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Wigner-Ville Distribution and Ambiguity Function of QPFT Signals

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Abstract. The quadratic phase Fourier transform(QPFT) has received my attention in recent years because of its applications in signal processing. At the same time the applications of Wigner-Ville distribution (WVD) and ambiguity function (AF) in signal analysis and image processing can not be excluded. In this paper we investigated the Wigner-Ville Distribution (WVD) and ambiguity function (AF) associated with quadratic phase Fourier transform (WVD-QPFT/AF-QPFT). Firstly, we propose the definition of the WVD-QPFT, and then several important properties of newly defined WVD-QPFT, such as nonlinearity, boundedness, reconstruction formula, orthogonality relation and Plancherel formula are derived. Secondly, we propose the definition of the AF-QPFT, and its with classical AF, then several important properties of newly defined AF-QPFT, such as non-linearity, the reconstruction formula, the time-delay marginal property, the quadratic-phase marginal property and orthogonal relation are studied. Further, a novel quadratic convolution operator and a related correlation operator for WVD-QPFT are proposed. Based on the proposed operators, the corresponding generalized convolution, correlation theorems are studied. Finally, a novel algorithm for the detection of linear frequency-modulated(LFM) signal is presented by using the proposed WVD-QPFT and AF-QPFT.

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1. Introduction

A superlative generalized version of the Fourier transform(FT) called quadratic-phase Fourier transform(QPFT) has been introduced by Castro et al[1, 2]. This novel transform has overthrown all the the applicable signal processing tools as it gives a unified analysis of both stationery and non-stationery signals in an easy and insightful way. The QPFT is actually a generalization of several well-known signal processing tools like Fourier, fractional Fourier and linear canonical transforms whose kernel is in the exponential form. Many researches have been carried on quadratic-phase Fourier transform(see[3, 4, 5, 6, 7]).

In the parametric time-frequency analysis the Wigner-Ville distribution (WVD) has a major role to play. It was Eugene Wigner who first introduced the concept WVD when He was dealing with the calculation of the quantum corrections. Later in 1948 J. Ville derived it independently as a quadratic representation of the local time-frequency energy of a signal. Authors in [10]-[25] introduced the generalized

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version of WVD known as the WVD associated with the linear canonical transform by replacing the Fourier kernel with the kernel of the LCT in the classical WVD. Later, several authors extend the WVD in offset linear canonical transform which is a time shifted and frequency modulated version of the linear canonical transform [23].

On the other hand, Woodward in 1953 introduced the ambiguity function(AF) which is also a important parametric time-frequency tool that has a major role in the analysis and processing of the non-stationary signals including the fields of optical information processing, radar signal processing and sonar technology,(refer to [25, 26, 27]). In recent times a lot of research work has been carried out by coupling the classical AF with other transforms. The WVD and AF in the domains of linear canonical transform and offset linear canonical transform have great flexibility and advantages over the classical WVD and AF respectively. These generalized versions of WVD and AF have great merits in detection of linear-frequency modulated (LFM) signals. Apart from this under low signal-to-noise ratios (SNRs) they have better detection performance. Therefore, it is rewarding and significant to obtain a new generalized WVD and AF.

On the other side the convolution has numerous important applications in various areas of Mathematics like numerical analysis, linear algebra and signal processing. Where as correlation like convolution is an another important tool in optics, signal processing and detection applications. Recently Bhat and Dar [28] have studied Wigner-Ville distributions in quaternion offset linear canonical domains. Here the authors obtained the classical properties like reconstruction formula, orthogonality relation and Plancherel formula. They also established the convolution and correlation theorems for the proposed transform. For more results on convolution and correlation, we refer to [14]-[15]. In [29], the authors proposed Winger distribution associated with quadratic phase transform (WVD-QPFT) as a generalization of classical WVD by substituting the kernel with the QPFT kernel. They established some vital properties like the marginal, shifting, conjugate-symmetry, anti-derivative, Moyal's and inversion formulae. Moreover they also studied the convolution and correlation theorems associated WVD-QPFT. More results in this direction can be found in [30]-[33].

Motivated by these novel concepts, we in this paper studied the Wigner-Ville Distribution associated with quadratic phase Fourier transform (WVD-QPFT). Firstly, we propose the new definition of the WVD-QPFT and establish its relation with the classical WVD. Then we establish some general properties of newly defined WVD-QPFT, such as boundedness, nonlinearity, orthogonality relation, reconstruction formula and Plancherel formula are derived. Secondly, we propose the definition of the AF-QPFT, and its relation with classical AF, then several important properties of newly defined AF-QPFT, such as non-linearity, the reconstruction formula, the time-delay marginal property, the quadratic-phase marginal property and orthogonal relation are studied. Third, a novel quadratic convolution operator depending on two parameters and a related correlation operator for WVD-QPFT are proposed. Moreover, based on the proposed operators, the corresponding generalized convolution, correlation theorems are studied. The crux of our paper lies in establishing a numerical example which justifies our results.

The paper is organised as follows. In Section 2, we provide some preliminary results required in subsequent sections. In Section 3, we provide the new definition of Wigner-Ville distribution associated with the quadratic phase Fourier transform (WVD-QPFT). Then we investigated several basic properties of the WVD-QPFT which are important for signal representation in signal processing. In Section 4, we propose the definition of ambiguity function associated the quadratic phase Fourier transform (AF-QPFT) and investigate several basic properties of the WVD-QPFT vital for signal processing. In Section 5, we first define the convolution and correlation for the QPFT. We then established the new convolution and correlation for the WVD-QPFT. Applications and conclusion are given in Section 6.

2. Preliminaries

2.1. Quadratic-phase Fourier transform. In this subsection we introduce the Quadratic-phase Fourier transform which is a neoteric addition to the classical integral transforms and we also gave its inversion formula and some other classical results which are already present in literature.

Definition 2.1. [4] Given a parameter $\mu = (A, B, C, D, E)$, the QPFT of any signal f is defined by

$$\mathcal{Q}_\mu[f](w) = \int f(t)\Lambda_\mu(t, w)dt, \tag{1}$$

where $\Lambda_\mu(t, w) = \frac{1}{\sqrt{2\pi}} e^{i\Omega(t,w)}$ and $\Omega(t, w)$ is known as the Quadratic-phase function, given by

$$\Omega(t, w) = (At^2 + Btw + Cw + Dt + Ew) \tag{2}$$

with $A, B, C, D, E \in \mathbf{R}, \quad B \neq 0.$

Theorem 2.1. [4] *The inversion formula of the quadratic-phase Fourier transform is given by*

$$f(t) = B \int \mathcal{Q}_\mu[f](w)\overline{\Lambda_\mu(t, w)}dw. \tag{3}$$

Using the inversion Theorem, we can get the Parseval's relation given by

$$\langle f, g\rangle = |B|\langle \mathcal{Q}_\mu[f], \mathcal{Q}_\mu[g]\rangle \tag{4}$$

Theorem 2.2. [4, 5] *Let $f, g \in L^2(\mathbb{R})$ and $\alpha, \beta, \tau \in \mathbb{R}$ then*

- $\mathbb{Q}_\mu[\alpha f + \beta g](w) = \alpha\mathbb{Q}_\mu[f](w) + \beta\mathbb{Q}_\mu[g](w)$.
- $\mathbb{Q}_\mu[f(t - \tau)](w) = exp\{-i(A\tau^2 + B\tau w + D\tau)\}\mathbb{Q}_\mu[e^{-2iA\tau t} f(t)](w)$.
- $\mathbb{Q}_\mu[f(-t)](w) = \mathbb{Q}_{\mu'}[f(t)](-w), \quad \mu' = (A, B, C, -D, -E)$.
- $\mathbb{Q}_\mu[e^{i\alpha t} f(t)](w) = exp\{i(\alpha^2 + 2\alpha Bw + \alpha EB)\frac{1}{B}\}\mathbb{Q}_\mu[f]\left(w + \frac{A}{B}\right)$.
- $\mathbb{Q}_\mu[\overline{f(t)}](w) = \overline{\mathbb{Q}_{-\mu}[f(t)](w)}$.

Theorem 2.3 (Convolution[6]). *If $f, g \in L^2(\mathbf{R})$ then*

$$\mathcal{Q}_\mu[f *_\mu g](w) = \sqrt{\frac{2\pi i}{B}} e^{-i(Cw^2+Ew)} \mathcal{Q}_\mu[f](w)\mathcal{Q}_\mu[e^{-2iA(\cdot)^2-iD(\cdot)} g](w). \tag{5}$$

Where

$$(f *_\mu g)(t) = \int_{\mathbf{R}} f(x)g(t - x)e^{-2iA(t^2-z^2)-iD(t-z)}dz. \tag{6}$$

2.2. Wigner-Ville distribution and ambiguity function. In this sub-section, we recall the definition of the classical Wigner-Ville distribution(WVD) and Ambiguity function(AF).

Definition 2.2 (Wigner-Ville Distribution[25, 26]). For any two signals $f, g \in L^2(\mathbf{R})$ the cross Wigner-Ville distribution (or transform) is defined as

$$\mathcal{W}_{f,g}(t,u) = \int_{\mathbf{R}} f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}e^{-\iota ux}dx \tag{7}$$

Definition 2.3 (Ambiguity function[26, 27]). For two signals $f, g \in L^2(\mathbf{R})$ the cross Ambiguity function is defined as

$$\mathcal{A}_{f,g}(t,u) = \int_{\mathbf{R}} f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}e^{-\iota ux}dx \tag{8}$$

The other way to define Ambiguity function is as[25, 26]

$$\mathcal{A}_{f,g}(x,u) = \int_{\mathbf{R}} f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}e^{-\iota ut}dt. \tag{9}$$

Fix the translation operator $\mathcal{T}_k$ by $\mathcal{T}_k f(x) = f(x-k)$ and the modulation by $\mathcal{M}_{u_0}$ by $\mathcal{M}_{u_0} f(x) = e^{\iota u_0 x} f(x)$. Let us now recall some of the fundamental properties of above transform. For any two functions $f, g \in L^2(\mathbf{R})$, we have

(1) $\overline{\mathcal{W}_{f,g}(t,u)} = \mathcal{W}_{f,g}(t,u)$.
(2) $\frac{1}{2\pi}\int_{\mathbf{R}} \mathcal{W}_{f,g}(t,u)du = |f(t)|^2$
(3) $\frac{1}{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} \mathcal{W}_{f,g}(t,u)dudt = \|f\|^2 = \|\hat{f}\|^2$
(4) $\mathcal{W}_{\mathcal{T}_k f, \mathcal{T}_k g}(t,u) = \mathcal{W}_{f,g}(t-k,u)$
(5) $\mathcal{M}_{\mathcal{M}_{w_0} f, \mathcal{M}_{u_0} g}(t,u) = \mathcal{W}_{f,g}(t, u-u_0)$
(6) $\frac{1}{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} \mathcal{W}_f(t,u)\overline{\mathcal{W}_g(t,u)}dudt = |<f,g>|^2$
(7) $f(t) = \frac{1}{2\pi\overline{g(0)}}\int_{\mathbf{R}} \mathcal{W}_f\left(\frac{t}{2}, w\right)e^{\iota ut}du$, with $\overline{g(0)} \neq 0$

3. Wigner-Ville distribution associated with the quadratic-phase Fourier transform

In this section we introduce the the definition of WVD associated with the quadratic-phase Fourier transform (WVD-QPFT). Further some basic properties of the WVD-QPFT which are important for signal representation in signal processing are investigated.

Definition 3.1. For $f, g \in L^2(\mathbf{R})$, the cross WVD-QPFT of f, g is defined by

$$\mathcal{W}^{\Lambda_\mu}_{f,g}(t,w) \;=\; \int_{\mathbf{R}} f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}\Lambda_\mu(x,w)dx. \tag{10}$$

where $\Lambda_\mu(x,w) = \frac{1}{\sqrt{2\pi}}e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}$.

The equation (10) can be reshaped as

$$\mathcal{W}_{f,g}^{\Lambda_\mu}(t,w) = \int_{\mathbf{R}} R_{f,g}(t,x)\Lambda_\mu(x,w)dx \tag{11}$$

where $R_{f,g}(t,w) = f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}$ is the correlation of f and g.

Remark 3.1. If $f = g$ then $\mathcal{W}_{f,f}^{\Lambda_\mu}(t,w) = \mathcal{W}_{f}^{\Lambda_\mu}(t,w)$ is called the auto WVD-QPFT. Otherwise it is called Cross WVD-QPFT

Remark 3.2. From above note we have

$$\begin{aligned}\mathcal{W}_{f}^{\Lambda_\mu}(t,w) &= \int_{\mathbf{R}} f\left(t+\frac{x}{2}\right)\overline{f\left(t-\frac{x}{2}\right)}\frac{1}{\sqrt{2\pi}}e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} f\left(t+\frac{x}{2}\right)\overline{f\left(t-\frac{x}{2}\right)}e^{i(Ax^2+Bxw+Dx+)}e^{i(Cw^2+Ew)}dx\end{aligned} \tag{12}$$

By setting $A = 0$ and noting the classical definition of WVD the L.H.S of above becomes

$$\begin{aligned}&= \frac{1}{\sqrt{2\pi}}e^{i(Cw^2+Ew)}\int_{\mathbf{R}} f\left(t+\frac{x}{2}\right)\overline{f\left(t-\frac{x}{2}\right)}e^{i(Bxw+Dx)}dx \\ &= \frac{1}{\sqrt{2\pi}}e^{i(Cw^2+Ew)}\mathcal{W}_f(t,-Bw-D).\end{aligned} \tag{13}$$

Thus (13) gives the relation between WVD-QPFT and classical WVD.

Remark 3.3. By varying the parameter $\mu = (A,B,C,D,E)$ the proposed transform (10) boils down to various well known integral transforms associated with WVD, viz: for $\mu = (0,-1,0,0,0)$, we get classical WVD and for $\mu = (\cot\theta, -\csc\theta, \cot\theta, 0, 0)$, we get fractional WVD.

Without loss of generality we will deal with the case $B \neq 0$, as in other cases proposed transform reduces to a chirp multiplications. Thus for any $f, g \in L^2(\mathbf{R})$ we have

$$\begin{aligned}\mathcal{W}_{f,g}^{\Lambda_\mu}(t,w) &= \int_{\mathbf{R}} f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}\Lambda_\mu(x,w)dx \\ &= \mathcal{Q}_\mu\left\{f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}\right\}(w) \\ &= \mathcal{Q}_\mu\{R_{f,g}(t,x)\}(w).\end{aligned} \tag{14}$$

Applying the inverse QPFT to (14), we get

$$\{R_{f,g}(t,x)\} = \mathcal{Q}_\mu^{-1}\{\mathcal{W}_{f,g}^{\Lambda_\mu}(t,w)\} \tag{15}$$

which implies

$$\begin{aligned}f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)} &= \mathcal{Q}_\mu^{-1}\{\mathcal{W}_{f,g}^{\Lambda_\mu}(t,w)\} \\ &= \frac{|B|}{\sqrt{2\pi}}\int_{\mathbf{R}} \mathcal{W}_{f,g}^{\Lambda_\mu}(t,w)e^{-i(Ax^2+Bxw+Cw^2+Dx+Ew)}dw.\end{aligned} \tag{16}$$

Now, we are going to discuss several basic properties of the WVD-QPFT given by (10). These properties are very important in signal representations.

Theorem 3.1 (Complex conjugation)**.** *For $f, g \in L^2(\mathbf{R})$, we have*

$$\overline{\mathcal{W}^{\Lambda_\mu}_{f,g}(t,w)} = \mathcal{W}^{\overline{\Lambda}_\mu}_{g,f}(t,w) \tag{17}$$

Proof. By using the definition of WVD-QPFT, we have

$$\begin{aligned}
\overline{\mathcal{W}^{\Lambda_\mu}_{f,g}(t,w)} &= \overline{\int_{\mathbf{R}} f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}\Lambda_\mu(x,w)dx} \\
&= \overline{\int_{\mathbf{R}} f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}\frac{1}{\sqrt{2\pi}}e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}dx} \\
&= \int_{\mathbf{R}} g\left(t-\frac{x}{2}\right)\overline{f\left(t+\frac{x}{2}\right)}\frac{1}{\sqrt{2\pi}}e^{-i(Ax^2+Bxw+Cw^2+Dx+Ew)}dx \\
&= \int_{\mathbf{R}} g\left(t-\frac{x}{2}\right)\overline{f\left(t+\frac{x}{2}\right)}\overline{\Lambda_\mu(x,w)}dx \\
&= \mathcal{W}^{\overline{\Lambda}_\mu}_{g,f}(t,w)
\end{aligned}$$

which completes the proof. □

Theorem 3.2 (Boundedness)**.** *Let $f, g \in L^2(\mathbf{R})$. Then*

$$\left|\mathcal{W}^{\Lambda_\mu}_{f,g}(t,u)\right| \le \sqrt{\frac{2}{\pi}}\|f\|_{L^2(\mathbf{R})}\|g\|_{L^2(\mathbf{R})} \tag{18}$$

Proof. Thanks to the Cauchy-Schwarz inequality in quaternion domain, we get

$$\begin{aligned}
|\mathcal{W}^{\Lambda_\mu}_{f,g}(t,w)|^2 &= \left|\int_{\mathbf{R}} f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}\frac{1}{\sqrt{2\pi}}e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}dx\right|^2 \\
&\le \left(\int_{\mathbf{R}}\left|f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}\frac{1}{\sqrt{2\pi}}e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}\right|dx\right)^2 \\
&= \left(\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}\left|f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}\right|dx\right)^2 \\
&\le \frac{1}{2\pi}\left(\int_{\mathbf{R}}\left|f\left(t+\frac{x}{2}\right)\right|^2dx\right)\left(\int_{\mathbf{R}}\left|\overline{g\left(t-\frac{x}{2}\right)}\right|^2dx\right) \\
&= \frac{1}{2\pi}\left(2\int_{\mathbf{R}}|f(w)|^2dw\right)\left(2\int_{\mathbf{R}}\left|\overline{g(y)}\right|^2dy\right) \\
&= \frac{2}{\pi}\|f\|^2_{L^2(\mathbf{R})}\|g\|^2_{L^2(\mathbf{R})}.
\end{aligned}$$

Equivallently

$$|\mathcal{W}^{\Lambda_\mu}_{f,g}(t,u)| \le \sqrt{\frac{2}{\pi}}\|f\|_{L^2(\mathbf{R})}\|g\|_{L^2(\mathbf{R})}$$

which completes the proof. □

We now discuss a property that indicates the WVD-QPFT does not satisfy the superposition principle.

Theorem 3.3 (Nonlinearity)**.** *Let f and g be two signals in $L^2(\mathbf{R})$. Then*

$$\mathcal{W}^{\Lambda_\mu}_{f+g} = \mathcal{W}^{\Lambda_\mu}_{f,f} + \mathcal{W}^{\Lambda_\mu}_{f,g} + \mathcal{W}^{\Lambda_\mu}_{g,f} + \mathcal{W}^{\Lambda_\mu}_{g,g} \tag{19}$$

Proof. By virtue of Definition 3.1 we have

$$
\begin{aligned}
\mathcal{W}^{\Lambda_\mu}_{f+g}(t,w) = &\int_{\mathbf{R}^2} \left[f\left(t+\frac{x}{2}\right)+g\left(t+\frac{x}{2}\right)\right]\overline{\left[f(t-\frac{x}{2})+g(t-\frac{x}{2})\right]} \\
&\times \frac{1}{\sqrt{2\pi}} e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}dx \\
=&\int_{\mathbf{R}^2} \Big[f\left(t+\frac{x}{2}\right)\overline{f(t-\frac{x}{2})}+f\left(t+\frac{x}{2}\right)\overline{g(t-\frac{x}{2})} \\
&+g(t+\frac{x}{2})\overline{f(t-\frac{x}{2})}+g(t+\frac{x}{2})\overline{g(t-\frac{x}{2})}\Big] \\
&\times \frac{1}{\sqrt{2\pi}} e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}dx \\
=&\int_{\mathbf{R}^2} f\left(t+\frac{x}{2}\right)\overline{f\left(t-\frac{x}{2}\right)}\frac{1}{\sqrt{2\pi}}e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}dx \\
&+\int_{\mathbf{R}^2} f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}\frac{1}{\sqrt{2\pi}}e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}dx \\
&+\int_{\mathbf{R}^2} g\left(t+\frac{x}{2}\right)\overline{f\left(t-\frac{x}{2}\right)}\frac{1}{\sqrt{2\pi}}e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}dx \\
&+\int_{\mathbf{R}^2} g\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}\frac{1}{\sqrt{2\pi}}e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}dx \\
=&\mathcal{W}^{\Lambda_\mu}_{f,f}+\mathcal{W}^{\Lambda_\mu}_{f,g}+\mathcal{W}^{\Lambda_\mu}_{g,f}+\mathcal{W}^{\Lambda_\mu}_{g,g}
\end{aligned}
$$

which completes the proof. □

Note the properties like Modulation, Shift, Dilation are similar to the classical QPFT so are avoided.

The next theorem guarantees that the original signal can be reconstructed back from the corresponding WVD-QPFT within a constant factor.

Theorem 3.4 (Reconstruction formula)**.** *For $f,g\in L^2(\mathbf{R})$ where $g(0)\neq 0$. We get the following inversion formula of the WVD-QPFT:*

$$
f(t) \quad = \quad \frac{1}{\overline{g(0)}}\frac{|B|}{\sqrt{2\pi}}\int_{\mathbf{R}^2}\mathcal{W}^{\Lambda_\mu}_{f,g}\left(\frac{t}{2},w\right)e^{-i(At_1^2+Bt_1w+Cw^2+Dt_1+Ew)}dw \tag{20}
$$

Proof. By (15), we have

$$
\{R_{f,g}(t,x)\} \quad = \mathcal{Q}^{-1}_{\Lambda_\mu}\{\mathcal{W}^{\Lambda_\mu}_{f,g}(t,w)\}
$$

which gives

$$
f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)} = |B|\int_{\mathbf{R}}\mathcal{W}^{\Lambda_\mu}_{f,g}(t,w)\overline{\Lambda_\mu(x,w)}dw.
$$

Now let $t_1=t+\frac{x}{2}$ and $t_2=t-\frac{x}{2}$, we get

$$
f(t_1)\overline{g(t_2)} = \frac{|B|}{\sqrt{2\pi}}\int_{\mathbf{R}}\mathcal{W}^{\Lambda_\mu}_{f,g}\left(\frac{t_1+t_2}{2},w\right)e^{-i(A(t_1-t_2)^2+B(t_1-t_2)w+Cw^2+D(t_1-t_2)+Ew)}dw
$$

By taking $t_2 = 0$, we have for any t_1

$$f(t_1) = \frac{|B|}{\sqrt{2\pi}} \frac{1}{g(0)} \int_{\mathbf{R}} \mathcal{W}^{\Lambda_\mu}_{f,g}\left(\frac{t_1}{2}, w\right) e^{-i(At_1^2+Bt_1w+Cw^2+Dt_1+Ew)}dw$$

which completes the proof. □

Theorem 3.5 (Orthogonality relation). *If $f_1, f_2, g_1, g_2 \in L^2(\mathbf{R})$ are real valued signals. Then*

$$\left\langle \mathcal{W}^{\Lambda_\mu}_{f_1,g_1}(t,w), \mathcal{W}^{\Lambda_\mu}_{f_2,g_2}(t,w)\right\rangle = 2\left\langle f_1, f_2\right\rangle \overline{\left\langle g_1, g_2\right\rangle} \tag{21}$$

Proof. By the definition of Wigner-Ville distribution associated with QPFT and innear product relation we have

$$\begin{aligned}
&\langle W^{\Lambda_\mu}_{f_1,g_1}(t,w), \mathcal{W}^{\Lambda_\mu}_{f_2,g_2}(t,w)\rangle \\
&= \int_{\mathbf{R}}\int_{\mathbf{R}} \mathcal{W}^{A_1,A_2}_{f_1,g_1}(t,w)\overline{\mathcal{W}^{A_1,A_2}_{f_2,g_2}(t,w)}dwdt \\
&= \int_{\mathbf{R}}\int_{\mathbf{R}} \mathcal{W}^{A_1,A_2}_{f_1,g_1}(t,w)\left[\overline{\int_{\mathbf{R}} f_2\left(t+\frac{x}{2}\right)\overline{g_2\left(t-\frac{x}{2}\right)}}\right. \\
&\qquad\qquad \left.\overline{\times \frac{1}{\sqrt{2\pi}} e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}dx}\right] dwdt \\
&= \int_{\mathbf{R}}\int_{\mathbf{R}}\int_{\mathbf{R}} \left[\mathcal{W}^{\Lambda_\mu}_{f_1,g_1}(t,w)\frac{1}{\sqrt{2\pi}}e^{-i(Ax^2+Bxw+Cw^2+Dx+Ew)}g_2\left(t-\frac{x}{2}\right)\right. \\
&\qquad\qquad \left.\times\, \overline{f_2\left(t+\frac{x}{2}\right)}\right] dwdtdx \\
&= \frac{1}{|B|}\int_{\mathbf{R}}\int_{\mathbf{R}}\left[\int_{\mathbf{R}} \mathcal{W}^{\Lambda_\mu}_{f_1g_1}(t,w)\frac{|B|}{\sqrt{2\pi}}e^{-i(Ax^2+Bxw+Cw+Dx+Ew)}dw\right] \\
&\qquad\qquad \times g_2\left(t-\frac{x}{2}\right)\overline{f_2\left(t+\frac{x}{2}\right)}dtdx
\end{aligned} \tag{22}$$

Now by using (16) in (22), we have

$$\begin{aligned}
&\left\langle \mathcal{W}^{\Lambda_\mu}_{f_1,g_1}(t,w), \mathcal{W}^{\Lambda_\mu}_{f_2,g_2}(t,w)\right\rangle \\
&= \frac{1}{|B|}\int_{\mathbf{R}}\int_{\mathbf{R}} f_1\left(t+\frac{x}{2}\right)\overline{g_1\left(t-\frac{x}{2}\right)}g_2\left(t-\frac{x}{2}\right)\overline{f_2\left(t+\frac{x}{2}\right)}dtdx
\end{aligned}$$

Using the change of variables $t+\frac{x}{2} = \eta$,and $t-\frac{x}{2} = \xi$ the equation becomes

$$\langle \mathcal{W}^{\Lambda_\mu}_{f_1,g_1}(t,w), \mathcal{W}^{\Lambda_\mu}_{f_2,g_2}(t,w)\rangle = \frac{2}{|B|}\int_{\mathbf{R}}\int_{\mathbf{R}} f_1(\eta)\overline{g_1(\xi)}g_2(\xi)\overline{f_2(\eta)}d\eta d\xi.$$

On interchanging the order of integrals, we obtain

$$\left\langle \mathcal{W}^{\Lambda_\mu}_{f_1,g_1}(t,w), \mathcal{W}^{\Lambda_\mu}_{f_2,g_2}(t,w)\right\rangle = \frac{2}{|B|}\left\langle f_1, f_2\right\rangle \overline{\left\langle g_1, g_2\right\rangle} \tag{23}$$

which completes the proof. □

Consequences of Theorem 3.5.

(1) If $g_1 = g_2 = g$, then
$$\langle \mathcal{W}^{\Lambda_\mu}_{f_1,g}(t,w), \mathcal{W}^{\Lambda_\mu}_{f_2,g}(t,w)\rangle = \frac{2}{|B|}\|g\|^2_{L^2(\mathbf{R})}\langle f_1, f_2\rangle. \tag{24}$$

(2) If $f_1 = f_2 = f$, then
$$\langle \mathcal{W}^{\Lambda_\mu}_{f,g_1}(t,w), \mathcal{W}^{\Lambda_\mu}_{f,g_2}(t,w)\rangle = \frac{2}{|B|}\|f\|^2_{L^2(\mathbf{R})}\langle g_1, g_2\rangle. \tag{25}$$

(3) If $f_1 = f_2 = f$ and $g_1 = g_2 = g$, then
$$\begin{aligned}\langle \mathcal{W}^{\Lambda_\mu}_{f,g}(t,), \mathcal{W}^{\Lambda_\mu}_{f,g}(t,w)\rangle &= \int_{\mathbf{R}}\int_{\mathbf{R}} |\mathcal{W}^{\Lambda_\mu}_{f,g}(t,w)|^2 du dt \\ &= \frac{2}{|B|}\|f\|^2_{L^2(\mathbf{R})}\|g\|^2_{L^2(\mathbf{R})}. \end{aligned}\tag{26}$$

We now give a result which shows that the signal energy is preserved by the WVD-QPFT

Theorem 3.6 (Plancherel's theorem for WVD-QPFT)**.** *For $f, g \in L^2(\mathbf{R})$, we have the equality*
$$\begin{aligned}\int_{\mathbf{R}}\int_{\mathbf{R}} |\mathcal{W}^{\Lambda_\mu}_{f,g}(t,w)|^2 du dt &= \|\mathcal{W}^{\Lambda_\mu}_{f,g}\|^2_{L^2(\mathbf{R}} \\ &= 2\|f\|^2_{L^2(\mathbf{R}}\|g\|^2_{L^2(\mathbf{R})} \end{aligned}\tag{27}$$

Proof. If we look at (26), the proof of the theorem follows. □

4. Ambiguity function associated with the quadratic-phase Fourier transform

In this section, we provide the definition of Ambiguity function associated with the quadratic-phase Fourier transform(AF-QPFT) which can be regarded as the generalization of AF to the quadratic-phase Fourier transform. Further some basic properties of the WVD-QPFT which are important for signal representation in signal processing are investigated.

Definition 4.1. For $f, g \in \mathbf{R}$, the cross AF-QPFT of f, g is defined by
$$\mathcal{A}^{\Lambda_\mu}_{f,g}(t,w) = \int_{\mathbf{R}} f\left(x + \frac{t}{2}\right)\overline{g\left(x - \frac{t}{2}\right)}\Lambda_\mu(x,w)dx. \tag{28}$$
where $\Lambda_\mu(x,w) = \frac{1}{\sqrt{2\pi}}e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}$.

The another way to define the AF-QPFT is as
$$\mathcal{A}^{\Lambda_\mu}_{f,g}(x,w) = \int_{\mathbf{R}} f\left(t + \frac{x}{2}\right)\overline{g\left(t - \frac{x}{2}\right)}\Lambda_\mu(t,w)dt. \tag{29}$$
where $\Lambda_\mu(t,w) = \frac{1}{\sqrt{2\pi}}e^{i(At^2+Btw+Cw^2+Dt+Ew)}$.

The equation (28) can be reshaped as
$$\mathcal{A}^{\Lambda_\mu}_{f,g}(t,w) = \int_{\mathbf{R}} h_{f,g}(x,t)\Lambda_\mu(x,w)dx \tag{30}$$
where $h_{f,g}(x,t) = f\left(x + \frac{t}{2}\right)\overline{g\left(x - \frac{t}{2}\right)}$ is the correlation of f and g.

Remark 4.1. If $f = g$ then $\mathcal{A}^{\Lambda_\mu}_{f,f}(t,w) = \mathcal{A}^{\Lambda_\mu}_{f}(t,w)$ is called the auto AF-QPFT. Otherwise it is called Cross AF-QPFT

The following theorem shows the relationship between AF-QPFT and classical AF.

Theorem 4.1. *Let $f, g \in L^2(\mathbf{R})$, then we have*

$$\mathcal{A}^{\Lambda_\mu}_{f}(t,w) = \frac{1}{\sqrt{2\pi}} e^{i(Cw^2+Ew)} \mathcal{A}_f(t, -Bw - D), \quad A = 0. \tag{31}$$

Proof. From Remark 4.1, we have

$$\begin{aligned}\mathcal{A}^{\Lambda_\mu}_{f}(t,w) &= \int_{\mathbf{R}} f\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)}\frac{1}{\sqrt{2\pi}}e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} f\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)}e^{i(Ax^2+Bxw+Dx+)}e^{i(Cw^2+Ew)}dx \end{aligned}\tag{32}$$

By setting $A = 0$ and noting the classical definition of WVD the L.H.S of above becomes

$$\begin{aligned}&= \frac{1}{\sqrt{2\pi}}e^{i(Cw^2+Ew)}\int_{\mathbf{R}} f\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)}e^{i(Bxw+Dx)}dx\\ &= \frac{1}{\sqrt{2\pi}}e^{i(Cw^2+Ew)}\mathcal{A}_f(t, -Bw - D). \end{aligned}\tag{33}$$

Which completes the proof. □

By varying the parameter $\mu = (A, B, C, D, E)$ the proposed transform (28) boils down to various well known integral transforms associated with AF, viz: for $\mu = (0, -1, 0, 0, 0)$, we get classical AF and for $\mu = (\cot\theta, -\csc\theta, \cot\theta, 0, 0)$, we get fractional AF.

Now, we discuss several basic properties of the AF-QPFT given by (28). These properties play important roles in signal representation.

Proposition 4.2. *Let $f, g \in L^2(\mathbf{R})$, then*

$$\mathcal{A}^{\Lambda_\mu}_{f(-t)}(t,w) = -\mathcal{A}^{\Lambda_{\mu'}}_{f(t)}(t,-w) \tag{34}$$

Proof. From (28), we have

$$\mathcal{A}^{\Lambda_\mu}_{f(-t)}(t,w) \;=\; \int_{\mathbf{R}} f\left(-x+\frac{t}{2}\right)\overline{f\left(-x-\frac{t}{2}\right)}\frac{1}{\sqrt{2\pi}}e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}dx$$

On setting $-x = x'$, we obtain

$$\begin{aligned}&\mathcal{A}^{\Lambda_\mu}_{f(-t)}(t,w)\\ &= -\int_{\mathbf{R}} f\left(x'+\frac{t}{2}\right)\overline{f\left(x'-\frac{t}{2}\right)}\frac{1}{\sqrt{2\pi}}e^{i(A(-x')^2+B(-x')w+Cw^2+D(-x')+Ew)}dx'\\ &= -\int_{\mathbf{R}} f\left(x'+\frac{t}{2}\right)\overline{f\left(x'-\frac{t}{2}\right)}\frac{1}{\sqrt{2\pi}}e^{i(Ax'^2+Bx'(-w)+C(-w)^2+(-D)x'+(-E)(-w))}dx'\\ &= -\mathcal{A}^{\Lambda_{\mu'}}_{f,g}(t,-w)\\ &= -\mathcal{A}^{\Lambda_{\mu'}}_{f(t)}(t,-w), \quad \mu' = (A, B, C, -D, -E).\end{aligned}$$

Which completes the proof. □

Theorem 4.3 (Non-Linearity of AF). *Let f and g be two functions in $L^2(\mathbf{R})$. Then*

$$\mathcal{A}^{\Lambda_\mu}_{f+g} = \mathcal{A}^{\Lambda_\mu}_{f,f} + \mathcal{A}^{\Lambda_\mu}_{f,g} + \mathcal{A}^{\Lambda_\mu}_{g,f} + \mathcal{A}^{\Lambda_\mu}_{g,g} \tag{35}$$

Proof. We avoid proof as it follows from definition of AF-QPFT. □

Theorem 4.4 (Reconstruction theorem for AF). *For $f, g \in L^2(\mathbf{R})$ where g does not vanish at 0. We get the following inversion formula of the AF-QPFT:*

$$f(t) = \frac{1}{\overline{g(0)}} \frac{|B|}{\sqrt{2\pi}} \int_{\mathbf{R}^2} \mathcal{A}^{\Lambda_\mu}_{f,g}(t,w)\, e^{-i\left(A\left(\frac{t}{2}\right)^2 + B\left(\frac{t}{2}\right)w + Cw^2 + D\left(\frac{t}{2}\right) + Ew\right)} dw \tag{36}$$

Proof. By definition of AF-QPFT 4.1, we have

$$\begin{aligned} \mathcal{A}^{\Lambda_\mu}_{f,g}(t,w) &= \int_{\mathbf{R}} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} \Lambda_\mu(x,w) dx \\ &= \mathcal{Q}_\mu \left\{ f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} \right\}(w) \\ &= \mathcal{Q}_\mu \{h_{f,g}(x,t)\}(w). \end{aligned} \tag{37}$$

Applying the inverse QPFT to (37), we get

$$\{h_{f,g}(x,t)\} = \mathcal{Q}^{-1}_\mu \{\mathcal{A}^{\Lambda_\mu}_{f,g}(t,w)\} \tag{38}$$

which implies

$$\begin{aligned} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} &= \mathcal{Q}^{-1}_\mu \{\mathcal{A}^{\Lambda_\mu}_{f,g}(t,w)\} \\ &= \frac{|B|}{\sqrt{2\pi}} \int_{\mathbf{R}} \mathcal{A}^{\Lambda_\mu}_{f,g}(t,w) e^{-i(Ax^2 + Bxw + Cw^2 + Dx + Ew)} dw \\ &= |B| \int_{\mathbf{R}} \mathcal{A}^{\Lambda_\mu}_{f,g}(t,w) \overline{\Lambda_\mu(x,w)} dw. \end{aligned}$$

On setting $x = \frac{t}{2}$, we obtain

$$\begin{aligned} f(t)\overline{g(0)} &= |B| \int_{\mathbf{R}} \mathcal{A}^{\Lambda_\mu}_{f,g}(t,w) \overline{\Lambda_\mu\left(\frac{t}{2}, w\right)} dw \\ f(t) &= \frac{|B|}{\overline{g(0)}} \int_{\mathbf{R}} \mathcal{A}^{\Lambda_\mu}_{f,g}(t,w) \overline{\Lambda_\mu\left(\frac{t}{2}, w\right)} dw. \end{aligned}$$

Which completes the proof. □

Theorem 4.5 (Marginal property). *The time delay marginal property of AF-QPFT can be expressed as*

$$\frac{|B|}{\sqrt{2\pi}} \int_{\mathbf{R}} \mathcal{A}^{\Lambda_\mu}_{f,f}(t,w) e^{-i(Cw^2 + Ew)} dw = f\left(\frac{t}{2}\right) \overline{f\left(\frac{-t}{2}\right)} \tag{39}$$

Proof. From 39, we have

$$\begin{aligned} f\left(x + \frac{t}{2}\right) \overline{f\left(x - \frac{t}{2}\right)} &= |B| \int_{\mathbf{R}} \mathcal{A}^{\Lambda_\mu}_{f,f}(t,w) \overline{\Lambda_\mu(x,w)} dw \\ &= \frac{|B|}{\sqrt{2\pi}} \int_{\mathbf{R}} \mathcal{A}^{\Lambda_\mu}_{f,f}(t,w) e^{-i(Ax^2 + Bxw + Cw^2 + Dx + Ew)} dw. \end{aligned}$$

On setting $x=0$, we have

$$f\left(\frac{t}{2}\right)\overline{f\left(\frac{-t}{2}\right)} = \frac{|B|}{\sqrt{2\pi}}\int_{\mathbf{R}}\mathcal{A}^{\Lambda_\mu}_{f,f}(t,w)e^{-i(Cw^2+Ew)}dw.$$

Which proves Theorem 4.5. □

Theorem 4.6 (Quadratic-phase marginal property). *The Quadratic-phase marginal property of the AF-QFT is given by*

$$\frac{1}{\sqrt{2\pi}}e^{iw(Cw+Ew)}\int_{\mathbf{R}}\mathcal{A}^{\Lambda_\mu}_{f,f}(t,w)e^{-i(\frac{A}{2}(x^2-\frac{t^2}{4}))}dt = Q_{\mu'}[f](w)Q_{\mu'}[\overline{f}](w), \tag{40}$$

where $\mu' = \left(\frac{A}{4},\frac{B}{2},C,\frac{D}{2},E\right)$.

Proof. By using Definition 4.1, we have

$$\int_{\mathbf{R}}\mathcal{A}^{\Lambda_\mu}_{f,f}(t,w)e^{-i(\frac{A}{2}(x^2-\frac{t^2}{4}))}dt$$
$$= \int_{\mathbf{R}}\int_{\mathbf{R}} f\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)}\frac{1}{\sqrt{2\pi}}e^{i(Ax^2+Bxw+Cw^2+Dx+Ew)}dxe^{-i\frac{A}{2}(x+\frac{t}{2})(x-\frac{t}{2})}dt$$

By making change of variable $x+\frac{t}{2}=u$ and $x-\frac{t}{2}=v$, above equation becomes

$$\int_{\mathbf{R}}\mathcal{A}^{\Lambda_\mu}_{f,f}(t,w)e^{-i(\frac{A}{2}(x^2-\frac{t^2}{4}))}dt$$
$$= \int_{\mathbf{R}}\int_{\mathbf{R}} f(u)\overline{f(v)}\frac{1}{\sqrt{2\pi}}e^{i\left(A(\frac{u+v}{2})^2+B(\frac{u+v}{2})w+Cw^2+D(\frac{u+v}{2})+Ew\right)}e^{-i\frac{A}{2}uv}dudv$$
$$= \int_{\mathbf{R}}\int_{\mathbf{R}} f(u)\overline{f(v)}\frac{1}{\sqrt{2\pi}}e^{i\left(\frac{A}{4}(u^2+2uv+v^2)+\frac{B}{2}(u+v)w+Cw^2+\frac{D}{2}(u+v)+Ew\right)}e^{-i\frac{A}{2}uv}dudv$$
$$= \sqrt{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} f(u)\overline{f(v)}\frac{1}{\sqrt{2\pi}}e^{i\left(\frac{A}{4}u^2+\frac{B}{2}uw+Cw^2+\frac{D}{2}u+Ew\right)}$$
$$\times\frac{1}{\sqrt{2\pi}}e^{i\left(\frac{A}{4}v^2+\frac{B}{2}vw+Cw^2+\frac{D}{2}v+Ew\right)}e^{i(A\frac{uv}{2}-Cw^2-Ew)}e^{-i\frac{A}{2}uv}dudv$$
$$= \sqrt{2\pi}e^{i(-Cw^2-Ew)}\int_{\mathbf{R}} f(u)\frac{1}{\sqrt{2\pi}}e^{i\left(\frac{A}{4}u^2+\frac{B}{2}uw+Cw^2+\frac{D}{2}u+Ew\right)}du$$
$$\times\int_{\mathbf{R}}\overline{f(v)}\frac{1}{\sqrt{2\pi}}e^{i\left(\frac{A}{4}v^2+\frac{B}{2}vw+Cw^2+\frac{D}{2}v+Ew\right)}dv$$
$$= \sqrt{2\pi}e^{i(-Cw^2-Ew)}Q_{\mu'}[f](w)Q_{\mu'}[\overline{f}](w).$$

Which completes the proof. □

Theorem 4.7 (Orthogonality relation for AF-QFPT). *If* $f_1,f_2,g_1,g_2\in L^2(\mathbf{R})$ *are real valued signals. Then*

$$\left\langle\mathcal{A}^{\Lambda_\mu}_{f_1,g_1}(t,w),\mathcal{A}^{\Lambda_\mu}_{f_2,g_2}(t,w)\right\rangle = \frac{2}{|B|}\langle f_1,f_2\rangle\overline{\langle g_1,g_2\rangle} \tag{41}$$

Proof. We avoid proof as it is similar to the proof of the Theorem 3.5. □

Now we move forward towards our main section that is convolution and correlation theorems for Wigner-Ville distribution associated with quadratic-phase Fourier transform.

5. Convolution and correlation theorem for WVD-QPFT

Convolution and correlation results are very significant in signal and image processing. In [29] authors introduced the convolution theorem for WVD-QPFT, but in this section we establish convolution for the proposed transform by using the two parameter convolution operator, thus it is more flexible and accurate. We then establish the new convolution and correlation for the WVD-QPFT, these theorems will open new gates to investigate the sampling and filtering theorems of the WVD-QPFT.

Definition 5.1. For any two functions $f, g \in L^2(\mathbf{R})$, we define the convolution operator of the QPFT as

$$(f \star g)(t) = \int_{\mathbf{R}} f(x)g(t-x)\Psi(t,x)dx. \tag{42}$$

Where $\Psi(t,x)$ is a weight function given by $e^{-2iA(t^2-x^2)-iD(t-x)}$ or $e^{-4iAx(t-x)}$ but for the best results we take $\Psi(t,x) = e^{-2iA(t^2-x^2)-iD(t-x)}$ consisting of two parameters.

As a consequence of the above definition, we get the following important theorem, which states how the convolution of two real-valued functions interacts with their QPFTs.

Theorem 5.1 (WVD-QPFT Convolution). *For any two quaternion functions $f, g \in L^2(\mathbf{R})$, the following result holds*

$$\mathcal{W}^{\Lambda_\mu}_{f\star g}(t,w) = \int_{\mathbf{R}} \mathcal{W}^{\Lambda_\mu}_{f}(u,w)\mathcal{W}_f(t-u,-Bw-D)e^{-2iA(2(t^2-u^2))-iD(2(t-u))}du$$

where $\mathcal{W}_f(t,w)$ is classical WVD.

Proof. Applying the Definition of the WVD-QPFT we have

$$\mathcal{W}^{\Lambda_\mu}_{f\star g}(t,w) = \int_{\mathbf{R}} (f\star g)(t+\frac{n}{2})(\overline{f}\star\overline{g})(t-\frac{n}{2})\Lambda_\mu(n,w)dn. \tag{43}$$

Now using Definition 5.1 in (43) we have

$$\begin{aligned}\mathcal{W}^{\Lambda_\mu}_{f\star g}(t,w) &= \int_{\mathbf{R}}\left\{\int_{\mathbf{R}} f(x)g(t+\frac{n}{2}-x)e^{-2iA((t+\frac{n}{2})^2-x^2)-iD((t+\frac{n}{2})-x)}dx\right\}\\ &\quad\times\left\{\int_{\mathbf{R}} f(y)g(t-\frac{n}{2}-y)e^{-2iA((t-\frac{n}{2})^2-y^2)-iD((t-\frac{n}{2})-y)}dy\right\}\\ &\quad\times\frac{1}{\sqrt{2\pi}}e^{i(An^2+Bnw+Cw^2+Dn+Ew)}dn\end{aligned}$$

Setting $x = u + \frac{p}{2}$, $y = u - \frac{p}{2}$ and $n = p + q$, we have

$$\begin{aligned}\mathcal{W}^{\Lambda_\mu}_{f\star g}(t,w) &= \int_{\mathbf{R}}\int_{\mathbf{R}}\int_{\mathbf{R}} f(u+\frac{p}{2})g(t+\frac{n}{2}-(u+\frac{p}{2}))\\ &\quad\times e^{-2iA((t+\frac{p+q}{2})^2-(u+\frac{p}{2})^2)-iD((t+\frac{p+q}{2})-(u+\frac{p}{2}))}\\ &\quad\times\overline{f(u-\frac{p}{2})g(t-\frac{n}{2}-(u-\frac{p}{2}))}\\ &\quad\times e^{-2iA((t-\frac{p+q}{2})^2-(u-\frac{p}{2})^2)-iD((t-\frac{p+q}{2})-(u-\frac{p}{2}))}\\ &\quad\times\frac{1}{\sqrt{2\pi}}e^{i(A(p+q)^2+B(p+q)w+Cw^2+D(p+q)+Ew)}dpdqdn\end{aligned}$$

$$= \int_{\mathbf{R}}\int_{\mathbf{R}}\int_{\mathbf{R}} f(u+\frac{p}{2})g(t+\frac{n}{2}-(u+\frac{p}{2}))\overline{f(u-\frac{p}{2})g(t-\frac{n}{2}-(u-\frac{p}{2}))}$$
$$\times e^{-2iA(2t^2+\frac{q^2}{2}+pq-2u^2)-iD(2t-2u)}$$
$$\times \frac{1}{\sqrt{2\pi}}e^{i(A(p+q)^2+B(p+q)w+Cw^2+D(p+q)+Ew)}dpdqdn$$
$$= \int_{\mathbf{R}}\int_{\mathbf{R}}\int_{\mathbf{R}} f(u+\frac{p}{2})g(t+\frac{n}{2}-(u+\frac{p}{2}))\overline{f(u-\frac{p}{2})g(t-\frac{n}{2}-(u-\frac{p}{2}))}$$
$$\times e^{-2iA(2t^2-2u^2)-iD(2t-2u)-iAq^2+-2iApq}$$
$$\times \frac{1}{\sqrt{2\pi}}e^{i(Ap^2+Bpw+Cw^2+Dp+Ew)}e^{i(Aq^2+Bqw+Dq)}e^{2iApq}dpdqdn$$
$$= \int_{\mathbf{R}}\int_{\mathbf{R}} f(u+\frac{p}{2})\overline{f(u-\frac{p}{2})}\frac{1}{\sqrt{2\pi}}e^{i(Ap^2+Bpw+Cw^2+Dp+Ew)}dp$$
$$\times \int_{\mathbf{R}} g(t-u+\frac{q}{2})\overline{g(t-u-\frac{q}{2})}e^{iq(Bw+D)}dq$$
$$\times e^{-2iA(2t^2-2u^2)-iD(2t-2u)}du$$
$$= \int_{\mathbf{R}} \mathcal{W}_f^{\Lambda_\mu}(u,w)\mathcal{W}_f(t-u,-Bw-D)e^{-2iA(2(t^2-u^2))-iD(2(t-u))}du$$

which completes the proof. □

Remark 5.1. [29] If we take $\Psi(t,x) = e^{-4iAx(t-x)}$ then convolution theorem for WVD-QPFT theorem 5.2 can be rewritten as:

$$\frac{1}{\sqrt{2\pi}}e^{-i(Cw^2+Ew)}\mathcal{W}_{f\star g}^{\Lambda_\mu}(t,w) = \int_{\mathbf{R}} \mathcal{W}_f^{\Lambda_\mu}(u,w)\mathcal{W}_g^{\Lambda_\mu}(t-u,w)e^{-4iAu(2(t-u))}du.$$

Next, we will derive the correlation theorem for the WVD-QPFT. Let us define the correlation for the QPFT.

Definition 5.2. For any two functions $f, g \in L^2(\mathbf{R})$, we define the correlation operator of the QPFT as

$$(f \circ g)(t) = \int_{\mathbf{R}} \overline{f(x)}g(t+x)\Phi(t,x)dx. \tag{44}$$

Where $\Phi(t,x)$ is a weight function given by $e^{2iA(t^2x^2)+iD(t+x)}$ or $e^{4iAx(t+x)}$.

Now, we reap a consequence of the above definition by taking $\Phi(t,x) = e^{4iAx(t+x)}$ below which gives the relationship between the correlation of two functions and the QPFT.

Theorem 5.2 (WVD-QPFT Correlation). *For any two quaternion functions $f, g \in L^2(\mathbf{R})$, the following result holds*

$$\frac{1}{\sqrt{2\pi}}e^{i(Cw^2-Ew)}\mathcal{W}_{f\circ g}^{\Lambda_\mu}(t,w) = \int_{\mathbf{R}} \mathcal{W}_f^{\Lambda'_\mu}(u,-w)\mathcal{W}_g^{\Lambda_\mu}(t+u,w)e^{4iAu(2(t+u))}du$$

where $\mu' = (A, B, C, -D, E)$.

Proof. Applying the definition of the WVD-QPFT we have

$$\mathcal{W}_{f\circ g}^{\Lambda_\mu}(t,w) = \int_{\mathbf{R}} (f\circ g)(t+\frac{n}{2})(\overline{f}\circ\overline{g})(t-\frac{n}{2})\Lambda_\mu(n,w)dn. \tag{45}$$