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Some New Integral Inequalities for Exponential Type P-functions

Mahir Kadakal, İmdat İşcan, and Huriye Kadakal

Abstract. In this paper, by using an identity we obtain some new Hermite-Hadamard type inequalities for functions whose first derivative in absolute value is exponential type P-function by using Hölder and power-mean integral inequalities. Then, the authors compare the results obtained with both Hölder, Hölder-İşcan integral inequalities and prove that the Hölder-İşcan integral inequality gives a better approximation than the Hölder integral inequality. Also, some applications to special means of real numbers are also given.

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1. Preliminaries and fundamentals

Let $\Psi : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$\Psi\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_r^s \Psi(u)du \leq \frac{\Psi(r)+\Psi(s)}{2}$$

for all $r, s \in I$ with $r < s$. Both inequalities hold in the reversed direction if the function Ψ is concave. This double inequality is well known as the Hermite-Hadamard inequality [6]. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral integral inequalities for appropriate particular selections of the mapping Ψ.

In [5], Dragomir et al. gave the following definition and related Hermite-Hadamard integral inequalities as follow:

Definition 1.1. *A nonnegative function $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be P-function if the inequality*

$$\Psi(\theta r + (1-\theta)s) \leq \Psi(r) + \Psi(s)$$

holds for all $r, s \in I$ and $\theta \in (0,1)$.

Theorem 1.1. *Let $\Psi \in P(I)$, $r, s \in I$ with $r < s$ and $\Psi \in L[r,s]$. Then*

$$\Psi\left(\frac{r+s}{2}\right) \leq \frac{2}{s-r}\int_r^s \Psi(u)du \leq 2\left[\Psi(r)+\Psi(s)\right].$$

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Definition 1.2 ([17])**.** Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $\Psi : I \to \mathbb{R}$ is an h-convex function, or that Ψ belongs to the class $SX(h, I)$, if Ψ is non-negative and for all $u, v \in I$, $\theta \in (0,1)$ we have

$$\Psi(\theta r + (1-\theta)s) \leq h(\theta)\Psi(r) + h(1-\theta)\Psi(s).$$

If this inequality is reversed, then Ψ is said to be h-concave, i.e. $\Psi \in SV(h, I)$. It is clear that, if we choose $h(\theta) = \theta$ and $h(\theta) = 1$, then the h-convexity reduces to convexity and definition of P-function, respectively.

Readers can look at [1, 17] for studies on h-convexity.

In [13], Kadakal and İşcan gave the following definition and related Hermite-Hadamard integral inequalities as follow:

Definition 1.3. A non-negative function $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ is called exponential type convex function if for every $r, s \in I$ and $\theta \in [0,1]$,

$$\Psi(\theta r + (1-\theta)s) \leq \left(e^{\theta} - 1\right)\Psi(r) + \left(e^{1-\theta} - 1\right)\Psi(s).$$

We note that every nonnegative convex function is exponential type convex function.

Theorem 1.2 ([13])**.** *Let $\Psi : [r,s] \to \mathbb{R}$ be a exponential type convex function. If $r < s$ and $\Psi \in L[r,s]$, then the following Hermite-Hadamard type inequalities hold:*

$$\frac{1}{2\left[\sqrt{e}-1\right]}\Psi\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_r^s \Psi(u)du \leq (e-2)\left[\Psi(r) + \Psi(s)\right].$$

In recent years many authors have studied error estimations of Hermite-Hadamard type inequalities; for refinements, counterparts, generalizations, for some related papers see [2, 3, 4, 5, 9, 10, 11, 12, 13, 14, 16].

In [15], Numan and İşcan gave the following definition and Hermite-Hadamard integral inequality:

Definition 1.4 ([15])**.** A non-negative function $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ is called exponential type P-function if for every $r, s \in I$ and $\theta \in [0,1]$,

$$\Psi(\theta r + (1-\theta)s) \leq \left(e^{\theta} + e^{1-\theta} - 2\right)\left[\Psi(r) + \Psi(s)\right].$$

We will denote by $ETP(I)$ the class of all exponential type P-functions on interval I. We note that, every exponential type P-function is a h-convex function with the function $h(\theta) = e^{\theta} + e^{1-\theta} - 2$. Also, every exponential type convex function is also a exponential type P-function, every P-function is also a exponential type P-function and every nonnegative convex function is also an exponential type P-function.

Theorem 1.3. *Let $\Psi : [r,s] \to \mathbb{R}$ be a exponential type P-function. If $r < s$ and $\Psi \in L[r,s]$, then the following Hermite-Hadamard type inequalities hold:*

$$\frac{1}{4\left[\sqrt{e}-1\right]}\Psi\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_r^s \Psi(u)du \leq (2e-4)\left[\Psi(r) + \Psi(s)\right].$$

Theorem 1.4 (Hölder-İşcan integral inequality [8])**.** *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable* *functions on* $[a,b]$ *then*

$$\int_a^b |f(x)g(x)|\,dx \leq \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}$$

2. Some new integral inequalities for exponential type P-functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value is exponential type P-function and then we will compare the results obtained with both Hölder, Hölder-İşcan integral inequalities and prove that the Hölder-İşcan integral inequality gives a better approximation than the Hölder integral inequality. In this section, we will denote by $L[r,s]$ the space of (Lebesgue) integrable functions on $[r,s]$.İşcan [7] used the following lemma:

Lemma 2.1 ([7])**.** *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°*, such that* $f' \in L[a,b]$*, where* $a,b \in I$ *with* $a<b$ *and* $\theta,\lambda \in [0,1]$*. Then the following equality holds:*

$$(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f\left((1-\lambda)a+\lambda b\right)-\frac{1}{b-a}\int_a^b f(x)dx$$
$$=(b-a)\left[-\lambda^2\int_0^1 (t-\theta)f'\left(ta+(1-t)\left[(1-\lambda)a+\lambda b\right]\right)dt\right.$$
$$\left.+(1-\lambda)^2\int_0^1 (t-\theta)f'\left(tb+(1-t)\left[(1-\lambda)a+\lambda b\right]\right)dt\right].$$

Theorem 2.2. *Let* $f : I \subseteq [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I°*, such that* $f' \in L[a,b]$*, where* $a,b \in I^{\circ}$ *with* $a<b$ *and* $\lambda,\theta \in [0,1]$*. If* $|f'|$ *is exponential type* P*-function on interval* $[a,b]$*, then the following inequality holds*

$$\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f\left((1-\lambda)a+\lambda b\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq 2(b-a)\left(2e^{\theta}+2e^{1-\theta}-2\theta^2+2\theta-e-2\right)$$
$$\times\left[\lambda^2 A\left(|f'(a)|,|f'(A_\lambda)|\right)+(1-\lambda)^2 A\left(|f'(b)|,|f'(A_\lambda)|\right)\right], \tag{1}$$

where $A_\lambda = A_\lambda(a,b) = (1-\lambda)a+\lambda b$*, and* $A(u,v) = A_{1/2}(u,v) = \frac{u+v}{2}$ *is the arithmetic mean*

Proof. Using Lemma 2.1[7]theorem.2.1 and the following inequalities

$$\begin{aligned} |f'(ta+(1-t)c_\lambda)| &\leq \left(e^t+e^{1-t}-2\right)\left[|f'(a)|+|f'(A_\lambda)|\right] \\ |f'(tb+(1-t)c_\lambda)| &\leq \left(e^t+e^{1-t}-2\right)\left[|f'(b)|+|f'(A_\lambda)|\right], \end{aligned}$$

we get

$$\begin{aligned}
&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f\left((1-\lambda)a+\lambda b\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq (b-a)\left[\begin{array}{c} \lambda^2\int_0^1|t-\theta|\left|f\right|'(ta+(1-t)A_\lambda)|\,dt \\ +(1-\lambda)^2\int_0^1|t-\theta|\left|f'(tb+(1-t)A_\lambda)\right|dt\end{array}\right] \\
&\leq (b-a)\left[\lambda^2\int_0^1|t-\theta|\left(e^t+e^{1-t}-2\right)\left[|f'(a)|+|f'(A_\lambda)|\right]dt\right. \\
&\quad\left.+(1-\lambda)^2\int_0^1|t-\theta|\left(e^t+e^{1-t}-2\right)\left[|f'(b)|+|f'(A_\lambda)|\right]dt\right] \\
&= (b-a)\left[\lambda^2\left[|f'(a)|+|f'(A_\lambda)|\right]\int_0^1|t-\theta|\left(e^t+e^{1-t}-2\right)dt\right. \\
&\quad\left.+(1-\lambda)^2\left[|f'(b)|+|f'(A_\lambda)|\right]\int_0^1|t-\theta|\left(e^t+e^{1-t}-2\right)dt\right] \\
&= (b-a)\left[\lambda^2\left[|f'(a)|+|f'(A_\lambda)|\right]\left(2e^\theta+2e^{1-\theta}-2\theta^2+2\theta-e-2\right)\right. \\
&\quad\left.+(1-\lambda)^2\left[|f'(b)|+|f'(A_\lambda)|\right]\left(2e^\theta+2e^{1-\theta}-2\theta^2+2\theta-e-2\right)\right] \\
&= 2(b-a)\lambda^2A\left(|f'(a)|,|f'(A_\lambda)|\right)\left(2e^\theta+2e^{1-\theta}-2\theta^2+2\theta-e-2\right) \\
&\quad+2(b-a)\left(1-\lambda\right)^2A\left(|f'(b)|,|f'(A_\lambda)|\right)\left(2e^\theta+2e^{1-\theta}-2\theta^2+2\theta-e-2\right) \\
&= 2(b-a)\left(2e^\theta+2e^{1-\theta}-2\theta^2+2\theta-e-2\right) \\
&\quad\times\left[\lambda^2A\left(|f'(a)|,|f'(A_\lambda)|\right)+(1-\lambda)^2A\left(|f'(b)|,|f'(A_\lambda)|\right)\right]
\end{aligned}$$

where

$$\int_0^1|t-\theta|\left(e^t+e^{1-t}-2\right)dt=2e^\theta+2e^{1-\theta}-2\theta^2+2\theta-e-2$$

This completes the proof of the theorem. □

Theorem 2.3. *Let $f: I\subseteq[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I°, such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$ and $\lambda,\theta\in[0,1]$. If $|f'|^q, q>1$ is exponential type P-function on interval $[a,b]$, then the following inequality holds*

$$\begin{aligned}
&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f\left((1-\lambda)a+\lambda b\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq (b-a)\left(2e-4\right)^{\frac{1}{q}}\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}} \\
&\quad\times\left[\lambda^2\left[|f'(a)|^q+|f'(A_\lambda)|^q\right]^{\frac{1}{q}}+(1-\lambda)^2\left[|f'(b)|^q+|f'(A_\lambda)|^q\right]^{\frac{1}{q}}\right], \qquad (2)
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $A_\lambda=(1-\lambda)a+\lambda b$.

Proof. Using Lemma 2.1[7]theorem.2.1, well known Hölder's integral inequality and the following inequalities

$$\begin{aligned} \left|f'\left(ta+(1-t)c_{\lambda}\right)\right|^{q} &\le \left(e^{t}+e^{1-t}-2\right)\left[|f'(a)|^{q}+|f'(A_{\lambda})|^{q}\right] \\ \left|f'\left(tb+(1-t)c_{\lambda}\right)\right|^{q} &\le \left(e^{t}+e^{1-t}-2\right)\left[|f'(b)|^{q}+|f'(A_{\lambda})|^{q}\right] \end{aligned}$$

which is the property of the exponential type P-function of $|f'|^q$, we get

$$\begin{aligned} &\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f\left((1-\lambda)a+\lambda b\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \le\ & (b-a)\int_0^1 |t-\theta|\,\lambda^2\left|f'\left(ta+(1-t)A_\lambda\right)\right|dt \\ &+(b-a)\int_0^1 |t-\theta|\,(1-\lambda)^2\left|f'\left(tb+(1-t)A_\lambda\right)\right|dt \\ \le\ & (b-a)\left(\int_0^1 |t-\theta|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \lambda^{2q}\left|f'\left(ta+(1-t)A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}} \\ &+(b-a)\,(1-\lambda)^2\left(\int_0^1 |t-\theta|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-\lambda)^{2q}\left|f'\left(tb+(1-t)A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}} \\ \le\ & (b-a)\left(\int_0^1 |t-\theta|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(e^t+e^{1-t}-2\right)\lambda^{2q}\left[\left|f'(a)\right|^q+\left|f'(A_\lambda)\right|^q\right]dt\right)^{\frac{1}{q}} \\ &+(b-a)\left(\int_0^1 |t-\theta|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(e^t+e^{1-t}-2\right)(1-\lambda)^{2q}\left[\left|f'(b)\right|^q+\left|f'(A_\lambda)\right|^q\right]dt\right)^{\frac{1}{q}} \\ =\ & (b-a)\,(2e-4)^{\frac{1}{q}}\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}} \\ &\times\left[\left(\lambda^{2q}\left[\left|f'(a)\right|^q+\left|f'(A_\lambda)\right|^q\right]\right)^{\frac{1}{q}}+\left((1-\lambda)^{2q}\left[\left|f'(b)\right|^q+\left|f'(A_\lambda)\right|^q\right]\right)^{\frac{1}{q}}\right], \end{aligned}$$

where

$$\begin{aligned} \int_0^1 |t-\theta|^p\,dt &= \frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}, \\ \int_0^1 \left(e^t+e^{1-t}-2\right)dt &= 2e-4. \end{aligned}$$

This completes the proof of the theorem. □

Theorem 2.4. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I°, such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a < b$ and $\lambda,\theta \in [0,1]$. If $|f'|^q$ is exponential type P-function on interval $[a,b]$ and $q \ge 1$, then the following inequality holds*

$$\begin{aligned} &\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f\left((1-\lambda)a+\lambda b\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \le\ & 2^{\frac{1}{q}}(b-a)\left(\theta^2-\theta+\frac{1}{2}\right)^{1-\frac{1}{q}}\left[2e^{\theta}+2e^{1-\theta}-2\theta^2+2\theta-e-4\right]^{\frac{1}{q}} \\ &\times\left[\lambda^2 A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(A_\lambda)\right|^q\right)+(1-\lambda)^2 A^{\frac{1}{q}}\left(\left|f'(b)\right|^q,\left|f'(A_\lambda)\right|^q\right)\right], \qquad (3) \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $A_\lambda = A_\lambda(a,b) = (1-\lambda)a+\lambda b$, and $A(u,v) = A_{1/2}(u,v) = \frac{u+v}{2}$ is the arithmetic mean.

Proof. From Lemma 2.1[7]theorem.2.1, well known power-mean integral inequality and the property of exponential type P-function of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f\left((1-\lambda)a+\lambda b\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\;& (b-a)\lambda^2\int_0^1 |t-\theta|\left|f'\left(ta+(1-t)A_\lambda\right)\right|dt \\
&+(b-a)\left(1-\lambda\right)^2\int_0^1 |t-\theta|\left|f'\left(tb+(1-t)A_\lambda\right)\right|dt \\
\leq\;& (b-a)\lambda^2\left(\int_0^1 |t-\theta|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t-\theta|\left|f'\left(ta+(1-t)A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+(b-a)\left(1-\lambda\right)^2\left(\int_0^1 |t-\theta|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t-\theta|\left|f'\left(tb+(1-t)A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\;& (b-a)\lambda^2\left(\int_0^1 |t-\theta|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t-\theta|\left(e^t+e^{1-t}-2\right)\left[\left|f'(a)\right|^q+\left|f'(A_\lambda)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&+(b-a)\left(1-\lambda\right)^2\left(\int_0^1 |t-\theta|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t-\theta|\left(e^t+e^{1-t}-2\right)\left[\left|f'(b)\right|^q+\left|f'(A_\lambda)\right|^q\right]dt\right)^{\frac{1}{q}} \\
=\;& (b-a)\lambda^2\left(\theta^2-\theta+\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left[\left|f'(a)\right|^q+\left|f'(A_\lambda)\right|^q\right]\left[2e^\theta+2e^{1-\theta}-2\theta^2+2\theta-e-4\right]\right)^{\frac{1}{q}} \\
&+(b-a)\left(1-\lambda\right)^2\left(\theta^2-\theta+\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left[\left|f'(b)\right|^q+\left|f'(A_\lambda)\right|^q\right]\left[2e^\theta+2e^{1-\theta}-2\theta^2+2\theta-e-4\right]\right)^{\frac{1}{q}} \\
=\;& 2^{\frac{1}{q}}(b-a)\left(\theta^2-\theta+\frac{1}{2}\right)^{1-\frac{1}{q}}\left[2e^\theta+2e^{1-\theta}-2\theta^2+2\theta-e-4\right]^{\frac{1}{q}} \\
&\times\left[\lambda^2A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(A_\lambda)\right|^q\right)+\left(1-\lambda\right)^2A^{\frac{1}{q}}\left(\left|f'(b)\right|^q,\left|f'(A_\lambda)\right|^q\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 |t-\theta|\,dt &= \theta^2-\theta+\frac{1}{2} \\
\int_0^1 |t-\theta|\left(e^t+e^{1-t}-2\right)dt &= 2e^\theta+2e^{1-\theta}-2\theta^2+2\theta-e-4
\end{aligned}$$

This completes the proof of the theorem. □

Corollary 2.5. *Under the assumption of Theorem 2.4theorem.2.4, If we take $q=1$ in the inequality (3equation.2.3), then we get the following inequality:*

$$\begin{aligned}
&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f\left((1-\lambda)a+\lambda b\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\;& 2(b-a)\left[2e^\theta+2e^{1-\theta}-2\theta^2+2\theta-e-4\right] \\
&\times\left[\lambda^2A\left(\left|f'(a)\right|,\left|f'(A_\lambda)\right|\right)+\left(1-\lambda\right)^2A\left(\left|f'(b)\right|,\left|f'(A_\lambda)\right|\right)\right].
\end{aligned}$$

This inequality coincides with the inequality (1equation.2.1).

Corollary 2.6. *Under the assumption of Theorem 2.4theorem.2.4, If we take $\theta = 1$ in the inequality (3equation.2.3), then we get the following inequality:*

$$\left| f\left((1-\lambda)a+\lambda b\right) - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leq (b-a)e^{\frac{1}{q}}\left(\frac{1}{2}\right)^{1-\frac{2}{q}} \left[\lambda^2 A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(A_\lambda)|^q\right) + (1-\lambda)^2 A^{\frac{1}{q}}\left(|f'(b)|^q, |f'(A_\lambda)|^q\right)\right].$$

Corollary 2.7. *Under the assumption of Theorem 2.4theorem.2.4 with $\theta = 1$, If we take $|f'(x)| \leq M, x \in [a,b]$ then we get the following Ostrowski type integral inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq M(b-a)e^{\frac{1}{q}}\left(\frac{1}{2}\right)^{1-\frac{2}{q}} \left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]$$

for each $x \in [a,b]$.

Proof. There exist $\lambda_x \in [0,1]$ such that $x = (1-\lambda_x)a + \lambda_x b$ for each $x \in [a,b]$. So, we take $\lambda_x = \frac{x-a}{b-a}$ and $1-\lambda_x = \frac{b-x}{b-a}$. Therefore, for each $x \in [a,b]$ we obtain the required inequality from the inequality (3equation.2.3). □

Corollary 2.8. *Under the assumption of Theorem 2.4theorem.2.4 with $\theta = 1$, then we have following generalized trapezoid type integral inequality*

$$\left| \lambda f(a) + (1-\lambda)f(b) - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leq e^{\frac{1}{q}}(b-a)\left(\frac{1}{2}\right)^{1-\frac{2}{q}} \left[\lambda^2 A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(A_\lambda)|^q\right) + (1-\lambda)^2 A^{\frac{1}{q}}\left(|f'(b)|^q, |f'(A_\lambda)|^q\right)\right],$$

where $c = (1-\lambda)a + \lambda b$.

Corollary 2.9. *Under the assumption of Theorem 2.4theorem.2.4 with $\lambda = \frac{1}{2}$ and $\theta = \frac{2}{3}$, then we have the following Simpson type integral inequality*

$$\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leq 2^{\frac{1}{q}}(b-a)\left(\frac{7}{18}\right)^{1-\frac{1}{q}} \left[2e^{\frac{2}{3}} + 2e^{\frac{1}{3}} - e - \frac{8}{9}\right]^{\frac{1}{q}}$$
$$\times \left[\frac{1}{4}A^{\frac{1}{q}}\left(|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right) + \frac{1}{4}A^{\frac{1}{q}}\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right].$$

Corollary 2.10. *Under the assumption of Theorem 2.4theorem.2.4 with $\lambda = \frac{1}{2}$ and $\theta = 1$, then we have the following midpoint type integral inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leq e^{\frac{1}{q}}(b-a)\left(\frac{1}{2}\right)^{2-\frac{2}{q}} \left[A^{\frac{1}{q}}\left(|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right) + A^{\frac{1}{q}}\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right].$$

Corollary 2.11. *Under the assumption of Theorem 2.4theorem.2.4 with* $\lambda = \frac{1}{2}$ *and* $\theta = 0$*, then we have the following trapezoid type integral inequality*

$$\left|\lambda f(a) + (1-\lambda)f(b) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq e^{\frac{1}{q}}(b-a)\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left[A^{\frac{1}{q}}\left(|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right) + A^{\frac{1}{q}}\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right].$$

Theorem 2.12. *Let* $f : I \subseteq [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I°*, such that* $f' \in L[a,b]$*, where* $a, b \in I^\circ$ *with* $a < b$ *and* $\lambda, \theta \in [0,1]$*. If* $|f'|^q, q > 1$ *is exponential type* P*-function on interval* $[a,b]$*, then the following inequality holds*

$$\left|(1-\theta)(\lambda f(a) + (1-\lambda)f(b)) + \theta f((1-\lambda)a + \lambda b) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq 2(b-a)(e-2)^{\frac{1}{q}}\left[\lambda^2\left[|f'(a)|^q + |f'(A_\lambda)|^q\right]^{\frac{1}{q}} + (1-\lambda)^2\left[|f'(b)|^q + |f'(A_\lambda)|^q\right]^{\frac{1}{q}}\right]$$
$$\times\left[\left(\frac{(p-\theta+2)\theta^{p+1} + (1-\theta)^{p+2}}{(p+1)(p+2)}\right)^{\frac{1}{p}} + \left(\frac{\theta^{p+2} + (p+\theta+1)(1-\theta)^{p+1}}{(p+1)(p+2)}\right)^{\frac{1}{p}}\right] \quad (4)$$

where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $A_\lambda = (1-\lambda)a + \lambda b$*.*

Proof. Using Lemma 2.1[7]theorem.2.1, Hölder-İşcan integral inequality and the following inequalities

$$\begin{aligned} |f'(ta + (1-t)c)|^q &\leq \left(e^t + e^{1-t} - 2\right)\left[|f'(a)|^q + |f'(A_\lambda)|^q\right] \\ |f'(tb + (1-t)c)|^q &\leq \left(e^t + e^{1-t} - 2\right)\left[|f'(b)|^q + |f'(A_\lambda)|^q\right] \end{aligned}$$

which is the property of the exponential type P-function of $|f'|^q$, we get

$$\left|(1-\theta)(\lambda f(a) + (1-\lambda)f(b)) + \theta f((1-\lambda)a + \lambda b) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq (b-a)\int_0^1 |t-\theta|\,\lambda^2\left|f'(ta + (1-t)A_\lambda)\right| dt$$
$$+ (b-a)\int_0^1 |t-\theta|(1-\lambda)^2\left|f'(tb + (1-t)A_\lambda)\right| dt$$
$$\leq (b-a)\left\{\left(\int_0^1 (1-t)|t-\theta|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\lambda^{2q}\left|f'(ta + (1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^1 t|t-\theta|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\lambda^{2q}\left|f'(ta + (1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}}\right\}$$
$$+ (b-a)\left\{\left(\int_0^1 (1-t)|t-\theta|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)(1-\lambda)^{2q}\left|f'(tb + (1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^1 t|t-\theta|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t(1-\lambda)^{2q}\left|f'(tb + (1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq (b-a)\left\{\left(\int_0^1 (1-t)\,|t-\theta|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left(e^t+e^{1-t}-2\right)\lambda^{2q}\left[\left|f'(a)\right|^q+\left|f'(A_\lambda)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 t\,|t-\theta|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left(e^t+e^{1-t}-2\right)\lambda^{2q}\left[\left|f'(a)\right|^q+\left|f'(A_\lambda)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}$$

$$+(b-a)\left\{\left(\int_0^1 (1-t)\,|t-\theta|^p\,dt\right)^{\frac{1}{p}}\right.$$

$$\times\left(\int_0^1 (1-t)\,(1-\lambda)^{2q}\left(e^t+e^{1-t}-2\right)\left[\left|f'(b)\right|^q+\left|f'(A_\lambda)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1 t\,|t-\theta|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left(e^t+e^{1-t}-2\right)(1-\lambda)^{2q}\left[\left|f'(b)\right|^q+\left|f'(A_\lambda)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}$$

$$=\ (b-a)\left\{\left(\frac{(p-\theta+2)\,\theta^{p+1}+(1-\theta)^{p+2}}{(p+1)\,(p+2)}\right)^{\frac{1}{p}}\left((e-2)\lambda^{2q}\left[\left|f'(a)\right|^q+\left|f'(A_\lambda)\right|^q\right]\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\theta^{p+2}+(p+\theta+1)\,(1-\theta)^{p+1}}{(p+1)\,(p+2)}\right)^{\frac{1}{p}}\left((e-2)\lambda^{2q}\left[\left|f'(a)\right|^q+\left|f'(A_\lambda)\right|^q\right]\right)^{\frac{1}{q}}\right\}$$

$$+(b-a)\left\{\left(\frac{(p-\theta+2)\,\theta^{p+1}+(1-\theta)^{p+2}}{(p+1)\,(p+2)}\right)^{\frac{1}{p}}\left((e-2)\,(1-\lambda)^{2q}\left[\left|f'(b)\right|^q+\left|f'(A_\lambda)\right|^q\right]\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\theta^{p+2}+(p+\theta+1)\,(1-\theta)^{p+1}}{(p+1)\,(p+2)}\right)^{\frac{1}{p}}\left((e-2)\,(1-\lambda)^{2q}\left[\left|f'(b)\right|^q+\left|f'(A_\lambda)\right|^q\right]\right)^{\frac{1}{q}}\right\}$$

$$=\ (b-a)(e-2)^{\frac{1}{q}}\left[\left(\frac{(p-\theta+2)\,\theta^{p+1}+(1-\theta)^{p+2}}{(p+1)\,(p+2)}\right)^{\frac{1}{p}}+\left(\frac{\theta^{p+2}+(p+\theta+1)\,(1-\theta)^{p+1}}{(p+1)\,(p+2)}\right)^{\frac{1}{p}}\right]$$

$$\times\left[\lambda^2\left[\left|f'(a)\right|^q+\left|f'(A_\lambda)\right|^q\right]^{\frac{1}{q}}+(1-\lambda)^2\left[\left|f'(b)\right|^q+\left|f'(A_\lambda)\right|^q\right]^{\frac{1}{q}}\right]$$

where

$$\begin{aligned}\int_0^1 (1-t)\,|t-\theta|^p\,dt &= \frac{(p-\theta+2)\,\theta^{p+1}+(1-\theta)^{p+2}}{(p+1)\,(p+2)}\\ \int_0^1 t\,|t-\theta|^p\,dt &= \frac{\theta^{p+2}+(p+\theta+1)\,(1-\theta)^{p+1}}{(p+1)\,(p+2)},\\ \int_0^1 (1-t)\left(e^t+e^{1-t}-2\right)dt &= \int_0^1 t\left(e^t+e^{1-t}-2\right)dt=e-2.\end{aligned}$$

This completes the proof of the theorem. □

Remark 2.1. The inequality (4equation.2.4) gives better results than the inequality (2equation.2.2). Let us show that

$$\left(\frac{(p-\theta+2)\,\theta^{p+1}+(1-\theta)^{p+2}}{(p+1)\,(p+2)}\right)^{\frac{1}{p}}+\left(\frac{\theta^{p+2}+(p+\theta+1)\,(1-\theta)^{p+1}}{(p+1)\,(p+2)}\right)^{\frac{1}{p}}$$

$$\leq 2^{\frac{1}{q}}\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}}.$$

Using the well known classic inequalities $u^{\lambda}+v^{\lambda}\leq 2^{1-\lambda}(u+v)^{\lambda}, u,v\in(0,\infty), \lambda\in(0,1]$, by sample calculation we get

$$\left(\frac{(p-\theta+2)\,\theta^{p+1}+(1-\theta)^{p+2}}{(p+1)\,(p+2)}\right)^{\frac{1}{p}}+\left(\frac{\theta^{p+2}+(p+\theta+1)\,(1-\theta)^{p+1}}{(p+1)\,(p+2)}\right)^{\frac{1}{p}}$$

$$\leq 2^{1-\frac{1}{p}}\left(\frac{(p-\theta+2)\,\theta^{p+1}+(1-\theta)^{p+2}+\theta^{p+2}+(p+\theta+1)\,(1-\theta)^{p+1}}{(p+1)\,(p+2)}\right)^{\frac{1}{p}}$$

$$=2^{\frac{1}{q}}\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}}$$

which is the required.

3. Applications for special means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers r,s with $s>r$:

1. The arithmetic mean

$$A:=A(r,s)=\frac{r+s}{2},\quad r,s\geq 0.$$

2. The weighted arithmetic mean

$$A_{\alpha}(r,s):=(1-\alpha)r+\alpha s,\ r,s\geq 0,\alpha\in[0,1].$$

3. The geometric mean

$$G:=G(r,s)=\sqrt{rs},\quad r,s\geq 0.$$

4. The weighted geometric mean

$$G_{\alpha}:=G_{\alpha}(r,s)=r^{1-\alpha}s^{\alpha},\quad r,s>0,\alpha\in[0,1].$$

3. The harmonic mean

$$H:=H(r,s)=\frac{2rs}{r+s},\quad r,s>0.$$

4. The logarithmic mean

$$L:=L(r,s)=\begin{cases}\frac{s-r}{\ln s-\ln r}, & r\neq s\\ r, & r=s\end{cases};\quad r,s>0.$$

5. The p-logarithmic mean

$$L_{p}:=L_{p}(r,s)=\begin{cases}\left(\frac{s^{p+1}-r^{p+1}}{(p+1)(s-r)}\right)^{\frac{1}{p}}, & r\neq s, p\in\mathbb{R}\backslash\{-1,0\}\\ r, & r=s\end{cases};\quad r,s>0.$$

6.The identric mean

$$I:=I(r,s)=\frac{1}{e}\left(\frac{s^{s}}{r^{r}}\right)^{\frac{1}{s-r}},\quad r,s>0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H\leq G\leq L\leq I\leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 3.1. *Let $\lambda, \theta \in [0,1]$, $r, s \in [0, \infty)$ with $r < s$ and $n \geq 2$. Then, the following inequalities are obtained:*

$$\left|A_\theta\left(A_{1-\lambda}(r^n, s^n), A_\lambda^n(r,s)\right) - L_n^n(r,s)\right| \leq 4n(s-r)$$
$$\left(e^\theta + e^{1-\theta} - \theta^2 + \theta - \frac{e}{2} - 1\right)\left[\lambda^2 A\left(r^{n-1}, A_\lambda^{n-1}\right) + (1-\lambda)^2 A\left(s^{n-1}, A_\lambda^{n-1}\right)\right].$$

Proof. The assertion follows from the inequalities (1equation.2.1) for the function

$$f(x) = x^n, \quad x \in [0, \infty).$$

□

Proposition 3.2. *Let $\lambda, \theta \in [0,1]$, $r, s \in (0, \infty)$ with $r < s$. Then, the following inequalities are obtained:*

$$\left|A_\theta\left(A_{1-\lambda}(r^{-1}, s^{-1}), A_\lambda^{-1}(r,s)\right) - L^{-1}(r,s)\right|$$
$$\leq 4(s-r)\left(e^\theta + e^{1-\theta} - \theta^2 + \theta - \frac{e}{2} - 1\right)\left[\lambda^2 H^{-1}\left(r^2, A_\lambda^2\right) + (1-\lambda)^2 H^{-1}\left(s^2, A_\lambda^2\right)\right].$$

Proof. The assertion follows from the inequalities (1equation.2.1) for the function

$$f(x) = x^{-1}, \quad x \in (0, \infty).$$

□

Proposition 3.3. *Let $\lambda, \theta \in [0,1]$, $r, s > 0$ with $r < s$. Then, the following inequalities are obtained:*

$$\left|\ln\left(\frac{G_{1-\lambda}^\theta A_\lambda^\theta}{I}\right)\right|$$
$$\leq 4(s-r)\left(e^\theta + e^{1-\theta} - \theta^2 + \theta - \frac{e}{2} - 1\right)\left[\lambda^2 H^{-1}\left(r, A_\lambda\right) + (1-\lambda)^2 H^{-1}\left(s, A_\lambda\right)\right].$$

Proof. The assertion follows from the inequalities (1equation.2.1) for the function

$$f(x) = \ln x, \quad x > 0.$$

□

4. Conclusion

In this paper, with the help of an identity, some new Hermite-Hadamard type integral inequalities are obtained using the Hölder and power-mean integral inequalities for functions whose first derivative in absolute value is an exponential type P-function. The authors can obtain new types of integral inequalities for exponential type P-functions using different identities. Then, the authors compare the obtained results with both Hölder and Hölder-İşcan integral inequalities and show that Hölder-İşcan integral inequality provides a better approximation than Hölder inequality.

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(Mahir Kadakal) Department of Customs Management, Faculty of Applied Sciences, Bayburt University, 69000, Bayburt, TÜRKİYE
E-mail address: mahirkadakal@gmail.com

(İmdat İşcan) Department of Mathematics, Faculty of Arts and Sciences, Giresun University, 28200, Giresun, TÜRKİYE
E-mail address: imdati@yahoo.com

(Huriye Kadakal) Bayburt University, Faculty of Education, Department of Primary Education, Baberti Campus, 69000 Bayburt, TÜRKİYE
E-mail address: huriyekadakal@hotmail.com