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LIE GEOMETRIC METHODS IN THE STUDY OF OPTIMAL CONTROL AND VARIATIONAL CALCULUS WITH ECONOMIC APPLICATIONS

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PREFACE

The present book is devoted to Lie algebroids geometry and its applications to optimal control and variational calculus. The framework of the differential geometry is very useful in modelling and understanding of a large class of natural phenomena. The Lie geometric methods are applied successfully in differential equations, optimal control theory or theoretical physics. In the most of cases the study is starting with a variational problem formulated for a regular Lagrangian (see [1]), on the tangent bundle TM over the manifold M and very often the whole set of problems is transferred on the dual space T^*M, endowed with a Hamiltonian function, via Legendre transformation. The case of a non-regular Lagrangians is also studied. The problem in this case is that the proposed Lagrangian formalism yields a singular Lagrangian description, which makes the Legendre transform ill-defined and thus no straightforward Hamiltonian formulation can be related. One of the motivations for the present work is the study of Lagrangian systems subjected to external constraints (holonomic or nonholonomic). These systems have a wide application in many different areas as optimal control theory, mathematical economics or sub-Riemannian geometry.

In the last years the investigations have led to a geometric framework which is covering these phenomena. It is precisely the underlying structure of a Lie algebroid on the phase space which allows a unified treatment. This idea was first introduced by A. Weinstein [125, 23] in order to define a Lagrangian formalism which is very useful for the various types of such systems.

The concept of Lie algebroids have been introduced into differential geometry since the early 1950, and also can be found in physics and algebra, under a wide variety of names. However, the fundamental concept has been introduced in sixties by J. Pradines [112] in relation with Lie groupoids. For every Lie groupoid there exists an associated Lie algebroid, like as for every Lie group there exists an associated Lie algebra. A Lie algebroid [73, 75] over a smooth manifold M is a real vector bundle (E, π, M) with a Lie algebra structure on its space of sections, and an application σ, named anchor, which induces a Lie algebra homomorphism from sections of E to vector fields on M. It is convenient to think a Lie algebroid as a substituent for the tangent bundle of M, an element e of E as a generalized velocity, and the actual velocity v on TM is obtained when applying the anchor to e, i.e., $\sigma(e) = v$.

The basic example of Lie algebroid over the manifold M is the tangent bundle TM itself, with the identity mapping as anchor. Every integrable distribution of TM is a Lie algebroid with the inclusion as anchor and induced Lie bracket, and every Lie algebra is a Lie algebroid over one point. An important Lie algebroid is the cotangent bundle of a Poisson manifold [60]. Being related to many areas of geometry, as connections theory [75, 40, 24, 31, 49, 76, 111, 95] cohomology [73, 75] foliations and pseudogroups, symplectic and Poisson geometry [66, 124, 120, 39, 121, 61, 33, 36, 97, 99, 107] the Lie algebroids are today the object of extensive studies. More precisely, Lie algebroids have applications in mechanical systems and optimal control theory [30, 77, 78, 49, 29, 44, 4, 92, 96, 98, 16] (distributional systems) and are a natural framework in which one can be developed the theory of differential operators (exterior derivative and Lie derivative) and differential equations.

In his papers [73, 75] K. Mackenzie has been achieved a unitary study of Lie groupoids and algebroids and together with P. Higgins [46] have introduced the notion of prolongation of a Lie algebroid over a smooth map, useful in the study of induced vector bundle by the Lie algebroid structure. Using the geometry of Lie algebroids, A. Weinstein [125] shows that is possible to give a common description of the most interesting classical mechanical systems. He developed a generalized theory of Lagrangian mechanics and obtained the equations of motions, using the Poisson structure on the dual of a Lie algebroid and Legendre transformation associated with a regular Lagrangian. In the last years the problems raised by A. Weinstein have been investigated by many authors. Thus, E. Martinez [69, 70, 72] obtained the same Euler-Lagrange equations using the symplectic formalism for Lagrangian and Hamiltonian, similarly with the J. Klein formalism [57] for the classical Lagrangian mechanics.

In the classical version of the tangent bundle ($E = TM$) the Klein's method is based on the vector bundle structure of TM and the existence of a vector-valued 1-form. Such a form does not exist for a general Lie algebroid [65] because of different dimensions of the horizontal and vertical distributions, and so Klein's approach is not applied directly. To overcome this dificulty, E. Martinez, M de Leon, J. Marero [69, 64] have proposed a modified version, in which the bundles tangent to E and E^* are replaced by the prolongations $\mathcal{T}E$ and $\mathcal{T}E^*$ (in sense Higgins and Mackenzie [46]). The nonholonomic Lagrangian systems and Hamiltonian mechanics on Lie algebroids are studied by a group of E. Martinez [64]. The first step in studying the mechanical control systems on Lie algebroids seems to be done by J. Cortes and E. Martinez [30], which also approached the problem of accessibility and controllability. A framework for nonholonomic systems, using a subbundle of a Lie algebroids is proposed by T. Mestdag and B. Langerock [77]. A start in the study of some problems of control affine systems and sub-Riemannian geometry, using the framework of Lie algebroids is due to D. Hrimiuc and L. Popescu [49, 96, 98].

Control theory is splitting in two major branches: the first is the control theory of problems described by partial differential equations where the objective functionals are mostly quadratic forms, and the second is the control theory of problems described by the parameter dependent ordinary differential equations. In this last case it is more frequent to deal with non-linear systems and non-quadratic objective functional. The mathematical models from the optimal control theory cover also the economic growth in both open and closed economies, exploatation of (non-) renewable resources, pollution control, behaviour of firms or differential games [38, 115, 116].

The geometric methods in the control theory have been applied by many authors (see [18, 56, 15, 71]). One of the most important issues in the geometric approach is the analysis of the solution to the optimal control problem as provided by Pontryagin's Maximum Principle; that is, the curve $c(t) = (x(t), u(t))$ is an optimal trajectory if there exists a lifting of $x(t)$ to the dual space $(x(t), p(t))$ satisfying the Hamilton equations, together with a maximization condition for the Hamiltonian with respect to the control variables $u(t)$.

In the paper [71] E. Martinez presents the Pontryagin Maximum Principle on Lie algebroids using the prolongation (in sense of Higgins and Mackenzie [46]) of the Lie algebroid over the vector bundle projection of a dual bundle. In this book we study some distributional systems with positive homogeneous cost, using the Pontryagin Maximum Principle at the level of a Lie algebroid.

"In spite of that, the control theory can be considered part of the general theory of differential equations, the problems that inspires it and some of the results obtained so far, have configured a theory with a strong and definite personality, that is already offering interesting returns to its ancestors. For instance, the geometrization of non-linear affine-input control theory problems by introducing Lie-geometrical methods into its analysis, started already by R, Brocket [18], is inspiring classical Riemannian geometry and creating what is called today sub-Riemannian geometry" [118, 85, 14, 2, 22, 9, 10, 11, 12, 13].

If M is a smooth n-dimensional manifold then a sub-Riemannian structure on M is a pair (D, g) where D is a distribution of rank m and g is a

Riemannian metric on D. A piecewise smooth curve on M is called horizontal if its tangent vectors are in D. The length of a horizontal curve c is defined by

$$L(c) = \int_I \sqrt{g(\dot{c}(t))}dt, \tag{1}$$

where g is a Riemannian metric on D. The distance between two points a and b is $d(a, b) = inf L(c)$, where the infimum is taken over all horizontal curves connecting a to b. The distance is assumed to be infinite if there is no horizontal curve that connects these two points. If locally, the distribution D of rank m is generated by X_i, $i = \overline{1, m}$ a sub-Riemannian structure on M is locally given by a control system

$$\dot{x} = \sum_{i=1}^{m} u_i(t) X_i(x), \tag{2}$$

of constant rank m, with the controls $u(.)$. The controlled paths are obtained by integrating the system (2) and are the geodesics in the framework of sub-Riemannian geometry. If D is assumed to be bracket generating, *i.e.* sections of D and iterated brackets span the entire tangent space TM, by a well-known theorem of Chow [25] the system (2) is controllable, that is for any two points a and b, there exists a horizontal curve which connects these points (M is assumed to be connected).

The concept of sub-Riemannian geometry can be extended to a more general setting, [49, 27, 28] by replacing the Riemannian metric with a positive homogeneous one. For the theory of optimal control this extension is equivalent to the change of the quadratic cost of a control affine system with a positive homogeneous cost. Also, the case of distribution D with non-constant rank is generating interesting examples (Grushin case [37, 49]).

The case when the distribution D generated by vector fields X_i, $i = \overline{1, m}$ is integrable is also studied. In this case the distribution determines a foliation on M and two points can be joined if and only if belongs to the same leaf. In order to find the optimal trajectory of the system one uses the Pontryagin Maximum Principle at the level of Lie algebroids, built different in the case of holonomic or nonholonomic distributions.

* * *

The book is organized in two parts. The first part entitled *The geometry of Lie algebroids* contains eight chapters. In the first chapter some preliminaries concerning geometrical structures on the total space of a vector bundle are presented [82]. We focus on the notions of nonlinear connection and covariant derivative. In the next chapter we present the notion of *Lie algebroid* including the cohomology and structure equations [73]. The notion of prolongation of a Lie algebroid over the vector bundle projection is studied.in the chapter three. The Ehresmann nonlinear connection $\mathcal{N} = -\mathcal{L}_{\mathcal{S}}J$ with the coefficients given by

$$\mathcal{N}_{\alpha}^{\beta} = \frac{1}{2}\left(-\frac{\partial \mathcal{S}^{\beta}}{\partial y^{\alpha}} + y^{\varepsilon}L_{\alpha\varepsilon}^{\beta}\right),$$

is investigated and the relations with the Ehresmann connections on tangent bundles TE and TM are pointed out. In the chapter four we introduce the notion of dynamical covariant derivative and metric nonlinear connection at the level of the Lie algebroid $\mathcal{T}E$. The Lagrangian formalism on Lie algebroids yields a canonical semispray [70]

$$\mathcal{S}^{\varepsilon} = g^{\varepsilon\beta}\left(\sigma_{\beta}^{i}\frac{\partial L}{\partial x^{i}} - \sigma_{\alpha}^{i}\frac{\partial^{2}L}{\partial x^{i}\partial y^{\beta}}y^{\alpha} - L_{\beta\alpha}^{\theta}y^{\alpha}\frac{\partial L}{\partial y^{\theta}}\right),$$

and a canonical Ehresman connection, which is a metric nonlinear connection. We also have the Lagrange equations on Lie algebroids given by [125]

$$\frac{dx^{i}}{dt} = \sigma_{\alpha}^{i}y^{\alpha}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial y^{\alpha}}\right) = \sigma_{\alpha}^{i}\frac{\partial L}{\partial x^{i}} - L_{\alpha\beta}^{\theta}y^{\beta}\frac{\partial L}{\partial y^{\theta}}.$$

In the case of positive homogeneous Lagrangian (Finsler function) we find a canonical Ehresmann connection which depends only on Finsler function and the structure functions of the Lie algebroid.

In the chapter five we deal with the prolongation of a Lie algebroid over the vector bundle projections of a dual bundle. We introduce the notions of dual adapted tangent structure $\mathcal{J}$ and $\mathcal{J}$-regular sections. These structures induce a canonical nonlinear connection $\mathcal{N} = -\mathcal{L}_{\rho}\mathcal{J}$ with the coefficients given by [50]

$$\mathcal{N}_{\alpha\beta} = \frac{1}{2}\left(t_{\alpha\gamma}\frac{\partial \rho_{\beta}}{\partial \mu_{\gamma}} - \sigma_{\alpha}^{i}t_{\gamma\beta}\frac{\partial \xi^{\gamma}}{\partial q^{i}} - \rho(t_{\alpha\beta}) + \xi^{\gamma}t_{\lambda\beta}L_{\gamma\alpha}^{\lambda}\right).$$

In the case of Hamiltonian formalism these coefficients become [103]

$$\begin{aligned}\mathcal{N}_{\alpha\beta} &= \frac{1}{2}(\sigma^i_\gamma\{g_{\alpha\beta},\mathcal{H}\} - \frac{\partial^2\mathcal{H}}{\partial q^i\partial\mu_\varepsilon}(\sigma^i_\beta g_{\alpha\varepsilon}+\sigma^i_\alpha g_{\beta\varepsilon}) + \\ &+ \mu_\gamma L^\gamma_{\varepsilon\kappa}\frac{\partial\mathcal{H}}{\partial\mu_\varepsilon}\frac{\partial g_{\alpha\beta}}{\partial\mu_\kappa} + \mu_\gamma L^\gamma_{\alpha\beta} + \frac{\partial\mathcal{H}}{\partial\mu_\delta}(g_{\alpha\varepsilon}L^\varepsilon_{\delta\beta}+g_{\beta\varepsilon}L^\varepsilon_{\delta\alpha})),\end{aligned}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket. The corresponding Hamilton equations on Lie algebroid are given by [125, 64]

$$\frac{dq^i}{dt} = \sigma^i_\alpha\frac{\partial\mathcal{H}}{\partial\mu_\alpha},\quad \frac{d\mu_\alpha}{dt} = -\sigma^i_\alpha\frac{\partial\mathcal{H}}{\partial q^i} - \mu_\gamma L^\gamma_{\alpha\beta}\frac{\partial\mathcal{H}}{\partial\mu_\beta}.$$

In the chapter six we introduce the notion of dynamical covariant derivative and metric nonlinear connection at the level of a Lie algebroid $\mathcal{T}E^*$. We prove that the canonical nonlinear connection induces by a regular Hamiltonian is a unique metric and symmetric nonlinear connection. In the chapter seven we investigate some aspects of the Lie algebroids geometry endowed with a Poisson structures, the so-called Poisson-Lie algebroids. In the last chapter of this part we present the notion of Leibniz algebroids as a weakened version of a Lie algebroid, where the bilinear operation on sections of the vector bundle is not necessarily skew-symmetric

Author's papers [49, 50, 91, 92, 94, 95, 96, 97, 99, 100, 103, 104, 105, 106, 107, 109] are used in writting this part.

The purpose of the second part entitled *Optimal Control* is to study the drift less control affine systems (distributional systems) with positive homogeneous cost, using the Pontryagin Maximum Principle at the level of a Lie algebroid in the case of constant rank of distribution.

We prove that the framework of Lie algebroids is better than cotangent bundle in order to solve some problems of drift less control affine systems. In the first chapter the known results on the optimal control systems are recalled by geometric viewpoint. In the next chapter the distributional systems are presented and the relation between the Hamiltonians on E^* and T^*M is given by

$$H(p) = \mathcal{H}(\mu),\quad \mu = \sigma^\star(p),\quad p\in T^*_xM,\quad \mu\in E^*_x.$$

We investigate the cases of holonomic and nonholonomic distributions with constant rank. In the holonomic case, we will consider the Lie algebroid being just the distribution whereas in the nonholonomic case (i.e., strong bracket generating distribution) the Lie algebroid is the tangent bundle with the basis given by vectors of distribution completed by the first Lie brackets. Also, the case of distribution D with non-constant rank is studied in the last two sections of the chapter and some interesting examples are given. In the last chapter we present the intrinsec relation between the distributional systems and sub-Riemannian geometry. Thus, the optimal trajectory of our distributional systems are the geodesics in the framework of sub-Riemannian geometry. We investigate two classical cases: Grusin plan and Heisenberg group, but equipped with positive homogeneous costs (Randers metric). We are using the Pontryagin Maximum Principle at the level of Lie algebroids, in the case of Heisenberg group and show that this idea is very useful in order to solve a large class of distributional systems. Author's papers [50, 96, 98, 101, 102, 106, 108] are used in writting this part.

In my opinion, the book is useful to a large class of readers: graduate students, mathematicians and to everybody else interested in the subject of differential geometry, differential equations, optimal control with economic applications. I want to address my thanks to all authors mentioned in this book and to everybody else I forgot to mention, without any intention, in the Bibliography.

Finally, I wish to address my thanks to the referees for many useful remarks and suggestions concerning this book. I should like to express the deep gratitude to professor D. Hrimiuc for the collaboration during the post-doctoral fellowship at the University of Alberta, Edmonton, Canada, where many ideas presented in this book have been started. Also, I want to address my thanks to Professor P. Stavre for support and guidance given me in life and in mathematics.

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1 THE GEOMETRY OF LIE ALGEBROIDS

The purpose of this first part is to study the geometry of a Lie algebroid and its prolongations over the vector bundles projections. A Lie algebroid [73, 75] over a smooth manifold M is a real vector bundle (E, π, M) with a Lie algebra structure on its space of sections, and an application σ, named the anchor, which induces a Lie algebra homomorphism from the sections of E to vector fields on M. For this reason, in the first chapter we present some results on the geometry of the total space of a vector bundle, including nonlinear connections and covariant derivatives. In the next chapter we give only the relevant formulas for Lie algebroid cohomology we shall need later, and refer the reader to the monograph [73] for further details.

The chapter three deals with the prolongation $\mathcal{T}E$ of a Lie algebroid over the vector bundle projection. We introduce the Ehresmann nonlinear connection on the Lie algebroid $\mathcal{T}E$ and study its properties [95, 92]. We show that the vertical part of the Lie brackets of horizontal sections from the basis represents the components of the curvature tensor of the nonlinear connection. We study the related connections and show that a connection on the tangent bundle TE induces a connection on the Lie algebroid $\mathcal{T}E$. We introduce an almost complex structure on Lie algebroids and prove that its integrability is characterized by zero torsion and curvature property of the connection. We present the notion of dynamical covariant derivative at the level of a Lie algebroid and show that the metric compatibility of the semispray and associated nonlinear connection gives the one of the so called Helmholtz conditions of the inverse problem of Lagrangian Mechanics. In the homogeneous case a canonical nonlinear connection associated to a Finsler function is determined. We study the linear connections on $\mathcal{T}E$ and determine the torsion and curvature.

In the chapter four we study the dynamical covariant derivative and metric nonlinear connection on $\mathcal{T}E$ [104]. We introduce the dynamical covariant derivative as a tensor derivation and study the compatibility conditions with a pseudo-Riemannian metric. In the case of SODE connection we find the expression of Jacobi endomorphism and its relation with curvature tensor.