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Mean sojourn time fraction in frozen, homogenous, isotropic and self similar electrostatic turbulence

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Abstract

In the framework of the general theory of random Hamiltonian dynamical systems the relation between the mean sojourn time fraction in an arbitrary domain and the projector to the subspace of the invariant function is established. In the particular case of the random formal Hamiltonian system related to the electrostatic drift motion in homogenous magnetic field, the limiting case, when the electrostatic potential is not differentiable is studied. By this result the general form of the projector to invariant states is established in the case of homogenous, isotropic and self similar electrostatic turbulence. We prove that with probability one all of the trajectories are either unbounded (that coresponds to sub, normal or super diffusion) either are degenerated to a single point, that means that in the physical case when the self similarity is approximate only the trajectories are closed curves with small area. Implications on the electron anomalous transport in tokamak are discussed.

MSC: 60H10, 34F05.

1 Introduction

1.1 The physical aspects

The problem of prediction of the statistical properties of the trajectories of particles moving in a random electromagnetic field with given statistical properties is one of the key problems in the non equilibrium statistical mechanics, with implication on the future technologies related to the magnetic confinement of the plasma, astrophysics[[5]].

Our problem is related to a generic problem that appears in various studies in fluid and plasma turbulence [2]: The motion of a particle in a two-dimensional incompressible static stochastic velocity field. Also in this case the equations can be written like random Hamiltonian system with one degree of freedom. Similar mathematical formalism describe simplified models stochastic magnetic field line dynamics. The applications concern a large class of physical processes and, in particular, the particle and energy transport in hot magnetized plasmas.

The problem of describing the statistical properties of the solutions of Eq.(1) was considered in the works [5], [3], mainly because it is a starting point in the study of the non autonomous generalization of Eq.(1), with long range correlations. The mathematically equivalent problem of the stochastic magnetic field line dynamics was studied in [4], [6], [7].

In realistic situations the random electromagnetic field cannot be modelled with a temporal white noise, consequently the particle motion cannot be modelled with classical stochastic differential equations in the Itô formalism. Moreover, due to the long range time correlations [[5]] the Corsin approximation cannot be used. For the study of the transversal particle transport in constant magnetic field, with random electric field with polynomial decay of the time correlation function, in the zero Larmor radius approximation, a new method was elaborated [[3], [4]], for the study of the diffusive behavior. In this article this problem, will be studied, in the opposite extreme limit of white noise model: the model with frozen turbulence approximation. This extreme limit is important because it is also a starting point of the DCT method.

The problem studied here, in the framework of the infinite ergodic theory, [[13]], is to obtain information on the mean sojourn time of a family of particles in a given domain, when the distribution of the initial position is given. This problem has a particular case: to estimate the mean sojourn time fraction, that means the time fraction spent in the domain where from the initial condition is randomly selected according to given initial distribution. This problem appears naturally for the application in magnetic confinement fusion, where it is desirable that a large fraction of the trajectories to have bounded trajectories.

The computational complexity of this class of problems higher then the lattice QCD with fermions. It is natural to consider as the analogue of the "Ising model" of non equilibrium statistical mechanics.

In the framework of the infinite ergodic theory, when the volume of the phase space is infinite , the mean sojourn time give a partial statistical information about geometry of the trajectories. We will prove that in very general situations, the problem of mean sojourn time fraction can be expressed as suitable matrix element of projection operator to the invariant states, that appears in the von Neumann mean ergodic theorem. In the particular case of the problem of particle motion in a constant magnetic field, in the frozen turbulence and drift approximation, the mean sojourn time fraction is related to the statistical properties of the equipotential surfaces, that again can be expressed and reformulated in the term of projector on subspace of measurable functions with respect to the $\sigma-$ algebra generated by random electrostatic potential. By this reformulation the problem of the mean sojourn time fraction can be treated in the extreme, but solvable limit when the electric field is *self similar, homogenous and isotropic (SHI)*, which is only Hölder continuos but not differentiable. A class of examples of SHI random fields, the fractional Brownian field, was already studied in [24]

We prove that under the SHI assumptions the mean sojourn time fraction of a trajectory is either 0 *(in this case the particle has unbounded trajectory) , either the particle is completely trapped in a single point.*

Recall that in physical situations, when the random field is not exactly self similar at small distances less than a critical distance δl*, this means that the area enclosed by a closed trajectory has the order of magnitude* $O(\delta l^2)$ *. This property is an interesting physical manifestation of the fractal nature of SHI random fields.*

Consequently if we consider the problem of the particle transport in homogenous, isotropic random electric field that is obtained by a regularization of the SHI model, with the very short wavelength components filtered out, the mean sojourn time fraction is related to the geometry of the level surfaces of an SHI random field with continuous realizations. We prove that almost surely the connected components of the level surfaces enclose infinite or zero area.

1.2 The physical problem

In order to illustrate the our initial problem, we consider an one degree of freedom, autonomous, random Hamiltonian systems. The typical interesting case is the charged particle motion, in the zero Larmor radius approximation, transversal to constant magnetic field $\mathbf{B}$, under the effect of the random, static, electric potential $\Phi_\omega(\mathbf{x})$. The potential that contains a set of random parameters, generically denoted by ω that are elements of a probability space Ω. More exactly we denote by $(\Omega, \mathcal{A}, P)$ the probability space and its $\sigma-$algebra related to the realizations of the random electric potential in $\mathbb{R}^2$, $\mathbf{x} \equiv (x_1, x_2) \in \mathbb{R}^2$, are Cartesian coordinates perpendicular to the magnetic field . Then for $\omega \in \Omega$ the scalar function $\Phi_\omega(\mathbf{x})$ is the random potential. In this limit the dynamics is specified by an one degree of freedom random Hamiltonian system, with Hamiltonian function $H_\omega(\mathbf{x}) = \frac{c}{|\mathbf{B}|}\Phi_\omega(\mathbf{x})$ see e.g. [5]

$$\frac{dx_{\omega,i}(t)}{dt} = e_{i,j}\frac{c}{|\mathbf{B}|}\frac{\partial\Phi_\omega(\mathbf{x}_\omega(t))}{\partial x_{\omega,j}} \tag{1}$$

Here $e_{i,j}$ is the two dimensional Levi-Civita symbol. In this case the trajectories are exactly the level sets $\Phi_\omega(\mathbf{x}) = E$. In typical cases the mean sojourn time of a family of trajectory in a finite domain will be zero when the level sets are open.

In this article the main result is that under SHI assumptions the mean sojourn time is either zero, that means that the with probability one the level sets are open curves, going to infinity, either.

The first problem is related to the fact that under SHI assumptions $\Phi_\omega(\mathbf{x})$ is non differentiable a.s. This aspect will be circumvented by reformulation of the initial problem in the term of statistical geometry of the level surfaces of $\Phi_\omega(\mathbf{x})$.

2 Intuitive approach: mean sojourn time, the von Neumann mean ergodic theorem, conditional expectation values

2.1 Deterministic case

2.1.1 Mean sojourn time fraction and the projector to the subspace of invariant states

In order to define the mean sojourn time the general framework of the Hamiltonian dynamics with random perturbations of the Hamiltonian function, we consider first an autonomous deterministic Hamiltonian dynamically system, for the sake of simplicity with phase space $\mathbf{M} = \mathbb{R}^{2N}$. Generalizations are obvious. We denote by $\mathbf{x} = \{x_1, ..., x_{2N}\} \equiv \{p_1, q_1, \cdot\cdot\cdot, p_N, q_N\}$ the canonical phase space coordinates, by $H(\mathbf{x})$ the Hamiltonian, by $\lambda(\mathbf{x})$ the invariant Liouville measure and by $\mathbf{x} \to g_H^t(\mathbf{x})$ the diffeomorphism group associated to $H(\mathbf{x})$ [8] Let $\rho(\mathbf{x})$ the probability density of the distribution of the initial positions of the trajectories. We will denote by $\chi_A(\mathbf{x})$ the characteristic function of the domain $A \subset \mathbf{M}$. We will denote by λ the Lebesgue measure and $L_p = L_p\left(\mathbb{R}^{2N}, d\lambda\right)$, $p \geq 1$. The proofs and the notations will be simplified by the use of Hilbert space formalism of ergodic theory [9]. In the Hilbert space L_2 we have the canonical scalar product, invariant under $g_H^t(\mathbf{x})$

$$\langle \varphi, \psi \rangle = \int_{\mathbf{M}} \varphi^* (\mathbf{x})\, \psi(\mathbf{x}) d\lambda(\mathbf{x}) \tag{2}$$

Remark 1 *In the following we suppose that $\rho(\mathbf{x})$ and $\chi_A(\mathbf{x})$ are square integrable, i.e. the probability density $\rho(\mathbf{x}) \in L_1 \cap L_2$ and $\lambda(A) < \infty$.*

For any $\phi(\mathbf{x}) \in L_2$ we define the unitary operator U_t (Koopman [9]) as follows

$$(U_t\varphi)(\mathbf{x}) = \varphi\left[g_H^t(\mathbf{x})\right] \tag{3}$$

We observe that, in particular, the function $(U_t\chi_A)(x) = \chi_A\left(g_H^t(\mathbf{x})\right) \in L_2$ describes the visit at the time t of the *finite* domain A. Consequently, the mean value $\frac{1}{T}\int_0^T (U_t\chi_A)(\mathbf{x})dt = \frac{1}{T}\int_0^T \chi_A\left[g_H^t(\mathbf{x})\right] dt$ is the mean sojourn time in A of the trajectory that started from the initial position $\mathbf{x}$. By averaging over initial positions $\mathbf{x}$ with the probability density $\rho(\mathbf{x}) \in L_1 \cap L_2$, the mean sojourn time will be

$$S_T(H, A, \rho) := \frac{1}{T}\int_0^T \langle U_t\chi_A, \rho\rangle\, dt \tag{4}$$

that represents the mean sojourn time in the domain A, during the time interval $(0, T)$, of the particles that started according to the distribution $\rho(\mathbf{x})$. For large time T the quantity $S_T(H, A, \rho)$ is one of the candidates, which can describe the trapping effect, or the degree of confinement in the domain $A \subset \mathbf{M}$.

By Liouville theorem U_t is unitary operator. We denote by $\mathcal{H}_{H,inv}$ the subspace of square integrable invariant functions with respect to the dynamics generated by the Hamiltonian function H . From von Neumann mean ergodic theorem [10], [12], [9] results that the strong limit

$$\lim_{t\to\infty}\frac{1}{T}\int_0^T U_t\varphi dt := \widehat{P}_H\varphi \tag{5}$$

exists (as a limit in L_2) for any $\varphi \in L_2$, and $\widehat{P}_H$ is the projection operator on the subspace $\mathcal{H}_{H,inv}$. We have the equivalencies

$$\varphi \in \mathcal{H}_{H,inv} \iff \widehat{P}_H \, \varphi = \varphi \iff U_t\varphi = \varphi \quad \forall\, t \tag{6}$$

$$\iff \varphi\left(g_H^t(\mathbf{x})\right) = \varphi\left(\mathbf{x}\right), \quad \forall\, t \tag{7}$$

It follows that

$$\lim_{t\to\infty} \frac{1}{T}\int_0^T \langle U_t\varphi_1, \varphi_2\rangle \, dt = \left\langle \varphi_1, \widehat{P}_H \; \varphi_2 \right\rangle \tag{8}$$

also exists for all $\varphi_1, \varphi_2 \in L_2$. Consequently the $T \to \infty$ limit in Eq.(4) exists when A is finite and $\rho(\mathbf{x})$ is square integrable. We will denote this limit by $S(H, A, \rho)$. According to Eq.(8) the latter limit can be rewritten in the terms of $\widehat{P}_H$

$$S(H, A, \rho) = \left\langle \chi_A, \widehat{P}_H \; \rho \right\rangle \tag{9}$$

From the definition 5 follows that the $\widehat{P}_H$ preserves the positivity: if $\varphi \in L_2$ and $\varphi(\mathbf{x}) \geq 0$ almost everywhere () then also $\left[\widehat{P}_H\varphi\right](\mathbf{x}) \geq 0$ almost everywhere .

Particular cases of the projector $\widehat{P}_H$ Suppose that there is some invariant domain $D \subset \mathbf{R}^{2N}$ under g_H^t: $g_H^t(D) = D$. Denote by $\widehat{P}_{H,D}$ the restriction of $\widehat{P}_H$ on the invariant subspace $L_2(D)$ of the L_2 functions with support in D.

In the case $E \geq 0$, $\mathbf{a} \in \mathbf{R}^{2N}$ we denote by $L_{H,E,\mathbf{a}} \subset \mathbf{R}^{2N}$ the interior of that connected component of the level subset $\{\mathbf{x}|\mathbf{x} \in \mathbf{R}^{2N},\ |H(\mathbf{x}) - H(\mathbf{a})| \leq E\}$ that contains the point $\mathbf{a}$ and by $\chi_{H,E,\mathbf{a}}(\mathbf{x})$ its characteristic function. In the must interesting cases, when $N = 1$ and there is no open domain in $\mathbf{R}^2$ where $H(\mathbf{x})$ is constant, the closure of the linear combination of functions $\chi_{H,E,\mathbf{a}}(\mathbf{x})$ span the whole invariant space $\mathcal{H}_{H,inv}$. Because $\chi_{H,E,\mathbf{a}}(\mathbf{x}) \in L_2$ iff $\lambda(L_{H,E,\mathbf{a}}) < \infty$, the following cases of invariant functions, invariant domains will appear in this study.

Generic case, confinement. If $\lambda\left(L_{H,E,\mathbf{a}}\right) < \infty$ then $\chi_{H,E,\mathbf{a}}(\mathbf{x}) \in \mathcal{H}_{H,inv} \subset L_2$. If at least such a domain exists then clearly the projection operator $\widehat{P}_H$ is non trivial.

Non generic case, confinement. In the case $E = 0$, if for some $\mathbf{a} \in L_{H,0,\mathbf{a}} \neq \emptyset$, that means , there is some open domain, containing $\mathbf{a}$ where $H(\mathbf{x})$ is constant, then in fact any open subset $D' \subset D = L_{H,0,\mathbf{a}}$ is an invariant domain and if D' is set sufficiently small such that $\lambda(D') < \infty$, then $\chi_{D'}(\mathbf{x}) \in \mathcal{H}_{H,inv}$. In this case it is clear that $\widehat{P}_{H,D} = 1_D$, where 1_D is the identity operator in $L_2(D)$

Conversely, if for some invariant domain $D \subset \mathbf{R}^{2N}$ we have for $\widehat{P}_{H,D} = 1_D$, then then any function from $L_2(D)$ is invariant, so $U_t \downharpoonright D = 1_D$ and the Hamiltonian function is constant in the domain D .

Infinite measure invariant domains, generic case, $E > 0$. Suppose that for some E and $\mathbf{a}$ we have $\lambda\left(L_{H,E,\mathbf{a}}\right) = \infty$. In this case, for the restriction of the projector the the invariant domain $D = L_{H,E,\mathbf{a}}$ we have $\widehat{P}_{H,D} = \widehat{0}_D$.

Infinite measure invariant domains, non generic case, $E = 0$. When $D = L_{H,0,\mathbf{a}}$ has non void interior with $\lambda(D) = \infty$, like in the previous case (2.1.1) all of the finite measure subsets $D' \subset D$ are invariant, and $\widehat{P}_{H,D} = \widehat{1}_D$. In a similar manner the converse is true: if for some infinite measure invariant domain D we have $\widehat{P}_{H,D} = \widehat{1}_D$ then $H(\mathbf{x})$ is constant in D.

Reformulations

$\widehat{P}_H$ **in the term of conditional expectation values** The previous formulation has two drawback. First, the existence of the globally defined map g_H^t requires special treatment. Moreover, in the typical cases we are interested on the limiting cases when the Hamilton function is not smooth, only Holder continuos. Now we try to extend $S(H, A, \rho)$, $\widehat{P_H}$ for some limiting cases, when $H(\mathbf{x})$ is only continuos, so the flow $g_H^t(\mathbf{x})$ cannot be defined. For an arbitrary continuous function $H(\mathbf{x})$, accordingly to the previous discussion (2.1.1), in the generic case we define $\mathcal{T}_{1,H}$ the family of open sets

$$\mathcal{T}_{1,H} = \left\{ L_{H,E,\mathbf{a}} | E > 0, \mathbf{a} \in \mathbf{R}^{2N} \right\}$$

and in order to include also the non generic case we define

$$\mathcal{T}_{2,H} = \left\{ \ interior(D) | \ D \subset L_{H,0,\mathbf{a}}, \mathbf{a} \in \mathbf{R}^{2N} \right\}$$

and $\mathcal{T}_H = \mathcal{T}_{1,H} \cup \mathcal{T}_{2,H}$. Denote by $\mathcal{A}_H$ the $\sigma-$algebra generated by $\mathcal{T}_H$. For $\psi \in L_1 \cap L_2$, the conditional expectation value with respect to the $\sigma-$algebra $\mathcal{A}_H$, $\varphi(\mathbf{x}) = \mathbb{E} \left[\psi(\mathbf{x}) \ | \mathcal{A}_H \right]$ defines a bounded operator in the Hilbert space L_2 that can be extended by continuity to a projector that projects on the sub-space of $\mathcal{A}_H$- measurable functions. In the particular case when $H(\mathbf{x})$ is smooth and $\widehat{P}_H$ can be defined by the evolution map we have

$$\varphi = \left[\widehat{P}_H \ \psi \right] (\mathbf{x}) \ \Leftrightarrow \varphi(\mathbf{x}) = \mathbb{E} \left[\psi(\mathbf{x}) \ | \mathcal{A}_H \right] \tag{10}$$

.This remark is important because the σ- algebra $\mathcal{A}_H$,the projector $\widehat{P}_H$ and the problem of the mean sojourn time can be defined safely also in the limiting case when $H(\mathbf{x})$ is only continuos and non differentiable.

Thus, according to Eqs.(4, **??**), in the large time limit, the mean sojourn time in the domain A , when the distribution of the initial positions is given by probability density function $\rho(\mathbf{x})$, can be written as

$$S(H, \chi_A, \rho) := \left\langle \chi_A, \widehat{P}_H \ \rho \right\rangle \tag{11}$$

The operator $\widehat{P}_H$ has all the properties of projection operator in the Hilbert space L_2. From Eq.(10) results

$$\varphi(\mathbf{x}) \geq 0 \Rightarrow (\widehat{P}_H \varphi)(\mathbf{x}) \geq 0 \quad a.e \tag{12}$$

$$\widehat{P}_H = \widehat{P}_{kH} \tag{13}$$

Thus the function $S(H, \chi_A, \rho)$, respectively the operator $\widehat{P}_H$ describe the geometrical property of the trajectories.