Annals of the University of Craiova
Mathematics and Computer Science Series
Vol. XLI, Issue 2, December 2014

The *Annals of the University of Craiova, Mathematics and Computer Science Series*, is edited by Department of Mathematics and Department of Computer Sciences, University of Craiova, Romania.

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Volume Editors: Vicenţiu Rădulescu, Mihaela Sterpu

Layout Editor: Mihai Gabroveanu

ISSN 1223-6934
Online ISSN 2246-9958

Printed in Romania: Universitaria Press, Craiova, 2014.
http://www.editurauniversitaria.ro

Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 41(2), 2014, Pages 154–165
ISSN: 1223-6934

Several characterizations of the 4–valued modal algebras

Aldo Victorio Figallo and Paolo Landini

Abstract. A. Monteiro, in 1978, defined the algebras he named tetravalent modal algebras, that will be called $4-$*valued modal algebras* in this work. These algebras constitute a generalization of the $3-$valued Lukasiewicz algebras defined by Moisil.

The theory of the $4-$valued modal algebras has been widely developed by I. Loureiro in [6, 7, 8, 9, 10, 11, 12] and by A. V. Figallo in [2, 3, 4, 5].

J. Font and M. Rius indicated, in the introduction to the important work [1], a brief but detailed review on the $4-$valued modal algebras.

In this work varied characterizations are presented that show the "closeness" this variety of algebras has with other well–known algebras related to the algebraic counterparts of certain logics.

2010 Mathematics Subject Classification. Primary 06D99; Secondary 06D30.
Key words and phrases. De Morgan algebras, tetravalente modal algebras, three–valued Łukasiewicz algebras, 4–valued modal algebras.

1. Introduction

In 1940 G. C. Moisil [13] introduced the notion of three–valued Łukasiewicz algebra. In 1963, A.Monteiro [14] characterized these algebras as algebras $\langle A, \wedge, \vee, \sim, \nabla, 1\rangle$ of type $(2, 2, 1, 1, 0)$ which verify the following identities:

(A1) $x \vee 1 = 1$,
(A2) $x \wedge (x \vee y) = x$,
(A3) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$,
(A4) $\sim\sim x = x$,
(A5) $\sim (x \vee y) = \sim x \wedge \sim y$,
(A6) $\sim x \vee \nabla x = 1$,
(A7) $x \wedge \sim x = \sim x \wedge \nabla x$,
(A8) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$.

L.Monteiro [15] proved that A1 follows from A2, $\cdots$, A8, and that A2, $\cdots$, A8, are independent.

From A2, $\cdots$, A5 it follows that $\langle A, \wedge, \vee, \sim, 1\rangle$ is a De Morgan algebra with last element 1 and first element $0 = \sim 1$.

In Lemma 1.1 we will indicate other properties valid in the variety of 4–valued modal algebras necessary for the development that follows.

Lemma 1.1. *In every* 4*-valued modal algebra* $\langle A, \wedge, \vee, \sim, \nabla, 1\rangle$ *we have :* A9-A17.

(A9) $x \leq \nabla x$,
(A10) $\nabla 1 = 1$,
(A11) $\nabla x \leq \nabla\nabla x$,
(A12) $\nabla x \vee \sim \nabla x = 1$,

Received September 28, 2013. Revised September 6, 2014.

(A13) $\nabla x \wedge \sim \nabla x = 0$,
(A14) $\nabla\nabla x = \nabla x$,
(A15) *If* $x \leq y$, *then* $\nabla x \leq \nabla y$,
(A16) $\nabla(\nabla x \vee \nabla y) = \nabla(x \vee y)$,
(A17) $\nabla x \vee \nabla y = \nabla(x \vee y)$,

the proof of which we will indicate in the section that follows.

In 1969 J. Varlet [16] characterized three–valued Łukasiewicz algebras by means of other operations. Let $\langle A, \wedge, \vee, *, +, 0, 1\rangle$ be an algebra of type $(2,2,1,1,0,0)$ where $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice with least element 0, greatest element 1 and the following properties are satisfied:
(V1) $x \wedge x^* = 0$,
(V2) $(x \wedge y)^* = x^* \wedge y^*$,
(V3) $0^* = 1$,
(V4) $x \vee x^+ = 1$,
(V5) $(x \vee y)^+ = x^+ \wedge y^+$,
(V6) $1^+ = 0$,
(V7) If $x^* = y^*$ and $x^+ = y^+$, then $x = y$.

About these algebras he proved that it is posible to define, in the sense of [14, 15] a structure of three-valued Łukasiewicz algebra by taking $\sim x = (x \vee x^*) \wedge x^+$ and $\nabla x = x^{**}$.

Furthermore it holds $x^* = \sim \nabla x$ and $x^+ = \nabla \sim x$. Therefore three–valued Łukasiewicz are double Stone lattices which satisfy the determination principle V7. Moreover V7 may be replaced by the identity

$$(x \wedge x^+) \wedge (y \vee y^*) = x \wedge x^+.$$

Later, in 1963, A. Monteiro [14] considered the 4–valued modal algebras $\langle A, \wedge, \vee, \sim, \nabla, 1\rangle$ of type $(2,2,1,1,0)$ which satisfy A2, $\cdots$, A7 as an abstraction of three–valued Łukasiewicz algebras.

In this paper we give several characterizations of the 4–valued modal algebras. In the first one we consider the operations $\wedge, \vee, \neg, \Gamma, 0, 1$ where $\neg x = \sim \nabla x$, $\Gamma x = \nabla \sim x$ are called strong and weak negation respectively.

2. A characterization of the 4–valued modal algebras

Before working on Theorem 2.1 we will indicate proofs from A9 through A16. Then,

(A9) $x \leq \nabla x$:
- (1) $(x \wedge \sim x) \vee \nabla x = (\nabla x \wedge \sim x) \vee \nabla x = \nabla x$, [A7]
- (2) $(x \vee \nabla x) \wedge (\sim x \vee \nabla x) = \nabla x$,
- (3) $(x \vee \nabla x) \wedge 1 = \nabla x$, [A6]
- (4) $x \vee \nabla x = \nabla x$,
- (5) $x \leq \nabla x$.

(A10) $\nabla 1 = 1$: [A9]
(A11) $\nabla x \leq \nabla\nabla x$: [A9]
(A12) $\nabla x \vee \sim \nabla x = 1$:
- (1) $\nabla x \wedge \sim \nabla x = \sim \nabla x \wedge \nabla\nabla x$, [A7]
- (2) $\sim \nabla x \vee \nabla x = \nabla x \vee \sim \nabla\nabla x$ [(1), A5]

$$= (\nabla x \vee \sim \nabla\nabla x) \wedge 1$$
$$= (\nabla x \vee \sim \nabla\nabla x) \wedge (\sim x \vee \nabla x) \quad [A6]$$

$$= ((\nabla x \vee \sim \nabla\nabla x) \wedge \sim x) \vee ((\nabla x \vee \sim \nabla\nabla x) \wedge \nabla x)$$
$$= ((\nabla x \vee \sim x) \wedge (\sim \nabla\nabla x \vee \sim x)) \vee \nabla x$$
$$= (1 \wedge (\sim \nabla\nabla x \vee \sim x)) \vee \nabla x \quad [A6]$$
$$= \sim \nabla\nabla x \vee \sim x \vee \nabla x$$
$$= 1. \quad [A6]$$

(A13) $\nabla x \wedge \sim \nabla x = 0$: [A12]

(A14) $\nabla\nabla x = \nabla x$:

$$\nabla\nabla x = \nabla\nabla x \wedge 1$$
$$= \nabla\nabla x \wedge (\nabla x \vee \sim \nabla x) \quad [A12]$$
$$= (\nabla\nabla x \wedge \nabla x) \vee (\nabla\nabla x \wedge \sim \nabla x)$$
$$= \nabla x \vee (\nabla\nabla x \wedge \sim \nabla x) \quad [A11]$$
$$= \nabla x \vee (\sim \nabla x \wedge \nabla x) \quad [A7]$$
$$= \nabla x.$$

(A15) If $x \leq y$, then $\nabla x \leq \nabla y$:

(1) $x \leq y$, [Hip.]
(2) $\sim y \leq \sim x$, [(1)]
(3) $\sim y \vee \nabla y \leq \sim x \vee \nabla y$, [(2)]
(4) $1 = \sim x \vee \nabla y$, [A6]
(5) $0 = x \wedge \sim \nabla y$, [A6]
(6) $\sim \nabla x = \sim \nabla x \vee (x \wedge \sim \nabla y)$ [(5)]
$= (\sim \nabla x \vee x) \wedge (\sim \nabla x \vee \sim \nabla y)$
$= 1 \wedge (\sim \nabla x \vee \sim \nabla y)$ [A6]
$= \sim \nabla x \vee \sim \nabla y$,
(7) $\nabla x = \nabla x \wedge \nabla y$, [(6)]
(8) $\nabla x \leq \nabla y$. [(7)]

(A16) $\nabla(\nabla x \vee \nabla y) = \nabla(x \vee y)$:

(1) $x \leq x \vee y$,
(2) $y \leq x \vee y$,
(3) $\nabla x \leq \nabla(x \vee y)$, [(1), A15]
(4) $\nabla y \leq \nabla(x \vee y)$, [(2), A15]
(5) $\nabla x \vee \nabla y \leq \nabla(x \vee y)$, [(3), (4)]
(6) $x \leq \nabla x$, [A9]
(7) $y \leq \nabla y$, [A9]
(8) $x \vee y \leq \nabla x \vee \nabla y$, [(6), (7)]
(9) $\nabla(x \vee y) \leq \nabla(\nabla x \vee \nabla y)$, [(8), A15]
(10) $\nabla(\nabla x \vee \nabla y) \leq \nabla\nabla(x \vee y)$, [(5), A15]
(11) $\nabla(\nabla x \vee \nabla y) \leq \nabla(x \vee y)$, [(10), A14]
(12) $\nabla(\nabla x \vee \nabla y) = \nabla(x \vee y)$. [(9), (11)]

(A17) $\nabla x \vee \nabla y = \nabla(x \vee y)$:

(1) $\sim (\nabla x \vee \nabla y) \wedge \nabla(\nabla x \vee \nabla y) = (\nabla x \vee \nabla y) \wedge \sim (\nabla x \vee \nabla y)$ [A7]
$= (\nabla x \vee \nabla y) \wedge \sim \nabla x \wedge \sim \nabla y$ [A5]
$= ((\nabla x \wedge \sim \nabla x) \vee (\nabla y \wedge \sim \nabla x)) \wedge \sim \nabla y$
$= \nabla y \wedge \sim \nabla x \wedge \sim \nabla y$ [A13]
$= 0$, [A13]
(2) $\sim (\nabla x \vee \nabla y) \wedge \nabla(x \vee y) = 0$, [(1), A16]
(3) $(\nabla x \vee \nabla y) \vee \sim \nabla(x \vee y) = 1$,
(4) $((\nabla x \vee \nabla y) \vee \sim \nabla(x \vee y)) \wedge \nabla(x \vee y) = \nabla(x \vee y)$, [(3)]
(5) $((\nabla x \vee \nabla y) \wedge \nabla(x \vee y)) \vee (\sim \nabla(x \vee y) \wedge \nabla(x \vee y)) = \nabla(x \vee y)$,
(6) $(\nabla x \vee \nabla y) \wedge \nabla(x \vee y) = \nabla(x \vee y)$, [(5), A13]

(7) $\nabla(x \vee y) \leq \nabla x \vee \nabla y$, [(6)]
(8) $\nabla x \vee \nabla y = \nabla(x \vee y)$. [(7), (5) of A16]

Theorem 2.1. *Let* $\langle A, \wedge, \vee, \neg, \Gamma, 0, 1\rangle$ *be an algebra of type* $(2, 2, 1, 1, 0, 0)$ *where* $\langle A, \wedge, \vee, 0, 1\rangle$ *is a bounded distributive lattice with least element* 0, *greatest element* 1 *and the operators* $\triangledown, \sim$ *are defined on* A *by means of the formulas:*
(D1) $\triangledown x = \neg\neg x$,
(D2) $\sim x = (x \vee \neg x) \wedge \Gamma x$.
Then (i) *and* (ii) *are equivalent:*
(i) $\langle A, \wedge, \vee, \sim, \triangledown, 1\rangle$ *is a 4–valued modal algebra.*
(ii) $\langle A, \wedge, \vee, \neg, \Gamma, 0, 1\rangle$ *verifies these properties:*
(B1) $\neg\neg 1 = 1$,
(B2) $x \wedge \neg x = 0$,
(B3) $x \vee \Gamma x = 1$,
(B4) $\neg x \wedge \Gamma\neg x = 0$,
(B5) $\Gamma x \vee \neg\Gamma x = 1$,
(B6) $\Gamma(x \wedge y) = \Gamma x \vee \Gamma y$,
(B7) $\neg(x \vee y) = \neg x \wedge \neg y$,
(B8) $\neg(x \wedge \neg y) = \neg x \vee \neg\neg y$,
(B9) $\Gamma(x \vee \Gamma y) = \Gamma x \wedge \Gamma\Gamma y$,
(B10) $(x \vee y) \wedge \Gamma(x \vee y) \leq x \vee \neg x$,
(B11) $x \wedge \Gamma x \wedge y \wedge \Gamma y \leq \Gamma(x \vee y)$.
Where $a \leq b$ *if and only if* $a \wedge b = a$ *or* $a \vee b = b$. *Moreover, the operators* $\neg, \Gamma$ *are defined on* A *by means of the formulas:*
(D3) $\neg x = \sim \triangledown x$,
(D4) $\Gamma x = \triangledown \sim x$.

Proof. (i) $\Longrightarrow$ (ii)
(B1) $\neg\neg 1 = 1$: [D1, A10]
(B2) $x \wedge \neg x = 0$:

$$\begin{aligned} x \wedge \neg x &= x \wedge \sim \triangledown x && [\text{D3}]\\ &= \sim (\sim x \vee \triangledown x) \\ &= \sim 1 && [\text{A6}]\\ &= 0. \end{aligned}$$

(B3) $x \vee \Gamma x = 1$:

$$\begin{aligned} x \vee \Gamma x &= x \vee \triangledown \sim x && [\text{D4}]\\ &= \sim\sim x \vee \triangledown \sim x \\ &= 1. && [\text{A6}] \end{aligned}$$

(B4) $\neg x \wedge \Gamma\neg x = 0$:

$$\begin{aligned} \neg x \wedge \Gamma\neg x &= \sim \triangledown x \wedge \triangledown \sim\sim \triangledown x && [\text{D3, D4}]\\ &= \sim \triangledown x \wedge \triangledown\triangledown x && [\text{A5}]\\ &= \sim \triangledown x \wedge \triangledown x && [\text{A14}]\\ &= 0. && [\text{A13}] \end{aligned}$$

(B5) $\Gamma x \vee \neg\Gamma x = 1$:

$$\begin{aligned} \Gamma x \vee \neg\Gamma x &= \triangledown \sim x \vee \sim \triangledown\triangledown \sim x && [\text{D3, D4}]\\ &= \triangledown \sim x \vee \sim \triangledown \sim x && [\text{A14}]\\ &= 1. && [\text{A12}] \end{aligned}$$

(B6) $\Gamma(x \wedge y) = \Gamma x \vee \Gamma y$:

$$\begin{aligned} \Gamma(x \wedge y) &= \triangledown \sim (x \wedge y) && [\text{D4}]\\ &= \triangledown(\sim x \vee \sim y) \\ &= \triangledown \sim x \vee \triangledown \sim y && [\text{A17}] \end{aligned}$$

$= \Gamma x \vee \Gamma y.$ [D4]

(B7) $\neg(x \vee y) = \neg x \wedge \neg y$:

$\neg(x \vee y) = \sim \nabla(x \vee y)$ [D3]
$= \sim (\nabla x \vee \nabla y)$ [A17]
$= \sim \nabla x \wedge \sim \nabla y$
$= \neg x \wedge \neg y.$ [D3]

(B8) $\neg(x \wedge \neg y) = \neg x \vee \neg\neg y$:

$\neg(x \wedge \neg y) = \sim \nabla(x \wedge \sim \nabla y)$ [D3]
$= \sim (\nabla x \wedge \nabla \sim \nabla y)$ [A8]
$= \sim \nabla x \vee \sim \nabla \sim \nabla y$
$= \neg x \vee \neg\neg y.$ [D3]

(B9) $\Gamma(x \vee \Gamma y) = \Gamma x \wedge \Gamma\Gamma y$:

$\Gamma(x \vee \Gamma y) = \nabla \sim (x \vee \nabla \sim y)$ [D4]
$= \nabla(\sim x \wedge \sim \nabla \sim y)$
$= \nabla \sim x \wedge \nabla \sim \nabla \sim y$ [A8]
$= \Gamma x \wedge \Gamma\Gamma y.$ [D4]

(B10) $(x \vee y) \wedge \Gamma(x \vee y) \leq x \vee \neg x$:

(1) $x \wedge \Gamma x = x \wedge \sim x$:

$x \wedge \Gamma x = x \wedge \nabla \sim x$ [D4]
$= \sim (\sim x) \wedge \nabla(\sim x)$ [A4]
$= \sim\sim x \wedge \sim x$ [A7]
$= x \wedge \sim x,$ [A4]

(2) $(x \vee y) \wedge \Gamma(x \vee y) \leq \sim x$:

$(x \vee y) \wedge \Gamma(x \vee y) = (x \vee y) \wedge \sim (x \vee y)$ [(1)]
$= (x \vee y) \wedge \sim x \wedge \sim y)$ [A5]
$\leq \sim x,$

(3) $\sim x \leq x \vee \neg x$:

$\sim x \wedge (x \vee \neg x) = \sim x \wedge (x \vee \sim \nabla x)$ [D3]
$= (\sim x \wedge x) \vee (\sim x \wedge \sim \nabla x)$
$= (\sim x \wedge \nabla x) \vee \sim (x \vee \nabla x)$ [A7]
$= (\sim x \wedge \nabla x) \vee \sim \nabla x$ [A9]
$= (\sim x \vee \sim \nabla x) \wedge (\nabla x \vee \sim \nabla x)$
$= \sim (x \wedge \nabla x)$ [A5, A12]
$= \sim x,$ [A9]

(4) $(x \vee y) \wedge \Gamma(x \vee y) \leq x \vee \neg x.$ [(2), (3)]

(B11) $x \wedge \Gamma x \wedge y \wedge \Gamma y \leq \Gamma(x \vee y)$:

$(x \wedge \Gamma x) \wedge (y \wedge \Gamma y) = (x \wedge \sim x) \wedge (y \wedge \sim y)$ [(1) of B10]
$= (x \wedge y) \wedge (\sim x \wedge \sim y)$
$\leq (x \vee y) \wedge \sim (x \vee y)$
$= (x \vee y) \wedge \Gamma(x \vee y)$ [(1) of B10]
$\leq \Gamma(x \vee y).$

(ii) $\Longrightarrow$ (i)

(B12) $\Gamma 0 = 1$:

(1) $x \vee \Gamma x = 1,$ [B3]
(2) $\Gamma 0 = 1.$ [(1)]

(B13) $\neg 1 = 0$:

(1) $1 \wedge \neg 1 = 0,$ [B2]
(2) $\neg 1 = 0.$ [(1)]

(B14) $\neg x \leq \Gamma x$:

$$\begin{aligned} \neg x \wedge \Gamma x &= (\neg x \wedge \Gamma x) \vee 0 \\ &= (\neg x \wedge \Gamma x) \vee (x \wedge \neg x) && [B2] \\ &= \neg x \wedge (\Gamma x \vee x) \\ &= \neg x \wedge 1 && [B3] \\ &= \neg x. \end{aligned}$$

(B15) $\neg 0 = 1$:

(1) $0 = 1 \wedge \neg 1,$ [B2]

(2) $\neg 0 = \neg(1 \wedge \neg 1)$ [(1)]
$= \neg 1 \vee \neg\neg 1$ [B8]
$= 1.$ [B1]

(B16) $\Gamma 1 = 0$:

$\neg 0 \wedge \Gamma\neg 0 = 0,$ [B4]
$1 \wedge \Gamma 1 = 0,$ [B15]
$\Gamma 1 = 0.$

(B17) $\Gamma x \wedge \Gamma\Gamma x = 0$:

$$\begin{aligned} \Gamma x \wedge \Gamma\Gamma x &= \Gamma(x \vee \Gamma x) && [B9] \\ &= \Gamma 1 && [B3] \\ &= 0. && [B16] \end{aligned}$$

(B18) $\neg x \vee \neg\neg x = 1$:

$$\begin{aligned} \neg x \vee \neg\neg x &= \neg(x \wedge \neg x) && [B8] \\ &= \neg 0 && [B2] \\ &= 1. && [B15] \end{aligned}$$

(B19) $\neg\neg x = \Gamma\neg x$:

(1) $\Gamma\neg x = \Gamma\neg x \wedge 1$
$= \Gamma\neg x \wedge (\neg x \vee \neg\neg x)$ [B18]
$= (\Gamma\neg x \wedge \neg x) \vee (\Gamma\neg x \wedge \neg\neg x)$
$= 0 \vee (\Gamma\neg x \wedge \neg\neg x)$ [B4]
$= \Gamma\neg x \wedge \neg\neg x,$

(2) $\Gamma\neg x \leq \neg\neg x,$ [(1)]

(3) $\neg\neg x \leq \Gamma\neg x,$ [B14]

(4) $\neg\neg x = \Gamma\neg x.$ [(2), (3)]

(B20) $\Gamma\Gamma x = \neg\Gamma x$:

(1) $\neg\Gamma x \leq \Gamma\Gamma x,$ [B14]

(2) $\Gamma x \wedge \Gamma\Gamma x = 0,$ [B17]

(3) $\Gamma x \vee \neg\Gamma x = 1,$ [B5]

(4) $\Gamma\Gamma x = \Gamma\Gamma x \wedge 1 = \Gamma\Gamma x \wedge (\Gamma x \vee \neg\Gamma x) = (\Gamma\Gamma x \wedge \Gamma x) \vee (\Gamma\Gamma x \wedge \neg\Gamma x) = \Gamma\Gamma x \wedge \neg\Gamma x,$ [(2)]

(5) $\Gamma\Gamma x \leq \neg\Gamma x,$ [(4)]

(6) $\Gamma\Gamma x = \neg\Gamma x.$ [(1), (5)]

(B21) $\neg x \wedge \Gamma\Gamma x = 0$:

(1) $\neg x \leq \Gamma x,$ [B14]

(2) $\neg x \wedge \Gamma\Gamma x \leq \Gamma x \wedge \Gamma\Gamma x,$ [(1)]

(3) $\neg x \wedge \Gamma\Gamma x = 0.$ [(2), B17]

(B22) $x \leq \neg\neg x$:

(1) $x \wedge 1 = x \wedge (\neg x \vee \neg\neg x)$ [B18]
$= (x \wedge \neg x) \vee (x \wedge \neg\neg x)$
$= x \wedge \neg\neg x,$ [B2]

(2) $x \leq \neg\neg x.$ [(1)]

(B23) $\Gamma\Gamma x \leq x$:

(1) $x = x \vee \Gamma 1$ [B16]
$= x \vee \Gamma(x \vee \Gamma x)$ [B3]
$= x \vee (\Gamma x \wedge \Gamma\Gamma x)$ [B9]
$= (x \vee \Gamma x) \wedge (x \vee \Gamma\Gamma x)$
$= x \vee \Gamma\Gamma x,$ [B3]
(2) $\Gamma\Gamma x \leq x.$ [(1)]

(B24) $\neg\neg\neg x = \neg x$:
(1) $\neg\neg\neg x = \neg(\neg\neg x)$
$= \neg(\neg\neg x \vee x)$ [B22]
$= \neg\neg\neg x \wedge \neg x,$ [B7]
(2) $\neg\neg\neg x \leq \neg x,$ [(1)]
(3) $\neg x \leq \neg\neg\neg x,$ [B22]
(4) $\neg\neg\neg x = \neg x.$ [(2), (3)]

(B25) $\Gamma\Gamma\Gamma x = \Gamma x$:
(1) $\Gamma\Gamma\Gamma x \leq \Gamma x,$ [B23]
(2) $\Gamma\Gamma\Gamma x = \Gamma(\Gamma\Gamma x)$
$= \Gamma(\Gamma\Gamma x \wedge x)$ [B23]
$= \Gamma\Gamma\Gamma x \vee \Gamma x,$ [B6]
(3) $\Gamma x \leq \Gamma\Gamma\Gamma x,$ [(2)]
(4) $\Gamma x = \Gamma\Gamma\Gamma x.$ [(1), (3)]

(B26) $\neg\Gamma x \leq x$:
(1) $\Gamma\Gamma x = \neg\Gamma x,$ [B20]
(2) $\Gamma\Gamma x \leq x,$ [B23]
(3) $\neg\Gamma x \leq x.$ [(1), (2)]

(B27) $\Gamma\Gamma\neg x = \neg x$:
$\Gamma\Gamma\neg x = \neg\Gamma\neg x$ [B20]
$= \neg\neg\neg x$ [B19]
$= \neg x.$ [B24]

(B28) $\Gamma\Gamma\Gamma\neg x = \neg\neg x$:
$\Gamma\Gamma\Gamma\neg x = \Gamma\neg x$ [B27]
$= \neg\neg x.$ [B19]

(B29) $\Gamma((x \vee \neg x) \wedge \Gamma x) = \neg\neg x$:
$\Gamma((x \vee \neg x) \wedge \Gamma x) = \Gamma(x \vee \neg x) \vee \Gamma\Gamma x$ [B6]
$= \Gamma(x \vee \Gamma\Gamma\neg x) \vee \Gamma\Gamma x$ [B27]
$= (\Gamma x \wedge \neg\neg x) \vee \Gamma\Gamma x$ [B9, B28]
$= (\Gamma x \vee \Gamma\Gamma x) \wedge (\neg\neg x \vee \Gamma\Gamma x)$
$= \neg\neg x.$ [B3, B22, B23]

(B30) $\neg\neg\Gamma x = \Gamma x$:
$\neg\neg\Gamma x = \Gamma\neg\Gamma x$ [B19]
$= \Gamma\Gamma\Gamma x$ [B20]
$= \Gamma x.$ [B25]

(B31) $\neg\neg\neg\Gamma x = \Gamma\Gamma x$:
$\neg\neg\neg\Gamma x = \Gamma\Gamma\Gamma\neg\neg\Gamma x$ [B28]
$= \Gamma\neg\neg\Gamma x$ [B25]
$= \neg\neg\neg\Gamma x$ [B19]
$= \neg\Gamma x$ [B24]
$= \Gamma\Gamma x.$ [B20]

(B32) $\neg((x \wedge \Gamma x) \vee \neg x) = \Gamma\Gamma x$:
$\neg((x \wedge \Gamma x) \vee \neg x) = \neg(x \wedge \Gamma x) \wedge \neg\neg x$ [B7]

$= \neg(x \wedge \neg\neg\Gamma x) \wedge \neg\neg x$ [B30]
$= (\neg x \vee \Gamma\Gamma x) \wedge \neg\neg x$ [B8, B31]
$= \Gamma\Gamma x.$ [B2, B22, B23]

(B33) $\Gamma\neg\neg x = \neg x$:
$\Gamma\neg\neg x = \neg\neg\neg x$ [B19]
$= \neg x.$ [B24]

Now we are able to prove the axioms A4, A6 and A7.

Axiom A4 $\sim\sim x = x$:

First, we observe that from B14 and D2 we obtain: (D3) $\sim x = (x \wedge \Gamma x) \vee \neg x$. Then

(1) $\sim\sim x = (((x \wedge \Gamma x) \vee \neg x) \wedge \Gamma((x \wedge \Gamma x) \vee \neg x)) \vee \neg((x \wedge \Gamma x) \vee \neg x),$ [D3]
(2) $\Gamma((x \wedge \Gamma x) \vee \neg x)) = \Gamma((x \vee \neg x) \wedge (\Gamma x \vee \neg x))$
$= \Gamma((x \vee \neg x) \wedge \Gamma x)$ [B14]
$= \neg\neg x,$ [B29]
(3) $\sim\sim x = (((x \wedge \Gamma x) \vee \neg x) \wedge \neg\neg x) \vee \Gamma\Gamma x$ [(1), (2), B32]
$= (((x \wedge \Gamma x) \wedge \neg\neg x) \vee (\neg x \wedge \neg\neg x)) \vee \Gamma\Gamma x$
$= ((x \wedge \Gamma x) \wedge \neg\neg x) \vee \Gamma\Gamma x$ [B2]
$= ((x \wedge \Gamma x) \vee \Gamma\Gamma x) \wedge (\neg\neg x \vee \Gamma\Gamma x)$
$= ((x \wedge \Gamma x) \vee \Gamma\Gamma x) \wedge \neg\neg x$ [B22, B23]
$= ((x \vee \Gamma\Gamma x) \wedge (\Gamma x \vee \Gamma\Gamma x)) \wedge \neg\neg x$
$= (x \vee \Gamma\Gamma x) \wedge \neg\neg x$ [B3]
$= x \wedge \neg\neg x$ [B23]
$= x.$ [B22]

Axiom A6 $\sim x \vee \nabla x = 1$:
$\sim x \vee \nabla x = (x \wedge \Gamma x) \vee \neg x \vee \neg\neg x$ [D3, D1]
$= (x \wedge \Gamma x) \vee 1 = 1.$ [B18]

Axiom A7 $x \wedge \sim x = \sim x \wedge \nabla x$:
$\sim x \wedge \nabla x = ((x \wedge \Gamma x) \vee \neg x) \wedge \neg\neg x$ [D3, D1]
$= (x \wedge \Gamma x \wedge \neg\neg x) \vee (\neg x \wedge \neg\neg x)$
$= (x \wedge \Gamma x) \vee 0$ [B22, B2]
$= (x \wedge \Gamma x) \vee (x \wedge \neg x)$ [B2]
$= ((x \wedge \Gamma x) \vee x) \wedge ((x \wedge \Gamma x) \vee \neg x)$
$= x \wedge \sim x.$ [D3]

(B34) If $x \leq y$ then $\neg y \leq \neg x$ and $\Gamma y \leq \Gamma x$:
(1) $x \leq y,$ [Hip.]
(2) $x \vee y = y,$ [(1)]
(3) $\neg(x \vee y) = \neg y,$ [(2)]
(4) $\neg x \wedge \neg y = \neg y,$ [(3), B7]
(5) $\neg y \leq \neg x,$ [(4)]
(6) $x \wedge y = x,$ [(1)]
(7) $\Gamma(x \wedge y) = \Gamma x,$ [(6)]
(8) $\Gamma x \vee \Gamma y = \Gamma x,$ [(7), B6]
(9) $\Gamma y \leq \Gamma x.$ [(8)]

(B35) $\sim \Gamma x = \Gamma\Gamma x$:
$\sim \Gamma x = \Gamma\Gamma x \wedge (\Gamma x \vee \neg\Gamma x)$ [D2]
$= \Gamma\Gamma x \wedge (\Gamma x \vee \Gamma\Gamma x)$ [B20]
$= \Gamma\Gamma x.$ [A2]

(B36) $\sim (\neg x \wedge \Gamma y) = \neg\neg x \vee \Gamma\Gamma y$:
(1) $\sim (\neg x \wedge \Gamma y) = \Gamma(\neg x \wedge \Gamma y) \wedge ((\neg x \wedge \Gamma y) \vee \neg(\neg x \wedge \Gamma y)),$ [D2]