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Sparse Filtering Under Norm-Bounded Exogenous Disturbances Using Observers

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***Abstract*—The paper considers the sparse filtering problem under arbitrary norm-bounded exogenous disturbances. We propose a simple and universal observer-based approach to its solution, based on the LMI technique and the method of invariant ellipsoids; it allows the use of a reduced number of system outputs. From a technical point of view of application, we reduce the original problem to semi-definite programming, which is easily solved numerically. The proposed simple approach is easy to implement and can be equally extended to systems in continuous and discrete time.**

***Index Terms*—linear system, filtering, sparsity, exogenous disturbances, linear matrix inequalities, invariant ellipsoids**

I. Introduction

In the modern literature, the term *sparse filtering* is mainly assigned to such areas as machine learning, pattern recognition, signal and image processing; see, for example, [1]–[5]. In many situations, the classical assumption that the disturbances are random is not justified. Frequently, it is known that the disturbances are bounded only. In this case, *guaranteed* estimates of states can be constructed. This approach was proposed in the works of Witzenhausen, Bertsekas and Rhodes, Schweppe [6]. At about the same time, similar problems were developed by such researchers as Kurzhansky [7]. A significant contribution to this circle of research was made by Chernousko [8].

In the papers [9], [10], the problem of filtering with nonrandom bounded exogenous disturbances was considered, but only for stationary problem statements. Moreover, a state estimate was sought such that its residual is guaranteed to be enclosed in a single so-called *invariant ellipsoid*. The filter was also sought as the linear stationary filter. In this class, the problem turned out to be completely solvable, so that it was possible to construct an optimal filter and state estimate. From a technical point of view, the LMI apparatus [11] was used in [9], [10]. The LMI technique has proven itself well in the analysis and design (see, e.g. [12], [13]), but has not been widely used in filtering problems. A systematic presentation of this technique is given in the monograph [14].

On the other hand, the *sparsity* ideas are widely used in the various fields (e.g., see [15], [16]), but not in control. We mention publications [17], [18] devoted to the sparse feedback design. In [19], a new approach to constructing a sparse feedback was proposed, which is associated with minimizing nonzero rows or nonzero columns of the matrix. Such matrices are called row-sparse and column-sparse, respectively.

This method is distinguished by simplicity: the initial problems are reduced to solving low-dimensional convex optimization problem, and for its numerical solution one can use standard tools, such as MATLAB-based package YALMIP [20] and `cvx` [21], [22]. We mention the versatility of the proposed approach as continuous- and discrete-time cases are considered uniformly, and it is applicable to both linear state and output feedback design. At last, we stress its extendability to the various robust formulations, as well as to the optimal control problems, etc.

This paper is a natural continuation of [9], [10], and [19]. It proposes an approach to the solution of the *sparse filtering problem*, that is, filtering using a reduced number of outputs in the presence of arbitrary bounded exogenous disturbances.

Throughout the following, $\|\cdot\|$ is the Euclidean norm of a vector and the spectral norm of a matrix, $^{\mathrm{T}}$ is the transposition symbol, $\mathbb{S}^{n\times n}$ is the class of symmetric real $n\times n$ matrices, I is the identity matrix of appropriate dimension, and all matrix inequalities are understood in the sense of the sign definiteness of the matrices.

The present paper is the revised and expanded version of talk [23] presented at the IEEE 25th International Conference on System Theory, Control and Computing (ICSTCC 2021). In particular, a number of additions have been made to the text of the article, and the list of references has been significantly expanded and updated.

II. Sparse Control

Let us recall the main ideas of the above mentioned approach to the construction of sparse control. Let $\Omega \in \mathbb{R}^{n\times p}$;

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we introduce into consideration the following matrix norms:

$$\|\Omega\|_{r_1} = \sum_{i=1}^{n} \max_{1\le j\le p} |\omega_{ij}|, \quad \|\Omega\|_{c_1} = \sum_{j=1}^{p} \max_{1\le i\le n} |\omega_{ij}|.$$

The following result stated in [19].

Theorem 1: If the problem

$$\min \|\Omega\|_{r_1} \quad \text{s.t.} \quad A\Omega = B,$$

where $A \in \mathbb{R}^{m\times n}$, $m < n$, $B \in \mathbb{R}^{m\times p}$, $\Omega \in \mathbb{R}^{n\times p}$, is feasible, then there exists a solution with no more than m nonzero rows.

A similar result can be stated for the c_1-norm.

The approach developed in [19] allows the regular design of sparse controls in various statements. In particular, consider the linear system in continuous time

$$\dot{x} = Ax + Bu \tag{1}$$

with state $x \in \mathbb{R}^n$ and control input $u \in \mathbb{R}^m$, i.e. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$; the pair (A, B) is controllable.

The goal is to construct a *sparse* stabilizing control

$$u = \Phi x,$$

in the sense of zero components of the control vector. So, we are interesting in finding the *row-sparse* stabilizing controller $\Phi \in \mathbb{R}^{m\times n}$, i.e. having zero rows.

The technique required to obtain this result will be used in the sequel. It is well known, the matrix $A + B\Phi$ is stable iff there exists matrix $\Omega \succ 0$ such that

$$(A + B\Phi)^{\mathrm{T}}\Omega + \Omega(A + B\Phi) \prec 0.$$

Pre-multiplying and post-multiplying this inequality by the matrix $\Xi = \Omega^{-1}$ we obtain the inequality

$$A\Xi + \Xi A^{\mathrm{T}} + B\Phi\Xi + \Xi\Phi^{\mathrm{T}}B^{\mathrm{T}} \prec 0.$$

Finally, introducing a new matrix variable $\Psi = \Phi\Xi$, we obtain the LMI

$$A\Xi + \Xi A^{\mathrm{T}} + B\Psi + \Psi^{\mathrm{T}}B^{\mathrm{T}} \prec 0, \quad \Xi \succ 0, \tag{2}$$

in the matrix variables $\Xi \in \mathbb{S}^{n\times n}$ and $\Psi \in \mathbb{R}^{m\times n}$. Therefore, any stabilizing gain matrix for system (1) is presented by the expression

$$\widehat{\Phi} = \widehat{\Psi}\widehat{\Xi}^{-1}$$

where the matrices $\widehat{\Xi}$ and $\widehat{\Psi}$ satisfy (2).

It is clear, right multiplication preserves the row-sparse structure of the matrix. Therefore, if the solution $\widehat{\Psi}$ of the linear matrix inequality (2) is row-sparse, then the gain matrix $\widehat{\Phi}$ is row-sparse. Hence, the row sparsity of the matrix Ψ can be achieved by minimizing its r_1-norm. Thus, the following statement holds.

Statement 1 ([19]): The solution $\widehat{\Xi}$ and $\widehat{\Psi}$ of the convex optimization problem

$$\min \|\Psi\|_{r_1} \quad \text{s.t.} \quad A\Xi + \Xi A^{\mathrm{T}} + B\Psi + \Psi^{\mathrm{T}}B^{\mathrm{T}} \prec 0, \quad \Xi \succ 0,$$

in the matrix variables $\Xi \in \mathbb{S}^{n\times n}$ and $\Psi \in \mathbb{R}^{m\times n}$, defines the row-sparse stabilizing gain matrix

$$\Phi_{\mathrm{sp}} = \widehat{\Psi}\widehat{\Xi}^{-1}$$

for system (1).

With Statement 1, we detect the stabilizing control inputs. These controls are determined by nonzero rows of the matrix Φ_{sp}. Evidently, we can not state that the resulting solution will be row-sparse, but it is expected by virtue of Theorem 1.

The author apply these ideas to the sparse filtering problem stated in the next section.

III. Continuous-Time Case

A. Filtering problem

Consider the dynamical system

$$\begin{aligned} \dot{x} &= Ax + B\nu, \quad x(0) = x_0, \\ y &= Cx + D\nu, \end{aligned} \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{l\times n}$, with state $x(t) \in \mathbb{R}^n$, observed output $y(t) \in \mathbb{R}^l$, and exogenous disturbances $\nu(t) \in \mathbb{R}^m$ satisfying the constraint

$$\|\nu(t)\| \le 1 \quad \text{for all } t \ge 0; \tag{4}$$

the pair (A, B) is controllable and the pair (A, C) is observable. Let the state x of system (3) be unavailable, and the information about the system is provided by its output y.

We construct a linear filter described by the differential equation

$$\dot{\widehat{x}} = A\widehat{x} + \mathcal{F}(y - C\widehat{x}), \quad \widehat{x}(0) = 0.$$

We emphasize that only the constant matrix $\mathcal{F} \in \mathbb{R}^{n\times l}$ is to be chosen.

The goal is to find the minimal (in the certain sense) invariant ellipsoid containing the residual

$$\rho(t) = x(t) - \widehat{x}(t).$$

The application of the ideology of invariant ellipsoids to control systems is described in [11], [14] in detail. Recall that the ellipsoid

$$\mathcal{V}_x = \left\{x \in \mathbb{R}^n: \quad x^{\mathrm{T}}\Xi^{-1}x \le 1\right\}, \quad \Xi \succ 0,$$

is called invariant for a dynamical system if the condition $x(0) \in \mathcal{V}_x$ yields $x(t) \in \mathcal{V}_x$ for all times $t \ge 0$. So, any trajectory of the system, starting from any point lying inside the ellipsoid $\mathcal{V}_x$, at any time instant will be in this ellipsoid for all admissible exogenous disturbances.

By virtue of the attractiveness property of an invariant ellipsoid, the filtering accuracy is asymptotic for large deviations, and the filtering accuracy is uniform in t for small deviations.

There are many invariant ellipsoids, the goal is to find the minimum one and, to minimize it over $\mathcal{F}$. It is convenient for us to assume that the *minimal* ellipsoid has the minimal trace of its matrix. In [9], the following result was stated.

Theorem 2: Let $\widehat{\Omega}$ and $\widehat{\Psi}$ be the solution of the optimization problem

$$\min \operatorname{tr} \Upsilon$$

under the constraints

$$\begin{pmatrix} A^{\mathrm{T}}\Omega + \Omega A - \Psi C - C^{\mathrm{T}}\Psi^{\mathrm{T}} + \mu\Omega & \Omega B - \Psi D \\ B_1^{\mathrm{T}}\Omega - B_2^{\mathrm{T}}\Psi^{\mathrm{T}} & -\mu I \end{pmatrix} \preceq 0,$$

$$\begin{pmatrix} \Upsilon & I \\ I & \Omega \end{pmatrix} \succeq 0, \quad \Omega \succ 0,$$

with the matrix variables $\Omega \in \mathbb{S}^{n\times n}$, $\Psi \in \mathbb{R}^{n\times l}$, $\Upsilon \in \mathbb{S}^{n\times n}$, and the scalar parameter $\mu > 0$.

Then the optimal filter matrix gives as

$$\widehat{\mathcal{F}} = \widehat{\Omega}^{-1}\widehat{\Psi},$$

and minimal invariant ellipsoid for the residual of (3) with $x_0 = 0$ defined by the matrix

$$\widehat{\Xi} = \widehat{\Omega}^{-1}.$$

B. Sparse filtering

We will seek a sparse solution of the filtering problem for system (3), (4). Note, that the filter matrix $\mathcal{F}$ has the form

$$\mathcal{F} = \Omega^{-1}\Psi.$$

Therefore, if the matrix Ψ be column-sparse, then the corresponding filter matrix $\mathcal{F}$ be column-sparse as well. In turn, the column sparsity of the matrix Ψ can be achieved by minimizing its c_1-norm.

Thus, we have the following algorithm, which involves the execution of three consecutive steps.

Algorithm 1:

Step 1. Solving the optimization problem

$$\min \operatorname{tr} \Upsilon \tag{5}$$

under the constraints

$$\begin{pmatrix} A^{\mathrm{T}}\Omega + \Omega A - \Psi C - C^{\mathrm{T}}\Psi^{\mathrm{T}} + \mu\Omega & \Omega B - \Psi D \\ B_1^{\mathrm{T}}\Omega - B_2^{\mathrm{T}}\Psi^{\mathrm{T}} & -\mu I \end{pmatrix} \preceq 0, \tag{6}$$

$$\begin{pmatrix} \Upsilon & I \\ I & \Omega \end{pmatrix} \succeq 0, \quad \Omega \succ 0, \tag{7}$$

in the matrix variables $\Omega \in \mathbb{S}^{n\times n}$, $\Psi \in \mathbb{R}^{n\times l}$, $\Upsilon \in \mathbb{S}^{n\times n}$, and the scalar parameter $\mu > 0$, we obtain the values Ω^*, Ψ^*, and Υ^* which define the matrix

$$\mathcal{F}^* = (\Omega^*)^{-1}\Psi^*$$

of the optimal filter, and the matrix

$$\Xi^* = (\Omega^*)^{-1}$$

of the minimal invariant ellipsoid for the residual, and the corresponding value

$$\mathcal{J}^* = \operatorname{tr} \Upsilon^*$$

of the cost function.

Step 2. Having the value $\mathcal{J}^*$, we implement the relaxation coefficient $\lambda > 1$ and consider c_1-optimization problem

$$\min \|\Psi\|_{c_1} \quad \text{s.t. (6), (7) and } \operatorname{tr} \Upsilon \preceq \lambda \mathcal{J}^* \tag{8}$$

in the matrix variables $\Omega \in \mathbb{S}^{n\times n}$, $\Psi \in \mathbb{R}^{n\times l}$, $\Upsilon \in \mathbb{S}^{n\times n}$, and the scalar parameter $\mu > 0$.

By virtue of the properties of the c_1-norm, one can expect the occurrence of zero columns in the solution $\widehat{\Psi}_0$ of this problem.

Step 3. We resolve the problem (5)–(7) with the additional constraint that the matrix variable Ψ has zero columns at the same places as the matrix $\widehat{\Psi}_0$. Its solution $\widehat{\Omega}$, $\widehat{\Psi}$ defines the column-sparse filter matrix

$$\widehat{\mathcal{F}} = \widehat{\Omega}^{-1}\widehat{\Psi}$$

and the matrix $\widehat{\Xi} = \widehat{\Omega}^{-1}$ of the corresponding invariant ellipsoid for the residual.

In section V, it will be shown by example that the proposed procedure leads to highly sparse matrices of the filter with small losses in terms of the cost criterion.

Remark 1: If we have a priori information about the initial condition $x(0) \in \mathcal{V}_0$ of the system, where

$$\mathcal{V}_0 = \{x \in \mathbb{R}^n : \quad x^{\mathrm{T}}\Xi_0^{-1}x \le 1\}.$$

Then, letting $\widehat{x}(0) = 0$, we can guarantee that $\rho(0) \in \mathcal{V}_0$. If we prescribe that

$$\mathcal{V}_0 \subset \mathcal{V},$$

then we can guarantee that $\rho(t) \in \mathcal{V}$ for all $t \ge 0$.

Accordingly, if we add the condition

$$\Omega \preceq \Xi_0^{-1}$$

to the constraints (6)–(7) in Algorithm 1, then we obtain not only asymptotic, but uniform estimate of the sparse filtering accuracy.

Remark 2: Often it is necessary to evaluate the quality of filtering not all coordinates of the state x, but only some of coordinates. Let us have the output

$$z = C_z x$$

and the goal is to make the residual of its estimate

$$\rho_z = z - \widehat{z} = C_z(x - \widehat{x})$$

as small as possible. The solution of this problem is achieved by replacing the first of the conditions (7) by

$$\begin{pmatrix} \Upsilon & C_z \\ C_z^{\mathrm{T}} & \Omega \end{pmatrix} \succeq 0.$$

IV. Discrete-Time Case

The analogous results can be established for the dynamical system

$$\begin{aligned} x_{k+1} &= Ax_k + B\nu_k, \\ y_k &= Cx_k + D\nu_k, \end{aligned} \tag{9}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{l\times n}$, with initial condition x_0, state $x_k \in \mathbb{R}^n$, observed output $y_k \in \mathbb{R}^l$, and exogenous disturbance $\nu_k \in \mathbb{R}^m$, satisfying the constraint

$$\|\nu_k\| \le 1 \quad \text{for all } k = 0, 1, 2, \dots \tag{10}$$

Let the pair (A, B) is controllable and the pair (A, C) is observable.

We construct a filter described by the difference equation

$$\widehat{x}_{k+1} = A\widehat{x}_k + \mathcal{F}(y_k - C\widehat{x}_k), \quad \widehat{x}_0 = 0,$$

for the state estimate $\widehat{x}_k$ with a fixed matrix $\mathcal{F} \in \mathbb{R}^{n\times l}$.

Introduce the residual

$$\rho_k = x_k - \widehat{x}_k.$$

The problem is to find the matrix $\mathcal{F}$ that ensures the minimality of the invariant ellipsoid $\mathcal{V}$ containing the residual ρ_k.

The following theorem holds.

Theorem 3 ([14]): Let $\widehat{\Omega}$ and $\widehat{\Psi}$ be the solution of the optimization problem

$$\min \operatorname{tr} \Upsilon$$

under the constraints

$$\begin{pmatrix} -\mu\Omega & (\Omega A - \Psi C)^{\mathrm{T}} & 0 \\ \Omega A - \Psi C & -\Omega & \Omega B - \Psi D \\ 0 & (\Omega B - \Psi D)^{\mathrm{T}} & -(1-\mu)I \end{pmatrix} \preceq 0,$$

$$\begin{pmatrix} \Upsilon & I \\ I & \Omega \end{pmatrix} \succeq 0, \quad \Omega \succ 0,$$

in the matrix variables $\Omega \in \mathbb{S}^{n\times n}$, $\Psi \in \mathbb{R}^{n\times l}$, $\Upsilon \in \mathbb{S}^{n\times n}$, and the scalar parameter $0 < \mu < 1$.

Then the optimal filter matrix is given by the expression

$$\widehat{\mathcal{F}} = \widehat{\Omega}^{-1}\widehat{\Psi},$$

and the matrix of the minimal invariant ellipsoid for the residual ρ_k for system (9) with $x_0 = 0$ is given by the expression

$$\widehat{\Xi} = \widehat{\Omega}^{-1}.$$

The search procedure for a sparse solution of the filtering problem for system (9), (10) also involves performing three consecutive steps.

Algorithm 2:

Step 1. We solve the optimization problem

$$\min \operatorname{tr} \Upsilon \tag{11}$$

under the constraints

$$\begin{pmatrix} -\mu\Omega & (\Omega A - \Psi C)^{\mathrm{T}} & 0 \\ \Omega A - \Psi C & -\Omega & \Omega B - \Psi D \\ 0 & (\Omega B - \Psi D)^{\mathrm{T}} & -(1-\mu)I \end{pmatrix} \preceq 0, \tag{12}$$

$$\begin{pmatrix} \Upsilon & I \\ I & \Omega \end{pmatrix} \succeq 0, \quad \Omega \succ 0, \tag{13}$$

with the matrix variables $\Omega \in \mathbb{S}^{n\times n}$, $\Psi \in \mathbb{R}^{n\times l}$, $\Upsilon \in \mathbb{S}^{n\times n}$, and the scalar paramater $\mu > 0$.

The values Ω^*, Ψ^*, and Υ^* define the matrix

$$\mathcal{F}^* = (\Omega^*)^{-1}\Psi^*$$

of the optimal filter, the matrix

$$\Xi^* = (\Omega^*)^{-1}$$

of the minimal invariant ellipsoid for the residual, and the optimal value

$$\mathcal{J}^* = \operatorname{tr} \Upsilon^*$$

of the cost function.

Step 2. Having the value $\mathcal{J}^*$, we implement the relaxation coefficient $\lambda > 1$ and solve the optimization problem

$$\min \|\Psi\|_{c_1} \quad \text{s.t. (12), (13), and } \operatorname{tr}\Upsilon \preceq \lambda\mathcal{J}^*$$

in the matrix variables $\Omega \in \mathbb{S}^{n\times n}$, $\Psi \in \mathbb{R}^{n\times l}$, $\Upsilon \in \mathbb{S}^{n\times n}$, and the scalar parameter $0 < \mu < 1$. Due to the properties of the c_1-norm, one can expect the appearance of zero columns in its solution $\widehat{\Psi}_0$.

Step 3. We resolve the original problem (11)–(13) where the same arrangement of zero rows is fixed in the matrix variable Ψ as in the column-sparse matrix $\widehat{\Psi}_0$. Its solution $\widehat{\Omega}$, $\widehat{\Psi}$ defines the column-sparse filter matrix

$$\widehat{\mathcal{F}} = \widehat{\Omega}^{-1}\widehat{\Psi}$$

and the matrix

$$\widehat{\Xi} = \widehat{\Omega}^{-1}$$

of the corresponding invariant ellipsoid for the residual.

Remarks 1 and 2 remain valid in the discrete-time statements.

As the results of numerical simulations show, the "payment" for using a reduced number of controls/outputs (i.e., a loss in terms of cost function) is usually very small.

V. Example

Consider the HE3 problem borrowed from COMPleib [24] benchmark library. This library contains various problems having a clear engineering origin and using to test the efficacy of the proposed approaches. The considered linearized system describes the dynamics of the Bell201A-1 helicopter.

The matrices of the considered system have the following form:

$$A = \begin{pmatrix} -0.0046 & 0.038 & 0.3259 & -0.0045 & -0.402 & -0.073 & -9.81 & 0 \\ -0.1978 & -0.5667 & 0.357 & -0.0378 & -0.2149 & 0.5683 & 0 & 0 \\ 0.0039 & -0.0029 & -0.2947 & 0.007 & 0.2266 & 0.0148 & 0 & 0 \\ 0.0133 & -0.0014 & -0.4076 & -0.0654 & -0.4093 & 0.2674 & 0 & 9.81 \\ 0.0127 & -0.01 & -0.8152 & -0.0397 & -0.821 & 0.1442 & 0 & 0 \\ -0.0285 & -0.0232 & 0.1064 & 0.0709 & -0.2786 & -0.7396 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.0676 \\ -1.1151 \\ 0.0062 \\ -0.017 \\ -0.0129 \\ 0.139 \\ 0 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 \\ 0.1 \\ 0 \\ 0 \\ 0.05 \\ 0 \end{pmatrix}$$

Here the state vector is given as

$$x = \begin{pmatrix} u_H & \sigma & h & v & p & r & \theta & \varphi \end{pmatrix}^{\mathrm{T}},$$

where u_H is the forward velocity, h is the pitch rate, σ is the vertical velocity, p is the roll rate, v is the lateral velocity, r is the yaw rate, φ is the roll angle, θ is the pitch angle, and the output vector is

$$y = \begin{pmatrix} \sigma & \theta & \varphi & r & h & p \end{pmatrix}^{\mathrm{T}}.$$

Setting $\Xi_0 = 0.1I$ and using Theorem 2, at the first step of Algorithm 1 we obtain the optimal filter matrix $\mathcal{F}^*$ and the corresponding invariant ellipsoid for the residual with matrix Ξ^* such that

$$\operatorname{tr} \Xi^* = 1.1381.$$

At the second step, solving the c_1-optimization problem (8) for $\lambda = 10$, we obtain the matrix $\widehat{\Psi}_0$ with two last columns of the order of 10^{-10}.

At the third step, fixing these rows as zero and resolving the original problem, we obtain the column-sparse filter matrix $\widehat{\mathcal{F}}$ and the invariant ellipsoid $\widehat{\Xi}$ for the residual with

$$\operatorname{tr} \widehat{\Xi} = 1.2131.$$

$$\mathcal{F}^* = \begin{pmatrix} -3.3724 & -0.6504 & 0.0449 & 0.7496 & 1.8176 & -0.6771 \\ 1.2395 & -10.5478 & -0.0284 & -0.2938 & -0.9630 & 0.2334 \\ 0.8538 & -0.1075 & -0.0023 & -0.1830 & -0.0243 & -0.1172 \\ -0.0429 & 0.0287 & 9.8102 & 0.3401 & -0.3943 & -0.4499 \\ -0.3655 & 0.0872 & 1.0006 & -0.1138 & -0.4393 & -0.5462 \\ 0.2890 & 1.2176 & 0.0021 & -0.4221 & 0.3213 & -0.0226 \\ 6.8280 & -0.7092 & -0.0062 & -0.9332 & 0.7106 & 0.0687 \\ 0.0073 & -0.0000 & 0.2784 & 0.0000 & 0.0000 & -0.0000 \end{pmatrix}$$

$$\Xi^* = \begin{pmatrix} 0.3714 & -0.1278 & -0.0139 & 0.0007 & 0.0036 & 0.0124 & -0.0377 & 0.0000 \\ -0.1278 & 0.1602 & 0.0065 & -0.0003 & -0.0017 & -0.0058 & 0.0177 & -0.0000 \\ -0.0139 & 0.0065 & 0.1007 & -0.0000 & -0.0002 & -0.0006 & 0.0019 & -0.0000 \\ 0.0007 & -0.0003 & -0.0000 & 0.1000 & 0.0000 & 0.0000 & -0.0001 & -0.0000 \\ 0.0036 & -0.0017 & -0.0002 & 0.0000 & 0.1000 & 0.0002 & -0.0005 & 0.0000 \\ 0.0124 & -0.0058 & -0.0006 & 0.0000 & 0.0002 & 0.1006 & -0.0017 & 0.0000 \\ -0.0377 & 0.0177 & 0.0019 & -0.0001 & -0.0005 & -0.0017 & 0.1052 & -0.0000 \\ 0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.1000 \end{pmatrix}$$

$$\widehat{\Psi}_0 = \begin{pmatrix} -0.4093 & -2.0441 & 0.1151 & -0.1159 & 0 & 0 \\ 0.4093 & -2.0441 & -0.2968 & -0.2514 & 0 & 0 \\ 0.4093 & 2.0441 & -1.0477 & -0.0129 & 0 & 0 \\ 0.0724 & -0.3055 & 1.9967 & 0.2514 & 0 & 0 \\ -0.4093 & -0.3780 & 1.9967 & -0.2514 & 0 & 0 \\ -0.4093 & 2.0441 & 1.1985 & 0.2514 & 0 & 0 \\ -0.2441 & 2.0441 & 1.9967 & 0.2514 & 0 & 0 \\ -0.0143 & -0.4028 & 1.9967 & 0.0245 & 0 & 0 \end{pmatrix}$$

$$\widehat{\mathcal{F}} = \begin{pmatrix} -1.4878 & 0.6754 & 0.0519 & 0.5895 & 0 & 0 \\ 0.7782 & -11.1508 & -0.3270 & -0.2482 & 0 & 0 \\ 0.7283 & 0.0624 & -0.0159 & 0.0733 & 0 & 0 \\ -0.8973 & -0.1698 & 9.9423 & 0.4216 & 0 & 0 \\ -0.8263 & -0.1289 & 1.0840 & -0.0998 & 0 & 0 \\ 0.3308 & 1.3900 & 0.0164 & -0.4074 & 0 & 0 \\ 8.0516 & -0.0006 & -0.5781 & -0.9786 & 0 & 0 \\ 0.1116 & 0.0000 & 0.3123 & -0.0170 & 0 & 0 \end{pmatrix}$$

$$\widehat{\Xi} = \begin{pmatrix} 0.4183 & -0.1314 & 0.0026 & -0.0146 & -0.0251 & 0.0179 & -0.0083 & -0.0147 \\ -0.1314 & 0.1571 & 0.0013 & 0.0007 & 0.0069 & -0.0076 & 0.0104 & 0.0050 \\ 0.0026 & 0.0013 & 0.1019 & -0.0044 & -0.0030 & -0.0000 & 0.0055 & -0.0010 \\ -0.0146 & 0.0007 & -0.0044 & 0.1105 & 0.0076 & -0.0004 & -0.0125 & 0.0026 \\ -0.0251 & 0.0069 & -0.0030 & 0.0076 & 0.1062 & -0.0011 & -0.0077 & 0.0024 \\ 0.0179 & -0.0076 & -0.0000 & -0.0004 & -0.0011 & 0.1010 & -0.0010 & -0.0007 \\ -0.0083 & 0.0104 & 0.0055 & -0.0125 & -0.0077 & -0.0010 & 0.1170 & -0.0022 \\ -0.0147 & 0.0050 & -0.0010 & 0.0026 & 0.0024 & -0.0007 & -0.0022 & 0.1011 \end{pmatrix}$$

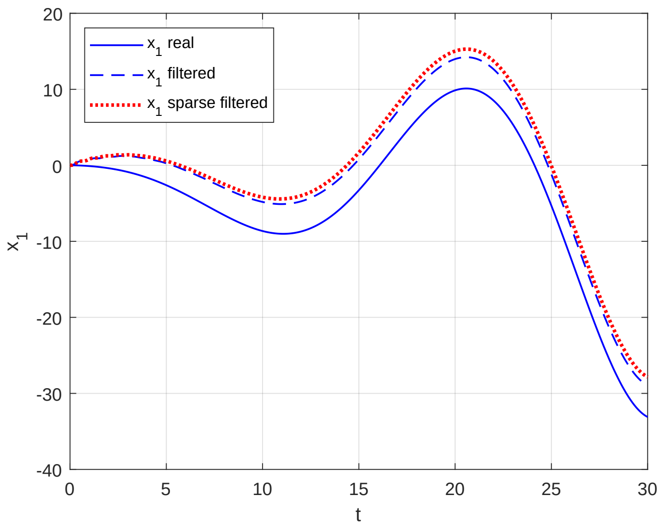

Fig. 1. Filtering the coordinate x_1.

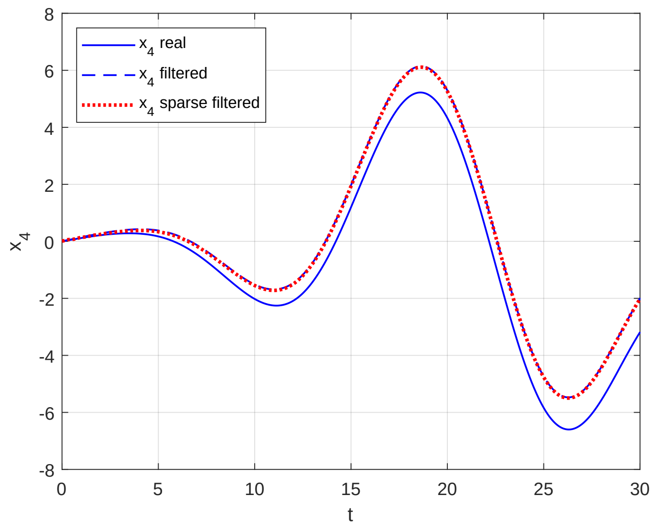

Fig. 2. Filtering the coordinate x_4.

Thus, we construct the sparse filter not using the outputs $y_5 = h$ (pitch rate) and $y_6 = p$ (roll rate), wherein the loss by the cost criterion is 6.5% only.

In the Fig. 1, the solid line depicts the trajectory $x_1(t) = u_H$ (forward velocity) of the system for some admissible exogenous disturbance, the dashed line depicts its optimal estimate $\widehat{x}_1(t)$, and the red dotted line depicts the result $\widetilde{x}_1(t)$ of using the proposed sparse filtering procedure.

For the coordinate $x_4 = v$ (lateral velocity), the sparse filtering accuracy is even higher, see Fig. 2.

The quality of filtering by other coordinates is also quite high.

From a computational point of view, the computations according to Algorithm 1 do not present any technical difficulties. At all its steps, we are dealing with convex optimization problems, for which the MATLAB-based cvx package mentioned in the introduction can be effectively used.

VI. Conclusion

We propose an approach to the sparse filtering problem under nonrandom bounded exogenous disturbances using an observer. The approach is based on the LMI technique and the method of invariant ellipsoids. Using of this concept made it possible to reduce the original problem to a semidefinite programming that can be easily solved numerically. The approach is simple and easily implementable; it equally covers both continuous- and discrete-time cases.

In the future, the author plans to expand the results obtained to the various robust formulations of the problem, in particular, to the system

$$\dot{x} = (A + F\Delta H)x + B\nu$$

subjected to norm-bounded matrix uncertainty $\Delta \in \mathbb{R}^{p\times q}$, $\|\Delta\| \le 1$, where F, H are given matrices of the appropriate dimensions.

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Global Position Feedback Tracking Control of a Serial Robot Manipulator with Revolute Joints

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***Abstract*—In this paper, we present the controller which globally stabilizes a non-stationary motion of a serial robot manipulator with revolute joints without velocity measurements. A family of desired manipulator motions is considered such that the first vertical link of the manipulator performs a given rotation, and the remaining links retain the given relative angular positions. It is proved that such motions of the manipulator can be made globally asymptotically stable using dynamic position feedback. The problem is solved taking into account the periodicity of the dynamics equations along the angular coordinates of the links. As an example, a numerical simulation of the three-link manipulator motion under the constructed controller is presented.**

***Index Terms*—stabilization control problem, serial robot manipulator, revolute joint, dynamic position feedback, Lyapunov function, cylindrical phase space**

I. Introduction

In control theory, trajectory tracking is a fundamental problem. Trajectory tracking of a multi-link robot manipulators is considered as challenging control problem due to nonlinearity and non-stationarity of the dynamics equations. The main approach to the solution of the trajectory tracking control problem for a serial robot manipulator is the construction of proportional derivative (PD) controller with feedforward [1], [2]. Note that the use of a PD controller requires the position and velocity measurements of the manipulator links. In practice, the use of tachometers is fraught with difficulties. This is, firstly, the noise of the signals of the measured speeds, and secondly, the installation of tachometers makes the robot heavier and increases its cost. In addition, in some practically important tasks, for example the installation of tachometers is impossible when the robot operates in an aggressive environment, in a hot cell, etc.

The majority of work for control design of robotic manipulators without velocity measurements uses the dynamic filters, see [2]–[5]. For results related to the use of velocity observers see [6], [7] and for nonlinear proportional integral controllers and Volterra integro-differential equations see [8]–[12]. Due to the complexity of the problem, results on the global trajectory tracking of robot manipulators without velocity measurements are scarce. Note that the problem on global output trajectory tracking control of Euler-Lagrange systems has been solved in [3] based on Lyapunov function method.

Motivated by the authors' early works for the trajectory tracking control problem of multi-link robot manipulators [13], [14], in this paper, we give the solution to the global trajectory tracking control problem without velocity measurements for the revolute joined robotic arms with a vertical first link. The key contributions of our paper can be written as follows:

1) We use the periodicity property of the robotic manipulators equipped with revolute joints. Due to this property, we construct the dynamic position feedback controller which is bounded in position term and ensures the global attractivity of the reference trajectory in a cylindrical phase space.

2) We ensure the global tracking of reference trajectories such that the first link rotation angle is unbounded and twice continuously differentiable function with both derivatives bounded, and other link rotation angles are constant.

Throughout this paper, the following notation is used. Symbol $|\cdot|$ indicates the vector norm in $\mathbb{R}^n$. Symbol $\|\cdot\|$ denotes the operator matrix norm corresponding to the vector norm $|\cdot|$. $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ denote the smallest and largest eigenvalues of some matrix respectively. Symbol $\mathcal{K}$ denotes the Hanh functions class.

The paper is organized as follows. In Section II we present the mathematical model of a robotic arm and define the problem setting. Our main result is stated in Section III. Example of a motion control for a three-link robot manipulator that illustrates our main results is presented in Section IV. Conclusions are provided in Section V.

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II. Mathematical Model of a Robotic Arm and Problem Formulation

We consider serial robotic arms described by Euler-Lagrange equations such as

$$A(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u, \tag{1}$$

where $q \in \mathbb{R}^n$, $\dot{q} \in \mathbb{R}^n$, and $\ddot{q} \in \mathbb{R}^n$ are the vectors of joint rotation angles, angular velocities, and angular accelerations respectively, $A(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ is the matrix of Coriolis and centrifugal terms, $g(q) \in \mathbb{R}^n$ represents the gravitational torques, $u \in \mathbb{R}^n$ is the vector of control torques.

Consider some properties of the robotic arms (1).

1. The matrix $A(q)$ satisfies the following inequalities

$$\begin{gathered} \|A(q)\| \le d_1 \quad \forall q \in \mathbb{R}^n, \\ d_2 \le \lambda_{min}(A(q)) \le \lambda_{max}(A(q)) \le d_3 \quad \forall q \in \mathbb{R}^n, \end{gathered}$$

where d_1, d_2, and d_3 are some positive reals.

2. The following holds

$$\begin{gathered} \dot{A}(q(t)) = C(q(t),\dot{q}(t)) + C^T(q(t),\dot{q}(t)) \\ \forall q \in \mathbb{C}^1,\ q:\ [0,+\infty) \to \mathbb{R}^n. \end{gathered}$$

3. $\forall e_1, e_2 \in \mathbb{R}^n$, the Coriolis matrix $C(e_1,e_2)$ satisfies the following inequalities

$$\begin{gathered} \lambda_{max}(C(e_1,e_2) + C^T(e_1,e_2)) \le \lambda_{c1}\|e_2\|, \\ \|C(e_1,e_2)\| \le \lambda_{c2}\|e_2\|, \end{gathered}$$

where $\lambda_{c1} > 0$ and $\lambda_{c2} > 0$ are some constants.

The focus of our paper is on the following properties of the revolute joined robotic arms with a vertical first link.

4. The inertia matrix $A(q)$ and potential energy $\Pi(q)$ of the manipulator do not depend on the first link rotation angle. This angle is said to be a cyclic coordinate and the rotation angles of the other links are said to be positional ones.

5. The matrices $A(q)$, $C(q,\dot{q})$ and the vector $g(q)$ in (1) are periodic functions of the variables $q_1, q_2, \ldots, q_n$ with some periods $h_i > 0$ $(i = 1, 2, \ldots, n)$ respectively. So, if $u = 0$, then, the system (1) has not only one equilibrium position $(q,\dot{q}) = (0,0)$ but a whole set of equilibrium positions $(q,\dot{q})$ such as $q = (h_1k_1, h_2k_2, \ldots, h_nk_n)^T$, $\dot{q} = 0$, where $k_j \in \mathbb{Z}$, $j = 1, 2, \ldots, n$.

For the mechanical system (1) assume that the output vector contains only link positions. Let find a position feedback controller u which moves the manipulator (1) from any initial position with any initial velocity to track a desired trajectory. Let us mathematically formulate the control problem.

Define the set Q of all desired trajectories of (1) such as

$$\begin{gathered} Q = \{q_r(t):\ [t_0,+\infty) \to \mathbb{R}^n: \\ \|\dot{q}_{r1}(t)\| \le q_{m1}, \quad \|\ddot{q}_{r1}(t)\| \le q_{m2}, \\ q_{ri} = constant, \quad i = 2, 3, \ldots, n\}, \end{gathered} \tag{2}$$

where $q_{r1}(t)$ is a twice differentiable function, $q_{mi} = constant > 0$ $(i = 1, 2)$, $t_0 = constant \ge 0$.

The problem consists in constructing a controller $u = u(t, q(t), q(t+s))$ $(s \in [-t, 0])$ such that the desired trajectory $q_r(t) \in Q$ of the manipulator (1) is uniformly asymptotically stable and globally attractive.

III. Robotic Arm Trajectory Tracking

Choose some desired trajectory $q_r(t) \in Q$ and denote the tracking errors as follows

$$e_q = q - q_r(t), \quad \dot{e}_q = \dot{q} - \dot{q}_r(t). \tag{3}$$

From (1) one can obtain the error dynamic equations

$$A_{st}(e_q)\ddot{e}_q + C_{st}(e_q, 2\dot{q}_r(t) + \dot{e}_q)\dot{e}_q = u - u_r(t, e_q), \tag{4}$$

where

$$\begin{gathered} A_{st}(e_q) = A(e_q + q_r(t)), \\ C_{st}(e_q, x) = C(e_q + q_r(t), x), \\ u_r(t, e_q) = A(q_r(t) + e_q)\ddot{q}_r(t) \\ +C(q_r(t) + e_q, \dot{q}_r(t))\dot{q}_r(t) + g(q_r(t) + e_q). \end{gathered} \tag{5}$$

Let us introduce a cylindrical phase space for (4) such as

$$\{(e_q, \dot{e}_q) \in \mathbb{K}^n \times \mathbb{R}^n\},$$

where $\mathbb{K}^n$ is given by

$$\mathbb{K}^n = \{x \in \mathbb{R}^n:\ x_1(\mathrm{mod} h_1), x_2(\mathrm{mod} h_2), \ldots, x_n(\mathrm{mod} h_n)\}.$$

Consider the controller u such as follows

$$\begin{gathered} u = u_r(t, e_q) + u_{st}(e_q, x), \\ u_{st}(e_q, x) = -K_p p(e_q) - K_x x, \\ \dot{x} = -a(x + b\dot{e}_q), \end{gathered} \tag{6}$$

where $a = constant > 0$ and $b = constant > 0$, $K_p, K_x \in \mathbb{R}^{n\times n}$ are some gain constant matrices, $x = x(t, t_0, e_{q0}, \dot{e}_{q0}, x_0)$ is a solution of a differential equation from (6), $p = p(e_q)$, $p: \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function such that $p(0) = 0$ and $p(e_q) = (p_1(e_{q1}), p_2(e_{q2}), \ldots, p_n(e_{qn}))^T$.

Using the integration by parts formula one can obtain the solution $x(t, t_0, x_0)$ of an ordinary differential equation in (6) with position measurements only. So, one can obtain

$$\begin{gathered} x(t, t_0, e_{q0}, \dot{e}_{q0}, x_0) = x_0 e^{-a(t-t_0)} - ab(e_q(t) \\ -e^{-a(t-t_0)}e_{q0} + a^2 b \int_{t_0}^{t} e_q(s)e^{-a(t-s)}ds). \end{gathered} \tag{7}$$

Using (4) and (6), one can easily obtain the closed-loop system such as

$$\begin{gathered} A_{st}(e_q)\ddot{e}_q + C_{st}(e_q, 2\dot{q}_r(t) + \dot{e}_q)\dot{e}_q \\ +K_p p(e_q) + K_x x = 0, \\ \dot{x} = -a(x + b\dot{e}_q). \end{gathered} \tag{8}$$

Note that using (7) one can obtain the first equation of (8) is functional differential [15].

The equilibrium positions of (8) are contained in the set

$$P = \{(e_q, \dot{e}_q, x) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n:\ p(e_q) = 0,\ \dot{e}_q = 0,\ x = 0\}. \tag{9}$$

Define the subset of (9) as follows

$$\begin{gathered} S = \{(e_q, \dot{e}_q, x) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n: \\ s(e_q) = 0,\ \dot{e}_q = 0,\ x = 0\}. \end{gathered} \tag{10}$$

where $s(e_q) = (s_1(e_{q1}), s_2(e_{q2}), \ldots, s_n(e_{qn}))^T$, $s_j(e_{qj}) = \int_0^{e_{qj}} p_j(z)dz$, $j = \overline{1, n}$.

Consider the following definitions of global attractivity, uniform stability, and uniform asymptotic stability properties of the sets (9) and (10).

Definition 1. *The solution set (9) of the closed-loop system (8) is said to be globally attractive if* $(\forall \varepsilon > 0)$ $(\forall t_0 \geq 0)$ $(\forall (e_{q0}, \dot{e}_{q0}, x_0)^T \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ $(\exists \sigma > 0)$ $(\forall t \geq t_0 + \sigma)$ $\|(p(e_q(t, t_0, e_{q0}, \dot{e}_{q0}, x_0)), \dot{e}_q(t, t_0, e_{q0}, \dot{e}_{q0}, x_0), x(t, t_0, e_{q0}, \dot{e}_{q0}, x_0))^T\| < \varepsilon$.

Definition 2. *The solution set (10) of the closed-loop system (8) is said to be uniformly stable if* $(\forall \varepsilon > 0)$ $(\exists \delta = \delta(\varepsilon) > 0)$ $(\forall t_0 \geq 0)$ $(\forall (e_{q0}, \dot{e}_{q0}, y_0) \in \{(x, \dot{e}_q, y) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$: $\|(p(e_q), \dot{e}_q, x)^T\| < \delta$ $)$ $(\forall t \geq t_0)$ $\|(s(e_q(t, t_0, e_{q0}, \dot{e}_{q0}, x_0)), \dot{e}_q(t, t_0, e_{q0}, \dot{e}_{q0}, x_0), x(t, t_0, e_{q0}, \dot{e}_{q0}, x_0))^T\| < \varepsilon$.

Definition 3. *The solution set (10) of the closed-loop system (8) is said to be uniformly asymptotically stable if it is uniformly stable and uniformly attractive. The uniform attractivity property seems that* $(\exists \Delta > 0)$ $(\forall \varepsilon > 0)$ $(\exists \sigma > 0)$ $(\forall t_0 \geq 0)$ $(\forall (e_{q0}, \dot{e}_{q0}, x_0) \in \{(e_q, \dot{e}_q, x) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$: $\|(p(e_q), \dot{e}_q, x)^T\| < \Delta$ $)$ $(\forall t \geq \sigma + t_0)$ $\|(s(e_q(t, t_0, e_{q0}, \dot{e}_{q0}, x_0)), \dot{e}_q(t, t_0, e_{q0}, \dot{e}_{q0}, x_0), x(t, t_0, e_{q0}, \dot{e}_{q0}, x_0))^T\| < \varepsilon$.

The following theorem presents the main contribution of this paper.

Theorem 1. *Let the controller (6) be such as*

$$K_p = wE, \quad K_x = -abE, \tag{11}$$

where w, a and b are some positive constants.

Then, the solution set P of the closed-loop system (8) is globally attractive and the solution set S is uniformly asymptotically stable.

Proof.

Consider the Lyapunov function candidate $V = V(e_q, \dot{e}_q, x)$ such as follows

$$V = \tfrac{1}{2}(\dot{e}_q)^T A_{st}(e_q)\dot{e}_q + w\sum_{i=1}^{n} s_i(e_{qi}) + \tfrac{1}{2}x^T x. \tag{12}$$

Note that $V(e_q, \dot{e}_q, x) \geq 0$ $\forall (t, e_q, \dot{e}_q, x) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. Moreover, there exists a function $\omega_1 \in \mathcal{K}$ such that

$$V(e_q, \dot{e}_q, x) \geq \omega_1(||(s(e_q), \dot{e}_q, x)^T||). \tag{13}$$

The time derivative of the Lyapunov function candidate V is calculated as

$$\begin{aligned} \dot{V} &= \tfrac{1}{2}(\dot{e}_q)^T \dot{A}_{st}(e_q)\dot{e}_q + (\dot{e}_q)^T A_{st}(e_q)\,(\ddot{e}_q) \\ &\quad + w(p(e_q))^T \dot{e}_q + x^T \dot{x} \\ &= \tfrac{1}{2}(\dot{e}_q)^T \dot{A}_{st}(e_q)(\dot{e}_q) + (\dot{e}_q)^T \\ &\quad \times(-C_{st}(e_q, 2\dot{q}_r(t) + \dot{e}_q)\dot{e}_q - K_p p(e_q) - K_x x) \\ &\quad + w(p(e_q))^T \dot{e}_q - a x^T x - ab\dot{e}_q^T x. \end{aligned} \tag{14}$$

From (14), one can obtain

$$\begin{aligned} \dot{V} &= (\dot{e}_q)^T (C_{st}(e_q, -\dot{q}_r(t))\dot{e}_q \\ &\quad + p^T(e_q)(wE - K_p)\dot{e}_q \\ &\quad + x^T(-abE - K_x)\dot{e}_q - a x^T x. \end{aligned} \tag{15}$$

In can easily see that $(\dot{e}_q)^T (C_{st}(e_q, -\dot{q}_r(t))\dot{e}_q = 0$. Then, from (15) using (11), one can get the following inequality

$$\dot{V} = -a x^T x \leq 0. \tag{16}$$

The set $\{\dot{V} = 0\}$ consists of the solutions of (8) such that $\{x = 0\}$. So, from (8) one can see that such solutions satisfy the following

$$\dot{e}_q = 0, \quad p(e_q) = 0. \tag{17}$$

Thus, one can conclude that the solution set (9) of (8) is globally attractive.

Note now that the function $V = V(e_q, \dot{e}_q, x)$ satisfies the inequalities

$$\omega_1(||(s(e_q), \dot{e}_q, x)^T||) \leq V \leq \omega_2(||(s(e_q), \dot{e}_q, x)^T||), \tag{18}$$

where $\omega_1, \omega_2 \in \mathcal{K}$.

Then, using (18) and Lyapunov stability theory, one can obtain that the solution set (10) is uniformly asymptotically stable. This completes the proof.

Note that the coefficients in (11) can be chosen as any positive constants, their value affects the rate of convergence of the real motion of the manipulator to the desired one.

Note also that the global trajectory control problem for robotic manipulators has been solved in [3]. The main differences between our result and the known one [3] are as follows. 1. In our paper, unbounded time functions can be chosen as reference trajectories. 2. In our paper, the problem has been solved in a cylindrical phase space, which made it possible to use a bounded proportional term in the controller. 3. The conditions of Theorem 1 do not coincide with ones from [3], and these conditions are not a special case of ones from [3].

IV. Global Tracking of a 3-DOF Robotic Manipulator

Consider the performance of the controller (6) for a 3-DOF robotic arm like as PUMA-560 (see, Fig. 1).

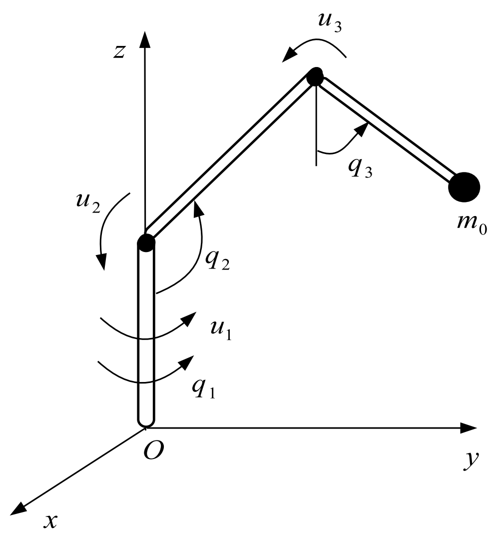

Fig. 1. Scheme of a 3-DOF robotic arm

Assume that the generalized coordinates $q_1 = \varphi_1$, $q_2 = \varphi_2$, and $q_3 = \varphi_3$ are the angular displacements of the cylindrical joints O_1, O_2, and O_3 respectively. The dynamics of a 3-DOF serial robot manipulator with cylindrical joints is defined by (1).

The components a_{ij} of $A(q)$ are given by:

$$a_{11} = I_1 + m_2 l_{c2}^2 \sin^2(q_2) + m_4(l_2 \sin(q_2) + l_{c3} \sin(q_3))^2,$$
$$a_{12} = a_{13} = a_{21} = a_{31} = 0,\ a_{22} = m_2 l_{c2}^2 + m_3 l_2^2,$$
$$a_{23} = a_{32} = m_4 l_2 l_{c3} \cos(q_2 - q_3)/2,\ a_{33} = m_4 l_{c3}^2.$$

where l_2 is the length of the second link; m_j is the mass of the link j; m_0 is the mass of a load; $m_4 = m_0 + m_3$; I_1 is the inertia moment of the first link with respect to Oz; l_{c2} and l_{c3} are the lengths of the intervals between the mass centers of the second link and the third one with a load and the rotation axes of these links accordingly.

The components c_{ij} of $C(q, \dot{q})$ are given by:

$$\begin{gathered} c_{11} = (m_2 l_{c2}^2 + m_4 l_2^2) \sin(2q_2)\dot{q}_2/2 \\ +m_4 l_2 l_{c3}(\sin(q_2)\cos(q_3)\dot{q}_3 + \cos(q_2)\sin(q_3)\dot{q}_2) \\ +m_4 l_{c3}^2 \sin(2q_3)\dot{q}_3/2, \\ c_{12} = -c_{21} = (m_2 l_{c2}^2 + m_4 l_2^2)\sin(2q_2)\dot{q}_1/2 \\ +m_4 l_2 l_{c3} \sin(q_3)\cos(q_2)\dot{q}_1, \\ c_{13} = -c_{31} = m_4 l_2 l_{c3} \sin(q_2)\cos(q_3)\dot{q}_1 \\ +m_4 l_{c3}^2 \sin(2q_3)\dot{q}_1/2, \\ c_{22} = c_{33} = 0,\ c_{23} = m_4 l_2 l_{c3} \sin(q_2 - q_3)\dot{q}_3/2, \\ c_{32} = -m_4 l_2 l_{c3} \sin(q_2 - q_3)\dot{q}_2/2. \end{gathered}$$

The components $g_j,\ j = 1, 2, 3$ of the vector $g(q)$ are as follows:

$$g_1 = 0,\ g_2 = (m_2 l_{c2} + m_4 l_2) g \sin q_2,$$
$$g_3 = m_4 l_{c3} g \sin q_3.$$

The functions $p_j : \mathbb{R} \to \mathbb{R},\ j = \overline{1,3}$ are given by

$$p_j(e_{qj}) = \sin\left(\frac{e_{qj}}{2}\right),\ j = \overline{1,3}.$$

The functions $s_j : \mathbb{R} \to \mathbb{R},\ j = \overline{1,3}$ are given by

$$s_i(e_{qj}) = 2\left(1 - \cos\left(\frac{e_{qj}}{2}\right)\right),\ j = \overline{1,3}.$$

One can easily see that the sets P and S can be written as

$$\begin{gathered} P = \{(e_q, \dot{e}_q, x) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 : \\ e_{qj} = 2\pi k_j\ (j = \overline{1,3}),\ k_j \in \mathbb{Z},\ \dot{e}_q = 0,\ x = 0\}, \\ S = \{(e_q, \dot{e}_q, x) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 : \\ e_{qj} = 4\pi k_j\ (j = \overline{1,3}),\ k_j \in \mathbb{Z},\ \dot{e}_q = 0,\ x = 0\}. \end{gathered}$$

The robot parameters are given as

$$\begin{gathered} I_1 = 0.1\, kg \cdot m^2, \\ m_2 = 13.8\, kg, \quad m_3 = 4.9\, kg, \quad m_0 = 3.1\, kg, \\ l_2 = 1.6\, m, \quad l_{c2} = 0.7\, m, \quad r_3 = 0.5\, m. \end{gathered}$$

The desired trajectory is chosen as

$$q_{1r}(t) = (3t)\, rad,\ q_{2r} = \pi/2\, rad,\ q_{3r} = \pi/4\, rad.$$

The controller is given by (6). The parameters of (6) are chosen such as

$$a = 10,\ b = 1,\ K_p = 2E,\ K_x = -10E. \tag{19}$$

Let the initial state and velocity of the manipulator be such as

$$\begin{gathered} q_1(0) = 3.0 + q_{1r}(0)\, rad,\ q_2(0) = -2.0 + q_{2r}\, rad\ , \\ q_3(0) = 2.1 + q_{3r}\, rad, \\ \dot{q}_1(0) = 40\, rad/s,\ \dot{q}_2(0) = -35\, rad/s, \\ \dot{q}_3(0) = 50\, rad/s. \end{gathered}$$

The simulation has been performed using Scilab 5.5.2 platform.

Figs. 2 – 4 show the tracking process for the desired trajectory. One can easily see the asymptotic convergence of the links trajectories to the desired ones plus $4\pi z$, where $z = (z_1, z_2, z_3)^T,\ z_j \in \mathbb{Z},\ j = 1, 2, 3.$

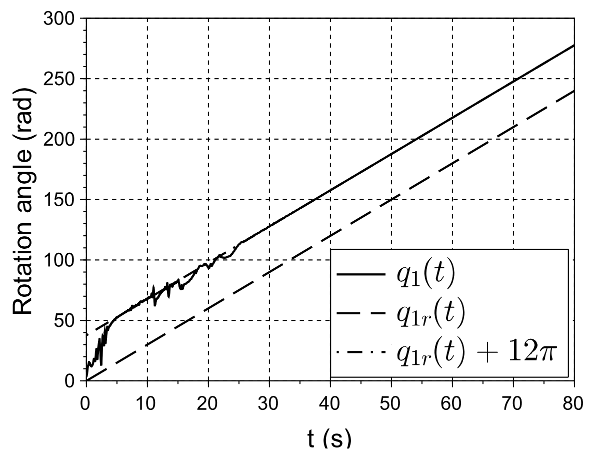

Fig. 2. Desired and actual angular coordinate for the first joint.

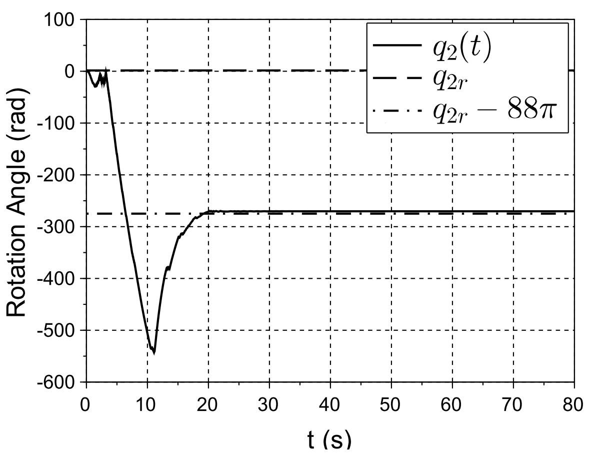

Fig. 3. Desired and actual angular coordinate for the second joint.

In Fig. 5 the time evolution of the stabilizing control torques has been shown. Thus, it can be seen from Figs. 2 – 4 that the solution to the global trajectory tracking control problem is obtained.

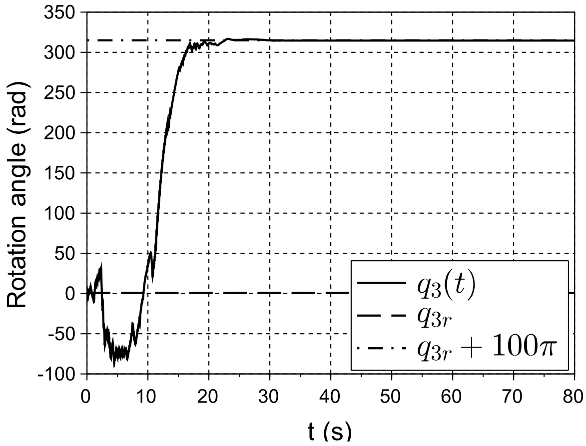

Fig. 4. Desired and actual angular coordinate for the third joint.

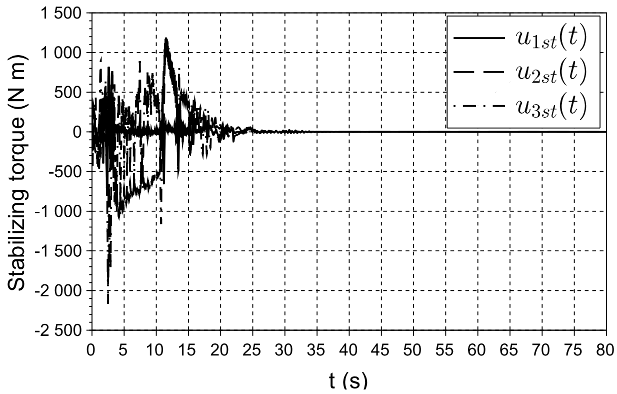

Fig. 5. Stabilizing control torques.

V. Conclusion

In this paper we have presented results that justify the design of a dynamic position feedback controller based on Lyapunov function method for a robotic arm trajectory tracking without velocity measurements. The first-order dynamic filter has been designed in order to compensate the absence of velocity measurements. We have shown that the controller with arbitrary small gain matrices provides the uniform asymptotic stability and global attractivity properties for the reference trajectories of a serial robot manipulator with revolute joints such that the first link rotates around vertical line and other links hold constant relative positions. It has been proved that the global trajectory tracking holds in a cylindrical phase space. In other words, from any initial state at any initial velocity, each link of the manipulator tends asymptotically to the motion displaced by a multiple of 2π from a desired one. The values of the gain matrices affect the rate of the real motion convergence to the given one of the manipulator. The theoretical results that we have presented for a multi-link robot manipulator have been demonstrated in numerical simulation of a three-link robotic arm like as PUMA-560.

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