\xi a) \tag{41}$$

where $\xi = g(e_i,e_i)$, as $\xi r = 0$ gives $\xi a = 0$. But we know that $\sum_{i=1}^{2n+1}\left\langle(\nabla_{e_i}Q)Y, e_i\right\rangle = \frac{1}{2}(Yr)$ which gives

$$\frac{1}{2}(Yr) \quad = \quad g(Y, Da) \tag{42}$$

as $Yr = 2$, so $(n-1)Ya = 0$ for $n > 1$ becomes $Ya = 0$, therefore a is constant.
This completes the proof. □

5. Bach tensor on η-Einstein K-paracontact manifolds for $(n > 1)$

Bach tensor for $2n+1$-dimensional manifold is given by

$$B(X,Y) = \frac{1}{2n-1}\left\{\sum_{i=1}^{2n+1}\epsilon_i(\nabla_{e_i}C)(e_i,X,Y) + \sum_{i,j=1}^{2n+1}\epsilon_i g(Qe_i,e_j)W(X,e_i,e_j,Y)\right\} \tag{43}$$

By lemma (4.2) we know that a and b are constants then equation (39) becomes

$$(\nabla_X Q)Y = b\left\{g(X,\varphi Y)\xi - \eta(Y)\varphi X\right\}. \tag{44}$$

We know that from the lemma (4.2) r is constant and simplifying the cotton tensor using (44)

$$C(X,Y)Z \quad = \quad bg(X,\varphi Y)\eta(Z) - b\eta(Y)g(\varphi X,Z) - bg(Y,\varphi X)\eta(Z) + bg(\varphi Y,Z)\eta(X).$$

Applying ∇_W on both side of the above equation gives

$$\begin{aligned}(\nabla_W C)(X,Y)Z =& b\nabla_W\left\{2g(X,\varphi Y)\eta(Z) + \eta(X)g(\varphi Y,Z) + \eta(Y)g(X,\varphi Z)\right\}\\ =& b2g(X,(\nabla_W\varphi)Y)\eta(Z) + bg(X,\varphi Y)g(W,\varphi Z) + bg((\nabla_W\varphi)Y,Z)\eta(X)\\ &+ bg(\varphi Y,Z)g(W,\varphi X) + bg(X,(\nabla_W\varphi)Z)\eta(Y) + bg(X,\varphi Z)g(W,\varphi Y).\end{aligned} \tag{45}$$

On contracting above equation over X and W gives

$$\begin{aligned}\sum_{i=1}^{2n+1}\epsilon_i(\nabla_{e_i}C)(e_i,Y)Z &= b\left\{\sum_{i=1}^{2n+1}\epsilon_i g(e_i,(\nabla_{e_i}\varphi)Y)\eta(Z+g(e_i,(\nabla_{e_i}\varphi)Z)\eta(Y)\right\}+2bg(\varphi Y,\varphi Z)\\ &= b\left\{\sum_{i=1}^{2n+1}\epsilon_i\langle R(\xi,e_i)Y,e_i\rangle\eta(Z)+g(R(\xi,e_i)Z,e_i)\eta(Y)\right\}+2bg(\varphi Y,\varphi Z)\\ &= b\{-S(Y,\xi)\eta(Z)-S(Z,\xi)\eta(Y)\}+2bg(\varphi Y,\varphi Z)\\ &= b\{4n\eta(Y)\eta(Z)+2g(\varphi Y,\varphi Z)\}.\end{aligned}$$

Now we calculate the right hand side of the Bach tensor that is

$$\sum_{i,j=1}^{2n+1}\epsilon_i g(Qe_i,e_j)g(W(X,e_i)e_j,Y) \quad = \quad -\sum_{i,j=1}^{2n+1}\epsilon_i g(Qe_i,W(X,e_i)Y).$$

By η-Einstein condition $Qe_i=ae_i+b\eta(e_i)\xi$, which gives

$$\begin{aligned}\sum_{i,j=1}^{2n+1}\epsilon_i g(Qe_i,e_j)g(W(X,e_i)e_j,Y) &= -\sum_{i,j=1}^{2n+1}\epsilon_i g(e_i+b\eta(e_i)\xi,W(X,e_i)Y)\\ &= \sum_{i=1}^{2n+1}\epsilon_i g(W(X,e_i)e_i,Y)+bg(W(X,\xi)\xi,Y).\end{aligned} \tag{46}$$

From the expression of Weyl tensor W we deduce the following relation

$$\begin{aligned}\sum_{i=1}^{2n+1}\epsilon_i\langle W(X,e_i)e_i,Y\rangle &= \sum_{i=1}^{2n+1}\epsilon_i(\langle R(X,e_i)e_i,Y\rangle-\frac{1}{2n-1}[g(Qe_i,e_i)g(X,Y)\\ &\quad -g(QX,e_i)g(e_i,Y)+g(e_i,e_i)g(QX,Y)-g(X,e_i)g(Qe_i,Y)]\\ &\quad +\frac{r}{2n(2n-1)}[g(e_i,e_i)g(X,Y)-g(X,e_i)g(e_i,Y)])\\ &=S(X,Y)-\frac{1}{2n-1}[rg(X,Y)-S(X,Y)+(2n+1)S(X,Y)\\ &\quad -S(X,Y)]+\frac{r}{2n(2n-1)}[(2n+1)g(X,Y)-g(X,Y)]\\ &=0.\end{aligned} \tag{47}$$

Taking inner product of $W(X,\xi)\xi$ with Y we get,

$$\begin{aligned}\langle W(X,\xi)\xi,Y\rangle &= \langle R(X,\xi)\xi,Y\rangle-\frac{1}{2n-1}(-2n\langle X,Y\rangle+2n\eta(X)\eta(Y)+\langle QX,Y\rangle\\ &\quad +2n\eta(X)\eta(Y))+\frac{r}{2n(2n-1)}(\langle X,Y\rangle-\eta(X)\eta(Y))\\ &= \langle\varphi\nabla_X\xi,Y\rangle+\frac{2n}{2n-1}(X,Y)-\frac{4n}{2n-1}\eta(X)\eta(Y)+\frac{r}{2n(2n-1)}(X,Y)\\ &\quad -\frac{r}{2n(2n-1)}\eta(X)\eta(Y)-\frac{1}{2n-1}S(X,Y)\end{aligned}$$

But $\langle\varphi X,\varphi Y\rangle=-\langle X,Y\rangle+\eta(X)\eta(Y)$, so we get

$$\langle W(X,\xi)\xi,Y\rangle=\frac{1}{2n-1}\left\{\left(1+\frac{r}{2n}\right)(X,Y)-\left(1+2n+\frac{r}{2n}\right)\eta(X)\eta(Y)\right\}-\frac{1}{2n-1}S(X,Y) \tag{48}$$

Using the value of $S(X,Y) = \left(1+\frac{r}{2n}\right)(X,Y) - \left(1+2n+\frac{r}{2n}\right)\eta(X)\eta(Y)$ in (48) gives

$$\langle W(X,\xi)\xi, Y\rangle = 0. \tag{49}$$

Therefore if g is Bach flat,

$$B(Y,Z) = 0 = \frac{b}{2n-1}\left\{4n\eta(Y)\eta(Z) + 2g(\varphi Y, \varphi Z)\right\}. \tag{50}$$

For $Y = Z = \xi$ we obtain $b = 0$. Hence we can state this result

Theorem 5.1. *Let M^{2n+1} be an η-Einstein K-paracontact manifold. If it has Bach flat then M^{2n+1} is an Einstein manifold.*

6. (κ, μ)-paracontact manifold, for $\kappa \neq -1$

In this section we deal with paracontact (κ, μ)-manifolds such that $\kappa > -1$ and $\kappa < -1$.
First for $\kappa > -1$, using (19) we calculate,

$$\begin{aligned}(\nabla_X Q)Y =& g(2(n-1)+\mu)(\nabla_X h)Y \\ &+ (2(n-1)+n(2\kappa-\mu))\{(\nabla_X\eta)Y\xi + \eta(Y)\nabla_X\xi\}\end{aligned} \tag{51}$$

Now considering the Cotton tensor on (κ, μ)-paracontact manifold as from (19), r is constant, which implies

$$C(X,Y)Z = g((\nabla_X Q)Y, Z) + g((\nabla_Y Q)X, Z). \tag{52}$$

Using equation (51) we obtain

$$\begin{aligned}C(X,Y)Z =& (2(n-1)+\mu)\{-(1+\kappa)(2g(X,\varphi Y)\eta(Z) + \eta(X)g(\varphi Y, X) \\ &- \eta(Y)g(\varphi X, Z)) + (1+\mu)(\eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z))\} \\ &+ 2(2(n-1)+n(2\kappa-\mu))g(X,\varphi Y)\eta(Z) + (2(n-1)+n(2\kappa-\mu)) \\ &\times\{\eta(Y)g(-\varphi X + \varphi hX, Z) - \eta(X)g(-\varphi Y + \varphi hY, Z)\}.\end{aligned} \tag{53}$$

Replacing X, Y, Z by $\varphi X, \varphi Y, \varphi Z$ respectively in the above equation then we get $C(\varphi X, \varphi Y)\varphi Z = 0$.
Similarly for $\kappa < -1$ we have from (20)

$$(\nabla_X Q)Y = g(2(n-1)+\mu)(\nabla_X h)Y + (2(n+1)+n(2\kappa-\mu))\{(\nabla_X\eta)Y\xi + \eta(Y)\nabla_X\xi\}$$

Now consider the Cotton tensor with r is constant and substitute $(\nabla_X Q)Y$ and $(\nabla_Y Q)X$ values in Cotton tensor then we get

$$\begin{aligned}C(X,Y)Z =& g((\nabla_X Q)Y, Z) + g((\nabla_Y Q)X, Z) \\ =& (2(n+1)+\mu)\{-(1+\kappa)(2g(X,\varphi Y)\eta(Z) + \eta(X)g(\varphi Y, X) \\ &- \eta(Y)g(\varphi X, Z)) + (1+\mu)(\eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z))\} \\ &+ 2(2(n+1)+n(2\kappa-\mu))g(X,\varphi Y)\eta(Z) + (2(n+1)+n(2\kappa-\mu)) \\ &\times\{\eta(Y)g(-\varphi X + \varphi hX, Z) - \eta(X)g(-\varphi Y + \varphi hY, Z)\}\end{aligned} \tag{54}$$

Replacing X, Y and Z by $\varphi X, \varphi Y$ and φZ respectively in the above equation, then $C(\varphi X, \varphi Y)\varphi Z = 0$.
Form the above two cases, when $\kappa \neq -1$ we obtain the following result;

Proposition 6.1. *On a (κ,μ)-paracontact metric manifold for $\kappa \neq -1$ the projection of the image of Cotton tensor $C/_{\varphi T_P(M^{2n+1}) X \varphi T_P(M^{2n+1})}$ in $\varphi T_p(M^{2n+1})$ is zero, i.e., $C(\varphi X,\varphi Y)\varphi Z = 0$, $\forall X, Y, Z \in T_P(M^{2n+1})$*

7. Vanishing Cotton tensor on (κ,μ)-paracontact manifold, for $\kappa \neq -1$

In this section we deal with paracontact (κ,μ)-manifolds such that $\kappa < -1$ and $\kappa > -1$ then we have the Cotton tensor $C(X,Y)Z = 0$.
For $\kappa > -1$, replacing Z by ξ in equation (54) then we get

$$C(X,Y)\xi = 0 = (2(n-1)+\mu)\{-(1+\kappa)(2g(X,\varphi Y))\} + 2(2(n-1)+n(n(2\kappa-\mu))g(X,\varphi Y)$$
$$(2(n-1)+\mu)(1+\kappa) + (2(n-1)+n(2n-\mu)) = 0 \tag{55}$$

Similarly, admitting ξ in the place of X in equation (54) gives,

$$\begin{aligned} C(\xi,Y)Z = 0 \quad &= \quad (2(n-1)+\mu)\{-(1+\kappa)g(\varphi Y,Z) + (1+\mu)g(\varphi hY,Z)\} \\ &\quad +(2(n-1)n(2\kappa-\mu))\{g(\varphi Y,Z) - g(\varphi hY,Z)\} \end{aligned} \tag{56}$$

Symmetrizing the above equation and replacing Y by hY we obtain

$$(1+\kappa)\{(2(n-1)+\mu)(1+\mu) - (2(n-1)+n(2\kappa-\mu)\} = 0$$

From equation (55) it gives,

$$(1+\kappa)\{(2(n-1)+\mu)(1+\mu) - (2(n-1)+\mu)(1+\kappa)\} = 0$$
$$\implies (1+\kappa)(\mu-\kappa)(2(n-1)+\mu) = 0$$

The above calculations leads this result.
Case(i) If $\mu \neq \kappa$ then $(2(n-1)+\mu) = 0$. Therefore M^{2n+1} is η-Einstein.
Case(ii) If $\mu = \kappa$ then from equation (55) $\mu = \kappa = 0$ or $\mu = \kappa = 0$. Therefore the we have the following result.

Lemma 7.1. *Let M^{2n+1} be a (κ,μ)-paracontact manifold, admitting vanishing Cotton tensor for $\kappa > -1$ then we have*
i). If $\mu \neq \kappa$ then M^{2n+1} is an η-Einstein manifold,
ii). If $(2(n-1)+\mu) \neq 0$ then $\mu = \kappa = 0$.

Next for $\kappa < -1$, Cotton tensor is

$$\begin{aligned} C(X,Y)Z =&(2(n+1)+\mu)\{(\nabla_X\eta)Y - (\nabla_Y\eta)X\} + (2(n+1)+n(\kappa-\mu))\{(\nabla_X\eta)Y\eta(Z) \\ &- (\nabla_X\eta)X\eta(Z)\} + (2(n-1)+n(2\kappa-\mu))\{\eta(Y)\nabla_X\xi - \eta(X)\nabla_Y\xi\} \\ =&(2(n+1)+\mu)\{-(1+\kappa)2g(X,\varphi Y)\eta(Z) + \eta(X)g(\varphi Y,Z) - \eta(Y)g(\varphi X,Z)\} \\ &+ (1+\mu)(\eta(X)g(\varphi hX,Z) - \eta(Y)g(\varphi hX,Z)) \\ &+ 2(2(n-1)+n(2\kappa-\mu))g(X,\varphi Y)\eta(Z) + (2(n-1)+n(2\kappa-\mu) \\ &\times \{\eta(Y)g(-\varphi X+\varphi hX,Z) - \eta(X)g(-\varphi Y+\varphi hY,Z)\} \end{aligned} \tag{57}$$

Substitute Z by ξ in the above equation become

$$C(X,Y)\xi = 0 = \{(2(n+1)+\mu)(1+\kappa) - (2(n-1)+n(2\kappa-\mu)\} \tag{58}$$

Replace X by ξ in the equation (57) gives

$$\begin{aligned} C(\xi,Y)Z = 0 =&(-2(n-1)+\mu)(1+\kappa)g(\varphi Y,Z) + (2(n-1)+\mu)(1+\mu)g(\varphi hY,Z) \\ &+ (2(n-1)+n(2\kappa+\mu))\{g(\varphi Y,Z) - g(\varphi hY,Z)\}. \end{aligned} \tag{59}$$

On symmetrizing the above equation we have

$$(1+\kappa)(2(n+1)+\mu)(\mu-\kappa)=0. \tag{60}$$

Therefore we can state the following lemma

Lemma 7.2. *Let M^{2n+1} be a (κ,μ) paracontact metric manifold for $\kappa<-1$, if M^{2n+1} has vanishing Cotton tensor for $\mu\neq\kappa$ then M^{2n+1} is an η- Einstein manifold.*

From case (i) of lemma (7.1) and lemma (7.2) we get the following result.

Theorem 7.3. *Let M^{2n+1} be a (κ,μ)-paracontact manifold for $\kappa\neq-1$. If M^{2n+1} has vanishing Cotton tensor for $\mu\neq\kappa$, then M^{2n+1} is an η-Einstein manifold.*

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Superstability of higher-order fractional differential equations

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Abstract. Using generalized Taylor's formula, this work investigate the superstability for a class of fractional differential equations with Caputo derivative. In this way, some interesting results are generalized.

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1. Introduction and preliminaries

One of the important main research area in the theory of Functional Equations (FE) is the Hyers-Ulam stability (HUS). In the past, in 1940, the researcher Ulam proposed a problem regarding the stability of FE to give conditions for a linear mapping near an approximately linear mapping be in the talk at the University of Wisconsin. In 1941, author in [7] solved it. Recently, by replacing FE with Differential Equations (DE), a generalization of Ulam's Problem (UP) has been made and many studies obtained the HUS of DE [17, 18].

Fractional differential equations (FDE) is an important research field, recent investigation has been recorded in this area, this includes stability [3, 12, 13, 20], finite-time stability (FTS) [15], stabilization [14], observer design [9, 14] and fault estimation [10]. Nevertheless, the concept of Fractional Derivative (FD) is not new and is much as old as DE. First of all, in 1695, L'Hospital proposed the question regarding FD in a letter written to Leibniz and connected his generalization of DE. In the past few years, many researchers have investigated on the study of HUS of FDE and published an important number of works [1, 4, 5, 19].

Authors in [3] have proposed a novel concept named superstability (SS) which is a special case of HUS, they have studied the stability of the following FE: $\xi(\chi_1+\chi_2) = \xi(\chi_1)\xi(\chi_2)$. It is important to know that the earliest works related to SS of DE appeared in [6, 8]. To the best of our knowledge, there is no works in the literature which treats the same concept for the fractional order systems.

In this work, we will study the SS of the following initial value problem

$$^{C}D_r^{p\lambda}E(x) + A(x)E(x) = 0, \tag{1}$$

with initial conditions (IC):

$$E(r) = {}^{C}D_r^{\lambda}E(r) = {}^{C}D_r^{2\lambda}E(r) = ... = {}^{C}D_r^{(p-1)\lambda}E(r) = 0, \tag{2}$$

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where $p \in \mathbb{N}^*$, ${}^C D_r^{s\lambda} E \in C\big([r, r+u]\big)$, for each $s \in \{0, 1, ..., p\}$, $A \in C\big([r, r+u]\big)$, $u > 0$ and ${}^C D_r^{s\lambda} = {}^C D_r^{\lambda}.{}^C D_r^{\lambda}...{}^C D_r^{\lambda}$ ($s-$times).

Motivated by [6, 8], we introduce the following definition.

Definition 1.1. Suppose that E satisfies:

$$|\psi\big(A, E, {}^C D_r^{\lambda} E, {}^C D_r^{2\lambda} E, ..., {}^C D_r^{p\lambda} E\big)| \leq \nu, \ \forall \omega \in [r, r+u], \tag{3}$$

for some $\nu \geq 0$ with IC therefore either

$$|E(\omega)| \leq \vartheta \nu, \ \forall \omega \in [r, r+u],$$

where $\vartheta > 0$, or

$$\psi\big(A, E, {}^C D_r^{\lambda} E, {}^C D_r^{2\lambda} E, ..., {}^C D_r^{p\lambda} E\big) = 0.$$

Then, we say that (1) has SS with IC.

Definition 1.2. [11] Given $0 < l < 1$. The Caputo fractional derivative of an absolutely continuous function f is defined as,

$${}^C D_c^{l} f(s) = \frac{1}{\Gamma(1-l)} \int_c^s (s-\tau)^{-l} f'(\tau) d\tau. \tag{4}$$

Theorem 1.1. [16]*(Generalized Taylor's formulat) Let $0 < \eta < 1$. Assume that ${}^C D_{r_1}^{t\eta} h \in C\big([r_1, r_2]\big)$, for each $t \in \{0, 1, ..., s\}$, with $s \in \mathbb{N}^*$, then we have*

$$h(x) = \sum_{t=0}^{s-1} {}^C D_{r_1}^{t\eta} h(r_1) \frac{(x-r_1)^{t\eta}}{\Gamma(t\eta+1)} + {}^C D_{r_1}^{s\eta} h(c) \frac{(x-r_1)^{s\eta}}{\Gamma(s\eta+1)},$$

with $c \in [r_1, x]$, for each $x \in (r_1, r_2]$.

2. Main theorem

In this section, we present our main result.

Theorem 2.1. *Assume that $\sup_{\chi \in [r, r+u]} |A(\chi)| < \dfrac{\Gamma(p\lambda+1)}{u^{p\lambda}}$. Then, (1) has the SS with IC (2).*

Proof. Let $\nu > 0$, and $E \in C\big([r, r+u]\big)$ such that ${}^C D_r^{t\lambda} E \in C\big([r, r+u]\big)$ for each $t \in \{0, 1, ...p\}$, if

$$|{}^C D_r^{p\lambda} E(x) + A(x) E(x)| \leq \nu$$

and

$$E(r) = {}^C D_r^{\lambda} E(r) = {}^C D_r^{2\lambda} E(r) = ... = {}^C D_r^{(p-1)\lambda} E(r) = 0.$$

Using Theorem 1.1, we get

$$E(x) = \sum_{t=0}^{p-1} {}^C D_r^{t\lambda} E(r) \frac{(x-r)^{t\lambda}}{\Gamma(t\lambda+1)} + {}^C D_r^{p\lambda} E(c) \frac{(x-r)^{p\lambda}}{\Gamma(p\lambda+1)},$$

with $c \in [r, x]$, for every $x \in (r, r+u]$. Thus

$$\begin{aligned} |E(x)| &= |{}^C D_r^{p\lambda} E(c) \frac{(x-r)^{p\lambda}}{\Gamma(p\lambda+1)}| \\ &\leq \sup_{\chi \in [r, r+u]} |{}^C D_r^{p\lambda} E(\chi)| \frac{u^{p\lambda}}{\Gamma(p\lambda+1)}. \end{aligned} \tag{5}$$

Then,

$$\begin{aligned} \sup_{\chi\in[r,r+u]} |E(\chi)| &\leq \frac{u^{p\lambda}}{\Gamma(p\lambda+1)}\Big[\sup_{\chi\in[r,r+u]} |{}^{C}D_r^{p\lambda}E(\chi)-A(\chi)E(\chi)| \\ &\quad + \sup_{\chi\in[r,r+u]} |A(\chi)| \sup_{\chi\in[r,r+u]} |E(\chi)|\Big] \\ &\leq \frac{u^{p\lambda}}{\Gamma(p\lambda+1)}\nu + \frac{u^{p\lambda}}{\Gamma(p\lambda+1)} \sup_{\chi\in[r,r+u]} |A(\chi)| \sup_{\chi\in[r,r+u]} |E(\chi)|. \end{aligned} \tag{6}$$

Hence,

$$\sup_{\chi\in[r,r+u]} |E(\chi)|\Big(1-\frac{u^{p\lambda}}{\Gamma(p\lambda+1)} \sup_{\chi\in[r,r+u]} |A(\chi)|\Big) \leq \frac{u^{p\lambda}}{\Gamma(p\lambda+1)}\nu.$$

Therefore, there exists $K>0$ such that

$$|E(x)| \leq K\nu,$$

for all $x \in [r, r+u]$.
This complete the proof. □

Remark 2.1. It is important to note that in [8] authors have obtained the SS results for DE with integer-order derivatives while in our case, the main result is obtained for fractional-order derivatives. In this sense, our work present a full generalization of the interesting results in [8].

3. Conclusion

In this paper, the generalized Taylor formula is used to demonstrate the SS of FDE of higher-order under certain conditions.

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