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Uniqueness of entire functions whose difference polynomials sharing a polynomial of certain degree with finite weight

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Abstract. The purpose of the paper is to study the possible uniqueness relation of entire functions when the difference polynomial generated by them sharing a non zero polynomial of certain degree. The result obtained in the paper will improve and generalize a number of recent results in a compact and convenient way.

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1. Introduction, Definitions and Results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let $a \in \mathbb{C}$. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notations of value distribution theory (see [6]). For a non-constant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to f, provided that $T(r,a) = S(r,f)$. The order of f is defined by

$$\sigma(f) = \limsup_{r \longrightarrow \infty} \frac{\log T(r,f)}{\log r}.$$

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros with the same multiplicities and we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities.

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We say that a finite value z_0 is called a fixed point of f if $f(z_0) = z_0$ or z_0 is a zero of $f(z) - z$.

For the sake of simplicity we also use the notation

$$m^* := \begin{cases} 0, & \text{if } m = 0 \\ m, & \text{if } m \in \mathbb{N} \end{cases}$$

Let $f(z)$ be a transcendental meromorphic function, n be a positive integer. During the last few decades many authors investigated the value distributions of $f^n f^{'}$. Specially in 1959, W.K. Hayman (see [5], Corollary of Theorem 9) proved the following theorem.

Theorem A. [5] Let f be a transcendental meromorphic function and $n(\geq 3)$ is an integer. Then $f^n f' = 1$ has infinitely many solutions.

The case $n = 2$ was settled by Mues [14] in 1979. Bergweiler and Eremenko [1] showed that $ff^{'} - 1$ has infinitely many zeros.

For an analog of the above results Laine and Yang investigated the value distribution of difference products of entire functions in the following manner.

Theorem B. [10] Let f be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n \geq 2$, $f^n(z)f(z+c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.

Afterwards, Liu and Yang improved Theorem B and obtained the next result.

Theorem C. [13] Let f be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n \geq 2$, $f^n(z)f(z+c) - p(z)$ has infinitely many zeros, where $p(z)$ is a non-zero polynomial.

Next we recall the uniqueness result corresponding to Theorem A, obtained by Yang and Hua [17] which may be considered a gateway to a new research in the direction of sharing values of differential polynomials.

Theorem D. [13] Let f and g be two non-constant entire functions, $n \in \mathbb{N}$ such that $n \geq 6$. If $f^n f^{'}$ and $g^n g^{'}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where $c_1, c_2, c \in \mathbb{C}$ satisfying $4(c_1 c_2)^{n+1} c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.

In 2001, Fang and Hong studied the uniqueness of differential polynomials of the form $f^n(f-1)f^{'}$ and proved the following uniqueness result.

Theorem E. [4] Let f and g be two transcendental entire functions, and let $n \geq 11$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f = g$.

In 2004, Lin and Yi extended the above result in view of the fixed point and they proved the following.

Theorem F. [12] Let f and g be two transcendental entire functions, and let $n \geq 7$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z CM, then $f = g$.

In 2010, Zhang got a analogue result in difference.

Theorem G. [19] Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant and $n \geq 7$ is an integer. If $f(z)^n(f(z)-1)f(z+c)$ and $g(z)^n(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$.

In 2010, Qi, Yang and Liu obtained the difference counterpart of Theorem D by proving the following theorem.

Theorem H. [15] Let f and g be two transcendental entire functions of finite order, and c be a nonzero complex constant; let $n \geq 6$ be an integer. If $f^n f(z+c)$ and $g^n g(z+c)$ share z CM, then $f \equiv t_1 g$ for a constant t_1 that satisfies $t_1^{n+1} = 1$.

Theorem I. [15] Let f and g be two transcendental entire functions of finite order, and c be a nonzero complex constant; let $n \geq 6$ be an integer. If $f^n f(z+c)$ and $g^n g(z+c)$ share 1 CM, then $fg \equiv t_2$ or $f \equiv t_3 g$ for some constants t_2 and t_3 that satisfy $t_3^{n+1} = 1$.

X.M. Li et. al. [11] [Theorem 1.1] replaced the fixed point sharing in the above two theorems to sharing a polynomial with $deg < \frac{n+1}{2}$.

So we see that there are many generalization in terms of difference operator. The purpose of this paper is to study the uniqueness problem for more general difference polynomials namely $f^n P(f) f(z+c)$ and $g^n P(g) g(z+c)$ sharing a non-zero polynomial so that improved version of all the above results can be unified under a single result. We also relax the nature of sharing with the notion of weighted sharing introduced in [8]- [9]. The following theorem is the main result of the paper.

Theorem 1. Let f and g be two transcendental entire functions of finite order, c be a non-zero complex constant and let $p(z)$ be a nonzero polynomial with $deg(p) \leq n-1$, $n(\geq 1)$, $m^*(\geq 0)$ be two integers such that $n > m^* + 5$. Let $P(\omega) = a_m\omega^m + a_{m-1}\omega^{m-1} + \ldots + a_1\omega + a_0$ be a nonzero polynomial. If $f^n P(f) f(z+c) - p$ and $g^n P(g) g(z+c) - p$ share $(0,2)$, then

(I) when $P(\omega) = a_m\omega^m + a_{m-1}\omega^{m-1} + \ldots + a_1\omega + a_0$ is a nonzero polynomial, one of the following three cases holds:

(I1) $f(z) \equiv t g(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 1, 2, \ldots, m$,

(I2) f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$,

(I3) $P(\omega)$ reduces to a nonzero monomial, namely $P(\omega) = a_i\omega^i \not\equiv 0$, for $i \in \{0, 1, \ldots, m\}$, if $p(z)$ is a nonzero constant b, then $f(z) = e^{\alpha(z)}$, $g = e^{\beta(z)}$, where $\alpha(z)$, $\beta(z)$ are two non-constant polynomials such that $\alpha + \beta \equiv d \in \mathbb{C}$ and $a_i^2 e^{(n+i+1)d} = b^2$;

(II) when $P(\omega) = \omega^m - 1$, then $f \equiv tg$ for some constant t such that $t^m = 1$;

(III) when $P(\omega) = (\omega - 1)^m (m \geq 2)$, one of the following two cases holds:

(III1) $f(z) \equiv g(z)$,

(III2) f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m \omega_1(z+c) - \omega_2^n(\omega_2 - 1)^m \omega_2(z+c)$;

(IV) when $P(\omega) \equiv c_0$, one of the following two cases holds:

(IV1) $f \equiv tg$ for some constant t such that $t^{n+1} = 1$,

(IV2) $f(z) = e^{\alpha(z)}$, $g = e^{\beta(z)}$, where $\alpha(z)$, $\beta(z)$ are two non-constant polynomials such that $\alpha + \beta \equiv d \in \mathbb{C}$ and $c_0^2 e^{(n+1)d} = b^2$.

We now explain following definitions and notations which are used in the paper.

Definition 1. [7] Let $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer p we denote by $N(r, a; f \mid \leq p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r,a;f\mid\leq p)$ we denote the corresponding reduced counting function.

In an analogous manner we can define $N(r,a;f\mid\geq p)$ and $\overline{N}(r,a;f\mid\geq p)$.

Definition 2. [9] Let k be a positive integer or infinity. We denote by $N_k(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m\leq k$ and k times if $m>k$. Then

$$N_k(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)+...+\overline{N}(r,a;f\mid\geq k).$$

Clearly $N_1(r,a;f)=\overline{N}(r,a;f)$.

Definition 3. [8, 9] Let k be a nonnegative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m\leq k$ and k+1 times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(>k)$ if and only if it is an a-point of g with multiplicity $n(>k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for any integer p, $0\leq p<k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

2. Lemmas

Lemma 1. [16] Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be meromorphic functions such that $T(r,a_i(z))=S(r,f)$ for $i=0,1,2,...,n$. Then

$$T(r,a_nf^n+a_{n-1}f^{n-1}+...+a_1f+a_0)=nT(r,f)+S(r,f).$$

Lemma 2. [2] Let $f(z)$ be a meromorphic function of finite order σ, and let c be a fixed nonzero complex constant. Then for each $\varepsilon>0$, we have

$$m(r,\frac{f(z+c)}{f(z)})+m(r,\frac{f(z)}{f(z+c)})=O(r^{\sigma-1+\varepsilon}).$$

Lemma 3. [2] Let f be a meromorphic function of finite order σ, $c\neq 0$ be fixed. Then for each $\varepsilon>0$, we have

$$T(r,f(z+c))=T(r,f)+O(r^{\sigma-1+\varepsilon})+O(\log r).$$

Lemma 4. Let f be an entire function of finite order σ, c be a fixed nonzero complex constant and let $n\in\mathbb{N}$ and $P(\omega)$ be defined as in Theorem 1. Then for each $\varepsilon>0$, we have

$$T(r,f^nP(f)f(z+c))=T(r,f^{n+1}P(f))+O(r^{\sigma-1+\varepsilon}).$$

Proof. By Lemma 2 we have

$$\begin{aligned} T(r, f^n P(f) f(z+c)) &= m(r, f^n P(f) f(z+c)) \\ &\le m(r, f^n P(f) f) + m(r, \frac{f(z+c)}{f(z)}) \\ &\le m(r, f^{n+1} P(f)) + O(r^{\sigma-1+\varepsilon}) \\ &= T(r, f^{n+1} P(f)) + O(r^{\sigma-1+\varepsilon}). \end{aligned}$$

Also we have

$$\begin{aligned} T(r, f^{n+1} P(f)) &= m(r, f^n P(f) f) \\ &\le m(r, f^n P(f) f(z+c)) + m(r, \frac{f(z)}{f(z+c)}) \\ &\le m(r, f^n P(f) f(z+c)) + O(r^{\sigma-1+\varepsilon}) \\ &\le T(r, f^n P(f) f(z+c)) + O(r^{\sigma-1+\varepsilon}). \end{aligned}$$

Therefore $T(r, f^n P(f) f(z+c)) = T(r, f^{n+1} P(f)) + O(r^{\sigma-1+\varepsilon})$. □

Remark 1. Under the condition of Lemma 4, by Lemma 1 we have $S(r, f^n P(f) f(z+c)) = S(r, f)$.

Lemma 5. [3] Let f be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then

$$N(r, 0; f(z+c)) \le N(r, 0; f(z)) + S(r, f), \quad N(r, \infty; f(z+c)) \le N(r, \infty; f) + S(r, f),$$

$$\overline{N}(r, 0; f(z+c)) \le \overline{N}(r, 0; f(z)) + S(r, f), \quad \overline{N}(r, \infty; f(z+c)) \le \overline{N}(r, \infty; f) + S(r, f),$$

Lemma 6. Let f be a transcendental entire function of finite order σ, c be a fixed nonzero complex constant, $n(\ge 1)$, $m^*(\ge 0)$ be two integers and let $a(z)(\not\equiv 0, \infty)$ be a small function of f. If $n > 1$, then $f^n P(f) f(z+c) - a(z)$ has infinitely many zeros.

Proof. Let $\Phi = f^n P(f) f(z+c)$. Now in view of Lemma 5 and the second theorem for small functions (see [18]) we get

$$\begin{aligned} T(r, \Phi) &\le \overline{N}(r, 0; \Phi) + \overline{N}(r, \infty; \Phi) + \overline{N}(r, a(z); \Phi) + (\varepsilon + o(1))\, T(r, f) \\ &\le \overline{N}(r, 0; f^n P(f)) + \overline{N}(r, 0; f(z+c)) + \overline{N}(r, a(z); \Phi) + (\varepsilon + o(1))\, T(r, f) \\ &\le 2\, \overline{N}(r, 0; f) + \overline{N}(r, 0; P(f)) + \overline{N}(r, a(z); \Phi) + (\varepsilon + o(1))\, T(r, f) \\ &\le (2 + m^*)\, T(r, f) + \overline{N}(r, a(z); \Phi) + (\varepsilon + o(1))\, T(r, f), \end{aligned}$$

for all $\varepsilon > 0$.
From Lemmas 1 and 4 we get

$$(n + m^* + 1)\, T(r, f) \le (2 + m^*)\, T(r, f) + \overline{N}(r, a(z); \Phi) + (\varepsilon + o(1))\, T(r, f).$$

Take $\varepsilon < 1$. Since $n > 1$ from above one can easily say that $\Phi - a(z)$ has infinitely many zeros.
This completes the Lemma. □

Lemma 7. [9] Let f and g be two non-constant meromorphic functions sharing $(1, 2)$. Then one of the following holds:

(i) $T(r, f) \le N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r, f) + S(r, g)$,
(ii) $fg \equiv 1$,

(iii) $f \equiv g$.

Lemma 8. [Hadamard Factorization Theorem] Let f be an entire function of finite order ρ with zeros $a_1, a_2, \ldots$, each zeros is counted as often as its multiplicity. Then f can be expressed in the form

$$f(z) = Q(z)e^{\alpha(z)},$$

where $\alpha(z)$ is a polynomial of degree not exceeding $[\rho]$ and $Q(z)$ is the canonical product formed with the zeros of f.

Lemma 9. Let f and g be two transcendental entire functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $p(z)$ be a nonzero polynomial such that $deg(p) \leq n-1$, where $n \in \mathbb{N}$. Let $P(\omega)$ be a nonzero polynomial defined as in Theorem 1. Suppose

$$f^n P(f) f(z+c) g^n P(g) g(z+c) \equiv p^2.$$

Then $P(\omega)$ reduces to a nonzero monomial, namely $P(\omega) = a_i \omega^i \not\equiv 0$, for $i \in \{0, 1, \ldots, m\}$. If $p(z) = b \in \mathbb{C} \setminus \{0\}$, then $f(z) = e^{\alpha(z)}$, $g = e^{\beta(z)}$, where $\alpha(z)$, $\beta(z)$ are two non-constant polynomials such that $\alpha + \beta \equiv d \in \mathbb{C}$ and $a_i^2 e^{(n+i+1)d} = b^2$.

Proof. Suppose

$$f^n P(f) f(z+c) g^n P(g) g(z+c) \equiv p^2. \tag{2.1}$$

We consider the following cases:
Case 1: Let $deg(p(z)) = l(\geq 1)$.
From the assumption that f and g are two transcendental entire functions, we deduce by (2.1) that $N(r, 0; f^n P(f)) = O(\log r)$ and $N(r, 0; g^n P(g)) = O(\log r)$.
First we suppose that $P(\omega)$ is not a nonzero monomial. For the sake of simplicity let $P(\omega) = \omega - a$ where $a \in \mathbb{C} \setminus \{0\}$. Clearly $\Theta(0; f) + \Theta(a; f) = 2$, which is impossible for an entire function. Thus $P(\omega)$ reduces to a nonzero monomial, namely $P(\omega) = a_i \omega^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$ and so (2.1) reduces to

$$a_i^2 f^{n+i} f(z+c) g^{n+i} g(z+c) \equiv p^2. \tag{2.2}$$

From (2.2) it follows that $N(r, 0; f) = O(\log r)$ and $N(r, 0; g) = O(\log r)$. Now by Lemma 8 we obtain that $f = h_1 e^{\alpha_1}$ and $f = h_2 e^{\beta_1}$, where h_1, h_2 are two nonzero polynomials and α_1 and β_1 are two non-constant polynomials.
By virtue of the polynomial $p(z)$, from (2.2) we arrive at a contradiction.
Case 2: Let $p(z) = b \in \mathbb{C} \setminus \{0\}$.
Then from (2.1) we have

$$f^n P(f) f(z+c) g^n P(f) g(z+c) \equiv b^2. \tag{2.3}$$

Now from the assumption that f and g are two non-constant entire functions, we deduce by (2.3) that $f^n P(f) \neq 0$ and $g^n P(g) \neq 0$. By Picard's theorem, we claim that $P(\omega) = a_i \omega^i \not\equiv 0$ for $i \in \{0, 1, \ldots, m\}$, otherwise the Picard's exception values are atleast three, which is a contradiction. Then (2.3) reduces to

$$a_i^2 f^{n+i} f(z+c) g^{n+i} g(z+c) \equiv b^2. \tag{2.4}$$

Hence by Lemma 8 we obtain that

$$f = e^{\alpha}, \quad g = e^{\beta}, \tag{2.5}$$

where $\alpha(z)$, $\beta(z)$ are two non-constant polynomials.
Now from (2.4) and (2.5) we obtain

$$(n+i)(\alpha(z)+\beta(z))+\alpha(z+c)+\beta(z+c)\equiv d_1,$$

where $d_1\in\mathbb{C}$, i.e.,

$$(n+i)(\alpha^{'}(z)+\beta^{'}(z))+\alpha^{'}(z+c)+\beta^{'}(z+c)\equiv 0. \tag{2.6}$$

Let $\gamma(z)=\alpha^{'}(z)+\beta^{'}(z)$. Then from (2.6) we have

$$(n+i)\,\gamma(z)+\gamma(z+c)\equiv 0. \tag{2.7}$$

We assert that $\gamma(z)\equiv 0$. It not suppose $\gamma(z)\not\equiv 0$. Note that if $\gamma(z)\equiv d_2\in\mathbb{C}$, from (2.7) we must have $d_2=0$. Suppose that $deg(\gamma)\geq 1$. Let $\gamma(z)=\sum\limits_{i=1}^{m} b_i z^i$, where $b_m\neq 0$. Therefore the co-efficient of z^m in $(n+i)\gamma(z)+\gamma(z+c)$ is $(n+1+i)b_m\neq 0$. Thus we arrive at a contradiction from (2.7). Hence $\gamma(z)\equiv 0$, i.e., $\alpha+\beta\equiv d\in\mathbb{C}$. Also from (2.4) we have $a_i^2e^{(n+i+1)d}=b^2$. This completes the proof. □

Lemma 10. Let f and g be two transcendental entire functions of finite order, $c\in\mathbb{C}\setminus\{0\}$ and $p(z)$ be a nonzero polynomial such that $deg(p)\leq n-1$, where $n\in\mathbb{N}$. Let $P(\omega)$ be defined as in Theorem 1 with at least two of a_i, $i=0,1,\ldots,m$ are nonzero. Then

$$f^nP(f)f(z+c)g^nP(g)g(z+c)\not\equiv p^2.$$

Proof. Proof of the Lemma follows from Lemma 9. □

Lemma 11. Let f, g be two transcendental entire functions of finite order, $c\in\mathbb{C}\setminus\{0\}$ and $n\in\mathbb{N}$ with $n>1$. If

$$f^nP(f)f(z+c)\equiv g^nP(g)g(z+c),$$

where $P(\omega)$ is defined as in Theorem 1 then

(I) when $P(\omega)=a_m\omega^m+a_{m-1}\omega^{m-1}+\ldots+a_1\omega+a_0$, one of the following two cases holds:

(I1) $f(z)\equiv tg(z)$ for a constant t such that $t^d=1$, where $d=GCD(n+m,\ldots,n+m-i,\ldots,n)$, $a_{m-i}\neq 0$ for some $i=1,2,\ldots,m$,

(I2) f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $R(\omega_1,\omega_2)=\omega_1^nP(\omega_1)\omega_1(z+c)-\omega_2^nP(\omega_2)\omega_2(z+c)$;

(II) when $P(\omega)=\omega^m-1$, then $f\equiv tg$ for some constant t such that $t^m=1$;

(III) when $P(\omega)=(\omega-1)^m(m\geq 2)$, one of the following two cases holds:

(III1) $f(z)\equiv g(z)$,

(III2) f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $R(\omega_1,\omega_2)=\omega_1^n(\omega_1-1)^m\omega_1(z+c)-\omega_2^n(\omega_2-1)^m\omega_2(z+c)$;

(IV) when $P(w)\equiv c_0$, then $f\equiv tg$ for some constant t such that $t^{n+1}=1$.

Proof. Suppose

$$f^nP(f)f(z+c)\equiv g^nP(g)g(z+c). \tag{2.8}$$

Since g is transcendental entire function, hence $g(z),g(z+c)\not\equiv 0$.
We consider following two cases.

Case 1. $P(\omega)\not\equiv c_0$.
Let $h=\frac{f}{g}$. If h is a constant, by putting $f=hg$ in (2.8) we get

$$a_mg^m(h^{n+m+1}-1)+a_{m-1}g^{m-1}(h^{n+m}-1)+\ldots+a_1g(h^{n+2}-1)+a_0(h^{n+1}-1)\equiv 0,$$

which implies that $h^d=1$, where $d=GCD(n+m+1,\ldots,n+m+1-i,\ldots,n+1)$, $a_{m-i}\neq 0$ for some $i\in\{0,1,\ldots,m\}$. Thus $f\equiv tg$ for a constant t such that $t^d=1$,where $d=GCD(n+m+1,\ldots,n+m+1-i,\ldots,n+1)$, $a_{m-i}\neq 0$ for some $i\in\{0,1,\ldots,m\}$.
If h is not a constant, then we know by (2.8) that f and g satisfying the algebraic equation $R(f,g)=0$, where $R(\omega_1,\omega_2)=\omega_1^nP(\omega_1)\omega_1(z+c)-\omega_2^nP(\omega_2)\omega_2(z+c)$.
We now discuss the following Subcases.
Subcase 1. $P(\omega)=\omega^m-1$.
Then from (2.8) we have

$$f^n(f^m-1)f(z+c)\equiv g^n(g^m-1)g(z+c). \tag{2.9}$$

Let $h=\frac{f}{g}$. Clearly from (2.9) we get

$$g^m[h^{n+m}h(z+c)-1]\equiv h^nh(z+c)-1. \tag{2.10}$$

First we suppose that h is non-constant. We assert that $h^{n+m}h(z+c)$ is non-constant. If not let $h^{n+m}h(z+c)\equiv c_1\in\mathbb{C}\setminus\{0\}$. Then we have

$$h^{n+m}\equiv\frac{c_1}{h(z+c)}.$$

Now by Lemmas 1 and 3 we get

$$(n+m)\,T(r,h)\leq T(r,h)+S(r,h),$$

which contradicts with $n>m+5$. Thus from (2.10) we have

$$g^m\equiv\frac{h^nh(z+c)-1}{h^{n+m}h(z+c)-1}. \tag{2.11}$$

Let z_0 be a zero of $h^{n+m}h(z+c)-1$. Since g is an entire function, it follows that z_0 is also a zero of $h^nh(z+c)-1$. Consequently z_0 is a zero of h^m-1 and so

$$\overline{N}(r,0;h^{n+m}h(z+c))\leq\overline{N}(r,0;h^m)\leq m\,T(r,h)+O(1).$$

So in view of Lemmas 1, 4, 5 and the second fundamental theorem we get

$$\begin{aligned}(n+m+1)\,T(r,h)&=T(r,h^{n+m}h(z+c))+S(r,h)\\&\leq\overline{N}(r,0;h^{n+m}h(z+c))+\overline{N}(r,1;h^{n+m}h(z+c))+S(r,h)\\&\leq 2\,N(r,0;h)+m\,T(r,h)+S(r,h)\\&\leq(m+2)\,T(r,h)+S(r,h),\end{aligned}$$

which contradicts with $n>1$.
Hence h is a constant. Since g is transcendental entire function, from (2.10) we have

$$h^{n+m}h(z+c)-1\equiv 0\Longleftrightarrow h^nh(z+c)-1\equiv 0$$

and so $h^m=1$. Thus $f(z)\equiv tg(z)$ for a constant t such that $t^m=1$.
Subcase 2. Let $P(\omega)=(\omega-1)^m$.
Then from (2.8) we have

$$f^n(f-1)^mf(z+c)\equiv g^n(g-1)^mg(z+c). \tag{2.12}$$

Let $h = \frac{f}{g}$. If $m = 1$, then the result follows from Subcase 1.
For $m \geq 2$: First we suppose that h is non-constant:
Then from (2.12) we can say that f and g satisfying the algebraic equation $R(f, g) = 0$, where

$$R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m \omega_1(z + c) - \omega_2^n(\omega_2 - 1)^m \omega_2(z + c).$$

Next we suppose that h is a constant:
Then from (2.12) we get

$$f^n f(z + c) \sum_{i=0}^{m} (-1)^i \, {}^mC_{m-i} \, f^{m-i} \equiv g^n g(z + c) \sum_{i=0}^{m} (-1)^i \, {}^mC_{m-i} g^{m-i}. \tag{2.13}$$

Now substituting $f = gh$ in (2.13) we get

$$\sum_{i=0}^{m} (-1)^i \, {}^mC_{m-i} \, g^{m-i}(h^{n+m+1-i} - 1) \equiv 0,$$

which implies that $h = 1$. Hence $f \equiv g$.
Case 2. $P(\omega) \equiv c_0$.
Let $h = \frac{f}{g}$. Then from (2.8) we have

$$h^n(z) \equiv \frac{1}{h(z + c)}. \tag{2.14}$$

Thus from Lemmas 1 and 3 we get

$$n\, T(r, h) = T(r, h(z + c)) + O(1) = T(r, h) + S(r, h),$$

which is a contradiction since $n \geq 2$. Hence h must be a constant, which implies that $h^{n+1} = 1$, thus $f = tg$ and $t^{n+1} = 1$.
This completes the the proof. □

3. Proofs of the Theorem

Proof of Theorem 1. Let $F = \frac{f^n P(f) f(z+c)}{p}$ and $G = \frac{g^n P(g) g(z+c)}{p}$. Then F and G share $(1, 2)$ except the zeros of $p(z)$. Now applying Lemma 7 we see that one of the following three cases holds.
Case 1. Suppose

$$T(r, f) \leq N_2(r, 0; F) + N_2(r, 0; G) + S(r, F) + S(r, G).$$

Now by applying Lemmas 1 and 7 we have

$$\begin{aligned} T(r,F) &\leq N_2(r,0;F)+N_2(r,0;G)+S(r,f)+S(r,g) \\ &= N_2(r,0;f^nP(f)f(z+c))+N_2(r,0;g^nP(g)g(z+c))+S(r,f)+S(r,g) \\ &\leq N_2(r,0;f^nP(f))+N_2(r,0;f(z+c))+N_2(r,0;g^nP(g))+N_2(r,0;g(z+c)) \\ &\quad +S(r,f)+S(r,g) \\ &\leq 2\,N(r,0;f)+N(r,0;P(f))+N(r,0;f(z+c))+2\,N(r,0;g)+N(r,0;P(g)) \\ &\quad +N(r,0;g(z+c))+S(r,f)+S(r,g) \\ &\leq (2+m^*)\,T(r,f)+N(r,0;f)+(2+m^*)\,T(r,g)+N(r,0;g)+S(r,f)+S(r,g) \\ &\leq (3+m^*)\,T(r,f)+(3+m^*)\,T(r,g)+S(r,f)+S(r,g) \\ &\leq (6+2m^*)\,T(r)+S(r) \end{aligned}$$

From Lemmas 1 and 4 we have

$$(n+m^*+1)\,T(r,f) \leq (6+2m^*)\,T(r)+S(r). \tag{3.1}$$

Similarly we have

$$(n+m^*+1)\,T(r,g) \leq (6+2m^*)\,T(r)+S(r). \tag{3.2}$$

Combining (3.1) and (3.2) we get

$$(n+m^*+1)\,T(r) \leq (6+2m^*)T(r)+S(r),$$

which contradicts with $n > 5+m^*$.
Case 2. $F \equiv G$.
Then we have

$$f^nP(f)f(z+c) \equiv g^nP(g)g(z+c)$$

and so the result follows from Lemma 11.
Case 3. $FG \equiv 1$.
Then we have

$$f^nP(f)f(z+c)g^nP(g)g(z+c) \equiv p^2$$

and so the result follows from Lemma 9.
This completes the proof. □

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New concepts of irregular-intuitionistic fuzzy graphs with applications

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Abstract. In this paper, some types of edge irregular intuitionistic fuzzy graphs such as neighbourly edge totally irregular intuitionistic fuzzy graphs, strongly edge irregular intuitionistic fuzzy graphs and strongly edge totally irregular intuitionistic fuzzy graphs are introduced. A comparative study between neighbourly edge irregular intuitionistic fuzzy graphs and neighbourly edge totally irregular intuitionistic fuzzy graphs is done. Likewise some properties of them are studied. Finally, we have given some interesting results a bout edge irregular IFGs that are very useful in computer science and networks.

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1. Introduction

In 1736, Euler first introduced the concept of graph theory. In the history of mathematics, the solution given by Euler of the well known Konigsberg bridge problem is considered to be the first theorem of graph theory. This has now become a subject generally regarded as a branch of combinatorics. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas such as logic, geometry, algebra, topology, analysis, number theory, information theory, artificial intelligence, operations research, optimization, neural networks, planning, computer science and etc [5], [7], [8], [10], [13].

Fuzzy set theory, introduced by Zadeh in 1965, is a mathematical tool for handling uncertainties like vagueness, ambiguity and imprecision in linguistic variables [30]. Research on theory of fuzzy sets has been witnessing an exponential growth; both within mathematics and in its application. Fuzzy set theory has emerged as a potential area of interdisciplinary research and fuzzy graph theory is of recent interest.

In 1983, Atanassov [3] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [30]. Atanassov added a new component(which determines the degree of non-membership) in the definition of fuzzy set. The fuzzy sets give the degree of membership of an element in a given set (and the nonmembership degree equals one minus the degree of membership), while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership which are more-or-less independent from each other, the only requirement is that the sum of these two degrees is not greater than 1. Intuitionistic fuzzy sets have been applied in a wide variety of

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