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RECENT TRENDS IN MAJORIZATION THEORY AND OPTIMIZATION

Applications to Wireless Communications

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Editura UNIVERSITARIA
Craiova, 2015

Editura PRO UNIVERSITARIA
Bucureşti, 2015

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Descrierea CIP a Bibliotecii Naţionale a României
ROVENŢA, IONEL
Recent trends in majorization theory and optimization : applications to wireless communications / Ionel Rovenţa. – Craiova : Universitaria ; Bucureşti : Pro Universitaria, 2015
Bibliogr.
ISBN 978-606-14-0925-9
ISBN 978-606-26-0303-8

621.395.345

Acknowledgement: The author was supported by the strategic grant POSDRU/159/1.5/ S/133255, Project ID 133255 (2014), cofinanced by the European Social Fund within the Sectorial Operational Program Human Resources Development 2007-2013.

CHAPTER 1

Introduction

The aim of this book is to present new theoretical results and applications concerning majorization theory.

Majorization tools and convex-concave arguments are used in order to study some optimization problems related to equilibrium problems, min-max theory and fixed point theory in a very general settings. We point out the relevance of these areas in the framework of applied mathematics, where the intersections between different research subjects and topics are numerous.

The possibility to use some convex analysis tools (majorization, convexity-concavity, min-max inequalities, equilibrium problems and fixed point theory) to treat some optimization problems (data rates in networks, traffic strikes, nonlinear optimization) is actual and modern.

Speaking about the optimization problems, we study some optimization properties revealed by convex analysis tools, in order to be applied in different areas of research. We study such optimization problems from an applied point of view, such as modeling communication networks and design of communication systems.

The possibility to use convex analysis in optimizations problems has been increased vigorously and such activity had a great influence on other areas of science. Convex analysis has grown in connection with the study of problems of optimization, equilibrium, control and stability of linear and nonlinear systems. These mathematical disciplines have no border and they rather have good effects on each other.

The concept of majorization appears in 1905, when Max Lorenz propose a graphical way to model the social differences in a finite population. Later on, Dalton (1920) and Hardy-Littlewood-Polya (1927, 1934), reveal some optimization properties, which led to the notion of Schur-convex function. Applications of majorization in 4G communications networks, are related to data transmission rates with huge dimensions, where the interferences between different links create a strangulations of data transmission rates.

An important amelioration was obtained when the optimal power distribution is studied as a nonlinear optimization problem, non-convex with constraints. The problem was solved by the identification of a Schur-convex structure in the objective function. It can be shown that the optimal power allocation is binary, in a sense that, the data are sent with maximal power or the data transmission is not allowed.

From this point of view, the results in literature are related to our results in the field of control theory. Our aim is to use our experience in the field of convex analysis, with special emphasize to the majorization concept and min-max inequalities, in order to obtain new optimal operating principles for some communications devices used in intelligent traffic lights.

The second approach of the optimization problems consists of a more theoretically point of view and deals with the study of min-max inequalities, equilibrium problems, weak majorization concepts in the context of metric spaces with global non-positive curvature (denoted global NPC spaces). Besides Hilbert spaces and manifolds, other important examples of global NPC spaces are the Bruhat-Tits buildings [**84**] (in particular, the trees). We mention that, Ky-Fan's inequality, Schauder's and Schaeffer's fixed point theorems and Hardy-Littlewood-Polya's majorization theorem have been extended to the context of global NPC spaces.

On the other hand, a new type of weak majorization is introduced and discussed in the present book. The subject of majorization in global NPC spaces was successfully studied using some ideas inspired from very recently research papers. Our aim is to extend and to use the weak majorization concept properties to the trees (which should model the optimal distribution in communication networks with high performances).

Other significant idea in this area is given by the objective which consists of defining a new weaker concept of point of Schur-convexity, inspired by the notion of point of convexity. The notion of point of convexity may be used in such a way to prove Ky-Fan's min-max type inequalities and fixed point theorems for weaker assumptions which involves the convexity at a point, even in the context of global NPC spaces.

Notice that some weaker or generalized types of convexity were successfully used in the study of existence and uniqueness of solutions of partial differential equations. In order to establish a sufficient condition for the existence of finite time blow-up solutions for an evolutionary problem, arising naturally in mechanics, biology and population dynamics, we introduced a new class of generalized convex functions [**63, 66**]. In this context, it is interesting to study the implications produced by the concept of point of convexity (convex-concave functions) in the study of existence and uniqueness of solutions for some partial differential equations.

By using the concept of majorization recently introduced in global NPC spaces another aim is to characterize the convex functions on the trees and to study the weak/strong majorization and the corresponding convexity inequalities on trees. We intend to implement such concepts on trees in order to obtain new optimality results concerning data transfer rates of information.

The new concept of majorization can be compared with another concept, which has relevant application on trees. On the other hand, it was pointed out that the Baker-Ericksen inequalities can be rewritten in terms of Schur-convexity, which are related to the concept of rank-one convexity of an energy. In applications from nonlinear elasticity we are dealing with different energies depending of the squared logarithm function, which is neither convex nor concave.

Nevertheless, a Hardy-Littlewood-Polya's type inequality holds in the first three dimensions. One of our results is related to rank-one convexity and polyconvexity of energies depending on the logarithmic strain tensor. This properties together with the coercivity of the energy allow us to obtain existence of the solution of the minimization problem in any finite dimension.

We intend to prove the existence of solutions for equilibrium problems similar with the ones studied recently, but more powerful ones. We will introduce a new concept of point of Schur-convexity in order to obtain more general optimal type results.

By using our experience into this field we intend to highlight the applicability of convex/concave results in PDE's problems which study the existence and multiplicity of solutions. More precisely, we will point out the relevance of our weaker convexity conditions which allow the concavity of the nonlinearity somewhere in the domain of definition, for the study of PDE's problems with nonlinearities which can be convex-concave functions.

Note that the problem of optimal power control for multiuser variable bit rate (VBR) video streaming in a cellular network with orthogonal channels can be also studied using majorization tools. The deterministic model for VBR video traffic that incorporates video frame and playout buffer characteristics cand be formulated as a constrained stochastic optimization problem. Then it is developed a majorization-based solution approach. For the case of a single VBR video session with relaxed peak power constraint, it is developed a power optimal algorithm with low complexity. It can be proved the power optimality of the proposed algorithm and the uniqueness of the global optimum, and demonstrate that the proposed algorithm is also smoothness optimal.

For the case of multiuser VBR video streaming, a heuristic algorithm that selectively suspends some video sessions when the peak power constraint is violated are studied. In addition to the traditional VBR video streaming application, we can also consider the case of interactive video streaming, and show that the proposed schemes can be easily adapted and applied. The proposed algorithms are evaluated with trace-driven simulations, and are shown to achieve considerable power savings and improved video quality over a conventional lazy scheme.

The relevance of this book is a consequence of the fact that we shall develop research in the modern and dynamic fields of optimization and convex analysis which will increase the level of knowledge and technology. Our results will lead to new insights into all the complex physical models that are analyzed and may offer to the engineering and IT community the possibility of imagining new and high-performance optimized devices.

From the view point of convex analysis applied to optimization problems we offer the possibility to the engineers to optimize the networks data rates in 4G networks and traffic strikes. Also, we shall promote international scientific cooperation by integrating in this book joint results with members from prestigious recognized research groups.

Part of this work was done during the author's visit to Lehrstuhl für Nichtlineare Analysis und Modellierung, Fakultät für Mathematik, Universität Duisburg-Essen, Campus Essen, Germany and he would like to thank Patrizion Neff and Ionel Dumitrel Ghiba for their kindness and warm hospitality. The author are also grateful to Professor Constantin P. Niculescu, the main collaborator into the field of convex analysis, for several interesting suggestions and comments.

The author was supported by the strategic grant POSDRU/159/1.5/S/133255, Project ID 133255 (2014), cofinanced by the European Social Fund within the Sectorial Operational Program Human Resources Development 2007-2013.

CHAPTER 2

Preliminaries on majorization theory

In last years, a lot of papers was dedicated to majorization theory, that was scattered in journals in a wide variety of fields. Indeed, many majorization concepts had been reinvented and used in different research areas, as Lorenz or dominance ordering in economics, optimization and graph theory.

Whenever the solution of a problem involves a discrete uniform distribution, the idea of a majorization proof was intensively used. Moreover, if a uniform allocation or distribution was in a sense optimal, then the concept of majorization frequently can be used to order allocations or distributions.

Naturally extensions of the majorization concept are possible and indeed many of them have been fruitfully introduced. The aim of this paper is to introduce a new majorization concept, from which derives multiple applications in different areas.

Let $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ be two vectors from $\mathbb{R}^n$.

Definition 1. *We say that x is majorized by y, denote it by $x \prec y$, if the rearrangement of the components of x and y such that $x_{[1]} \geq x_{[2]} \geq ... \geq x_{[n]}$, $y_{[1]} \geq y_{[2]} \geq ... \geq y_{[n]}$ satisfy $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$, $(1 \leq k \leq n-1)$ and $\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$.*

The notion of Schur-convex function has been introduced by I. Schur in 1923 and has interesting applications in analytic inequalities, elementary quantum mechanics and quantum information theory. See [**53**].

Definition 2. *The function $F : A \to \mathbb{R}$, where $A \subset \mathbb{R}^n$, is called Schur-convex if $x \prec y$ implies $F(x) \leq F(y)$. Any such function F is called Schur-concave if $-F$ is Schur-convex.*

An important source of Schur-convex functions can be found in Merkle [**57**]. Guan [**36, 37**] prove that all symmetric elementary functions and the symmetric means of order k are Schur-concave functions. Other families of Schur-convex functions are studied in [**21, 22, 23, 24, 77**].

In [**89**] a class of analytic inequalities for Schur-convex functions that are made of solutions of a second order nonlinear differential equation are studied. These analytic inequalities are used to infer some geometric inequalities, such as isoperimetric inequality. Li and Trudinger [**49**] consider a special class of inequalities for elementary symmetric functions that are relevant to the study of partial differential equations associated with curvature problems.

Recall here a classical result concerning the study of Schur-convexity for the case of smooth functions. See [**80**].

THEOREM 1. *Let $F(x) = F(x_1, ..., x_n)$ be a symmetric function with continuous partial derivatives on $I^n = I \times I \times ... \times I$, where I is an open interval. Then $F : I^n \to \mathbb{R}$ is Schur-convex if and only if the following inequality*

$$(x_i - x_j)\left(\frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial x_j}\right) \geq 0, \tag{0.1}$$

holds on I^n, for each $i, j \in \{1, .., n\}$. It is strictly Schur-convex if inequality (0.1) *is strict for $x_i \neq x_j$, $1 \leq i, j \leq n$. Any such function F is Schur-concave if the inequality* (0.1) *is reversed.*

In [**75**] and [**77**] we consider a class of Schur-concave functions with some *measure properties*. The isoperimetric inequality and Brunn-Minkowsky's inequality for such kind of functions are presented. Applications in geometric programming and optimization theory are also derived.

About 100 years ago, the properties concerning such notions as length, area, volume, as well as the probability of events, were abstracted under the banner of the word *measure*. We review the notion of measure using this word in an unusual way. More exactly, we study some measure properties of a special class of Schur-concave functions which will be reveal via some discrete versions of isoperimetric inequality and Brunn-Minkowsky's inequality.

We present a discrete version of isoperimetric inequality related to a special class of Schur-concave functions. The reason we discuss about isoperimetric inequality in the context of Schur-concave functions is given by the well known property of every Schur-concave function F, which is the essential property of the *volume measure*,

$$F(x_1, ..., x_n) \leq F\left(\frac{x_1 + ... + x_n}{n}, ..., \frac{x_1 + ... + x_n}{n}\right). \tag{0.2}$$

In other words, by using F as an *area measure*, the inequality (0.2) says that from all polygons with n edges and the sum of all edges constant, the regular polygon, with equal edges, has the biggest area.

1. A discrete isoperimetric inequality

In this section we define some *volume measures* by using a family of Schur-concave functions. Some discrete versions of isoperimetric inequality and Brunn-Minkowsky's inequality will confirm that our approach is correct. Recall here a well known result concerning Brunn-Minkowski's inequality for *convex bodies*, which are non-empty compact, convex subsets of $\mathbb{R}^n$.

THEOREM 2. *Let $\lambda \in (0,1)$ and let K, L be two convex bodies. Then we have*

(1.1)
$$(Vol_n((1-\lambda)k + \lambda L))^{1/n} \geq (1-\lambda)(Vol_n(K))^{1/n} + \lambda (Vol_n(L))^{1/n},$$

with equality when K and L are identically up to a translation.

We replace the volume measure $Vol_n(K)$ by a Schur-concave function of the form $F_n(x_1, ..., x_n) = f(x_1) + ... + f(x_n)$, where f is a nonnegative concave function. In the rest of the paper, F_n *will be called the n-dimensional volume function.*

THEOREM 3. *Let $\lambda \in [0,1]$. Then for each nonnegative concave function f we have*

$$(F_n((1-\lambda)x + \lambda y))^{1/n} \geq (1-\lambda)(F_n(x))^{1/n} + \lambda (F_n(y))^{1/n} \ (x, y \in \mathbb{R}^n),$$

where $F_n(x_1, ..., x_n) = f(x_1) + ... + f(x_n)$.

PROOF. Let $g(x_1, ..., x_n) = (x_1 + ... + x_n)^{1/n}$ defined on $\mathbb{R}^n_+$, which is globally concave and nondecreasing in each variable. We need to prove is the concavity of the function $g(f(x_1), ..., f(x_n))$, which holds since we have a composing between a nondecreasing globally concave function and another concave function. □

Let us consider a more difficult problem concerning the classical isoperimetric inequality for convex bodies from $\mathbb{R}^N$.

THEOREM 4. *(See [**59**]) Let K be a convex subset from $\mathbb{R}^n$ and B a closed ball from $\mathbb{R}^n$. Then we have*

$$\left(\frac{Vol_n(K)}{Vol_n(B)}\right)^{\frac{1}{n}} \leq \left(\frac{S_{n-1}(K)}{S_{n-1}(B)}\right)^{\frac{1}{n-1}}, \tag{1.2}$$

with equality if and only if K is a ball. Here, $S_{n-1}(K)$ means the area of the surface of a convex body K.

We replace the volume measure Vol_n with F_n, which is the corresponding n-dimensional volume function. Notice that the well known