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On the topological generalized crossed modules

Mustafa Habil Gürsoy

Abstract. In this paper, we define the topological versions of the concepts of crossed module over generalized groups, which is called generalized crossed module, and generalized group-groupoid. We construct the categories of the topological generalized crossed modules and their homomorphisms and of the topological generalized group-groupoid and homomorphisms between them. We also obtain some characterizations related to the concepts of topological generalized crossed module and topological generalized group-groupoid. Finally, at the end of the work, we prove that the category of topological generalized crossed modules is equivalent to that of topological generalized group-groupoids whose object sets are commutative topological generalized groups.

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1. Introduction

The idea of generalized group has been initiated by Molaei in [12], which is an important generalization of the notion of group. The most important difference of a generalized group from a group is that it has one unit element for each element. For this reason, each group is actually a generalized group. Generalized groups have been studied by many mathematicians [2, 4, 7, 11].

The concept of crossed module is another fundamental element of this study. The term of crossed module over groups first defined by Whitehead [3] was later studied on other algebraic structures [5, 8]. In the present work, we consider crossed module over generalized groups. This is a generalization of crossed module over groups.

In the category theory, a groupoid can be defined as a category in which every arrow is invertible. After the topological and differentiable versions of the groupoid were introduced by Ehresmann [1], this concept has been extensively studied by many mathematicians with different approaches [10, 11, 15]. Structured groupoid is one of these different approaches. A structured groupoid is a groupoid endowed with another algebraic structure [6, 9, 10, 11, 15]. The concept of group-groupoid defined by Brown and Spencer [15] is the most known structured groupoid. Then, Gursoy and et.al. described the notion of generalized group-groupoid using the concept of Molaei's generalized group [8].

Brown and Spencer proved that the categories of the group-groupoids and crossed modules over groups are equivalent. Then, Porter considered this equivalency in terms of groups with operations and internal groupoids [16]. After these studies, a more

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general case of equivalences showed in [15] and [16] was proved in [8] by using the concept of Molaei's generalized group.

In the present work, we first give the definitions of crossed module over topological generalized groups and topological generalized group-groupoid. Also, we prove that the categories of topological generalized crossed modules and topological generalized group-groupoids in which object sets are commutative topological generalized groups are equivalent. In other words, we present the topological versions of the results given by Gursoy and et.al..

2. Preliminaries

We present here basic concepts based on generalized groups that will be used in the study.

2.1. Generalized groups. We give here some basic informations related to generalized group.

Definition 2.1. [12] Let G be a non-empty set. For an operation on G, we say G a generalized group if the following axioms are verified.
i) $(ab)c = a(bc)$, for all $a, b, c \in G$
ii) For each $a \in G$, there is only one element $e(a) \in G$ such that $e(a)a = a = ae(a)$
iii) For each $a \in G$, there is $a^{-1} \in G$ such that $a^{-1}a = e(a) = aa^{-1}$.

The following lemma gives the characteristic properties of generalized groups.

Lemma 2.1. [12] *If G is a generalized group and $a \in G$, then*
i) there is a unique element $a^{-1} \in G$.
ii) $e(a) = e(a^{-1})$ and $e(e(a)) = e(a)$.
iii) $(a^{-1})^{-1} = a$.

Clearly every group is a generalized group but not conversely. However, the following lemma expresses the relationship between the group and generalized group.

Lemma 2.2. [13] *Each commutative generalized group is a group.*

The following two examples are very appropriate for us to understand the structure of a generalized group.

Example 2.1. [13] Let $G = IR \times (IR \setminus \{0\})$. Then G with the multiplication $(a, b) \cdot (c, d) = (bc, bd)$ is a generalized group. For any element $(a, b) \in G$, the identity is defined by $(a/b, 1)$, and inverse is defined by $(a/b^2, 1/b)$.

Example 2.2. [11] Let G be a generalized group with the multiplication m. Then, $G \times G$ is a generalized group with the multiplication

$$m_1((a, b), (c, d)) = (m(a, c), m(b, d)).$$

For $(a, b) \in G \times G$, the identity is defined by $e_1(a, b) = (e(a), e(b))$ and the inverse is defined by $(a, b)^{-1} = (a^{-1}, b^{-1})$.

Definition 2.2. [12] Let G be a generalized group. It is called normal generalized group if $e(ab) = e(a)e(b)$ for all $a, b \in G$.

Definition 2.3. [12] Let $f : G_1 \to G_2$ be a map between the generalized groups. Then, the map f is called a generalized group homomorphism if $f(ab) = f(a)f(b)$ for all $a, b \in G_1$.

Definition 2.4. [12] Let G be a generalized group and $H \subset G$. Then, H is called a generalized subgroup of G, if $ab^{-1} \in H$ for all $a, b \in H$.

Definition 2.5. [12] Let N be a generalized subgroup of the generalized group G. N is called a generalized normal subgroup if there exist a generalized group H and a homomorphism $f : G \to H$ such that for all $a \in G$, $N_a = \emptyset$ or $N_a = ker f_a$, where $N_a = N \cap G_a$, $G_a = \{g \in G \mid e(g) = e(a)\}$ and $f_a = f\mid_{G_a}$.

Now let's give a theorem from [4] which is necessary for the proof of Theorem 5.2.

Theorem 2.3. *Let G be a normal generalized group satisfying $e(a)b^{-1} = b^{-1}e(a)$, for all $a, b \in G$. Then, we have $(ab)^{-1} = b^{-1}a^{-1}$.*

Let us express the concept of generalized action used in the structure of the generalized crossed module. This definition belongs to Molaei.

Definition 2.6. [13] Let G be a generalized group and X a set. A generalized action of G on X is a map $\dot{}: G \times X \to X$ such that the following conditions hold:
i) $(g_1g_2)\dot{}x = g_1\dot{}(g_2\dot{}x)$, for all $g_1, g_2 \in G$ and $x \in X$.
ii) For all $x \in X$, there is $e(g) \in G$ satisfying $e(g)\dot{}x = x$.

Example 2.3. [13] Let G be generalized group in Example 2.1. IR is a generalized group with multiplication $ab = b$ for any elements a, b. Then, since

$$f : G \to IR, (a, b) \mapsto \frac{a}{b}$$

is a homomorphism of generalized groups, the function

$$: G \times IR \to IR, ((a, b), c)) \mapsto \left(\frac{ac}{b}\right)$$

is clearly a generalized action.

We define the generalized action of a generalized group G on another generalized group H. Because we need it to define generalized crossed module.

Definition 2.7. [8] Let G and H be two generalized groups. A generalized action of G on H is a map

$$\dot{}: G \times H \to H,\ (g, h) \mapsto g\dot{}h$$

such that the following conditions are verified.
i) $(g_1g_2)\dot{}h = g_1\dot{}(g_2\dot{}h)$, $\forall g_1, g_2 \in G$ and $h \in H$
ii) $g\dot{}(h_1h_2) = (g\dot{}h_1)(g\dot{}h_2)$, $\forall g \in G$ and $h_1, h_2 \in H$
iii) For all $h \in H$, there exists an element $e(g) \in G$ such that $e(g)\dot{}h = h$.
iv) $g\dot{}e(h) = e(h)$, $g \in G$,$h \in H$.

Example 2.4. [8] A generalized group G acts on itself with the product $g \cdot h = h$.

Let us control the conditions above.
i) For all $g, h, k \in G$, $g \cdot (h \cdot k) = g \cdot k = k = (gh) \cdot k$.
ii) For all $g, h, k \in G$, $(g \cdot h)(g \cdot k) = hk = g \cdot (hk)$.
iii) There exists element $e(g) \in G$ for all $h \in G$. Because, $e(g) \cdot h = h$ for all $h \in G$.
iv) For all $g_1, g_2 \in G$, we have the equality $g_1 \cdot (e(g_2)) = e(g_2)$.

For further details on generalized groups and generalized actions we refer to [8, 12, 13].

Let us give the definition of a topological generalized group.

Definition 2.8. [2] A generalized group G is called topological generalized group if
1) G is a Hausdorff space
2) The maps $m : G \times G \to G$, $m(g,h) = gh$ and $n : G \to G$, $n(g) = g^{-1}$ are continuous.

Example 2.5. [2] The generalized group $G = IR \times (IR \setminus \{0\})$ in Example 2.1 with the topology induced by a Euclidean metric is a topological generalized group.

Definition 2.9. [2] Let G_1 and G_2 be two topological generalized group. A topological generalized group homomorphism is a continuous generalized group homomorphism $f : G_1 \to G_2$.

2.2. Generalized crossed modules. In this section, we recall the concept of generalized crossed module given in [8].

Definition 2.10. [8] Let G be a generalized group acting on a generalized group H, and let $\eta : H \to G$ be a generalized group homomorphism. Then, a generalized crossed module is a triple (H, G, η) such that the followings are hold.
GCM1) $\eta(h \cdot g) = g\eta(h)g^{-1}$, $\forall h \in H$ and $\forall g \in G$
GCM2) $\eta(h_1) \cdot h_2 = h_1h_2h_1^{-1}$, $\forall h_1, h_2 \in H$.

Example 2.6. [8] Let G be a generalized group and the set

$$I(G) = \big\{f_a \mid f_a : G \to G,\ f_a(b) = aba^{-1},\ a, b \in G\big\}$$

be generalized group of the inner automorphisms of G. In this case, we obtain a generalized crossed module with the generalized group homomorphism

$$\eta : G \longrightarrow I(G),\ \ a \longmapsto \eta(a) = f_a$$

and the generalized action

$$\begin{aligned} I(G) \times G &\longrightarrow G \\ (f_a, b) &\longmapsto f_a \cdot b = f_a(b) = aba^{-1} \end{aligned}$$

It is easily shown that the conditions GCM1) and GCM2) are hold.

Example 2.7. [8] Let G be a generalized group and N be generalized normal subgroup of G. Then we obtain a generalized crossed module $(G, N, \eta = i)$ with the inclusion $\eta = i : N \to G$, $n \mapsto n$, and the generalized action

$$G \times N \to N,\ (g, n) \mapsto g \cdot n = gng^{-1}.$$

Definition 2.11. [8] Let (H, G, η) and $(H^{'}, G^{'}, \eta^{'})$ be generalized crossed modules, let $\tau : H \to H^{'}$ and $\mu : G \to G^{'}$ be generalized group homomorphisms. Then, the pair $(\tau, \mu) : (H, G, \eta) \to (H^{'}, G^{'}, \eta^{'})$ is called a generalized crossed module homomorphism if the following conditions are satisfied:
i) $\mu\eta(h) = \eta^{'}\tau(h)$, $h \in H$
ii) $\tau(g \cdot h) = \mu(g) \cdot \tau(h)$, $h \in H$ and $g \in G$.

Therefore, we obtain the category GCM of generalized crossed modules.

2.3. Generalized group-groupoids. In this section, we give some basic concepts related to the groupoid and group-groupoid. Afterwards, we recall the concept of generalized group-groupoid given in [8].

Definition 2.12. [14] A groupoid consists of the set G of the arrows and the set G_0 of the objects, together with the source map $s : G \to G_0$, target map $s : G \to G_0$, the object map $\epsilon : G_0 \to G$, $x \mapsto \epsilon(x) = \widetilde{x} = 1_x$, the inverse map $i : G \to G$, $a \mapsto i(a) = \bar{a}$, and a partial composition $(a, b) \mapsto a \circ b$ defined on the pullback $G_2 = G * G = \{(a, b) \mid t(a) = s(b)\}$. These maps verify the following conditions:

G1) $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in G$, where $t(a) = s(b)$ and $t(b) = s(c)$.
G2) For any $a \in G$, $\epsilon(s(a)) \circ a = a \circ \epsilon(t(a)) = a$.
G3) For any $a \in G$, $a \circ i(a) = \epsilon(s(a))$, $i(a) \circ a = \epsilon(t(a))$.

The maps in the definition above are called structure maps of groupoid. For a groupoid G over G_0 and $x, y \in G_0$, we have the sets $St_G x = s^{-1}(x)$, $CoSt_G y = t^{-1}(y)$ and $St_G x \cap CoSt_G y = G(x, y)$. Also, the set $G(x, x)$ is a group with the partial composition in G. We say it *vertex group* at x.

Example 2.8. [14] A group is a groupoid with only one object.

Example 2.9. [14] Any set G is a groupoid over itself with $s = t = id_G$.

Example 2.10. [14] For a set X, the product $X \times X$ is a groupoid over X. The maps s and t are the natural projections onto the second and first factors, respectively. The object map is $x \mapsto (x, x)$ and the partial composition is given by $(x, y) \circ (y, z) = (x, z)$. The inverse map is $i(x, y) = (y, x)$.

Definition 2.13. [14] Let G and H be groupoids over G_0 and H_0, respectively. A groupoid homomorphism $G \to H$ is a pair of (f, f_0) of maps $f : G \to H$, $f_0 : G_0 \to H_0$ such that $s_H \circ f = f_0 \circ s_G$, $t_H \circ f = f_0 \circ t_G$ and $f(a \circ b) = f(a) \circ f(b)$, $\forall (a, b) \in G_2$.

We sometimes denote the groupoid homomorphism (f, f_0) by f. Therefore, we obtain the category *Gpd* of the groupoids.

Definition 2.14. [14] A topological groupoid is a groupoid G over G_0 such that G and G_0 are topological spaces and the structure maps are continuous.

Let us give the definition of generalized group-groupoid which is a generalized group object in the category of groupoids.

Definition 2.15. [14] A generalized group-groupoid G is a groupoid where G_0 and G both have generalized group structures such that the following maps are groupoid homomorphisms:
i) $m : G \times G \to G$, $m(g, h) = gh$ (multiplication)
ii) $e : * \to G$, where $*$ is a singleton (unit)
iii) $n : G \to G$, $n(g) = g^{-1}$ (inverse).

Also there exists an interchange law between the multiplication of generalized group and the composition of groupoid:

$$(h_1 \circ g_1)(h_2 \circ g_2) = (h_1 h_2) \circ (g_1 g_2).$$

Example 2.11. [8] If G is a generalized group, then $G \times G$ is a generalized group-groupoid over G. Indeed, from Example 2.10 we know that $G \times G$ is a groupoid.

Furthermore, since G is a generalized group, $G \times G$ is also a generalized group with the operation $(x, y)(z, t) = (xz, yt)$. For an element (x, y), the identity and inverse are defined by $(e(x), e(y))$ and (y^{-1}, x^{-1}), respectively.

On the other hand, the maps m, e, n for the generalized group $G \times G$ are groupoid homomorphisms. Namely;

For the multiplication m of $G \times G$, we have

$$\left((z,y)(z',y')\right) \circ \left((y,x)\,(y',x')\right) = \left(zz',yy'\right) \circ \left(yy',xx'\right) = \left(zz',xx'\right)$$

and

$$((z,y) \circ (y,x)) \left((z',y') \circ (y',x')\right) = (zx)\,(z'x') = \left(zz',xx'\right).$$

Hence, the multiplication of $G \times G$ is a groupoid homomorphism. It is easily shown that the maps e and n are groupoid homomorphisms. So, $G \times G$ is a generalized group-groupoid.

Definition 2.16. [8] Let G and H be generalized group-groupoids. A generalized group-groupoid homomorphism $f : G \to H$ is a homomorphism of underlying groupoids such that generalized group structure is preserved.

Therefore, we obtain the category $GG-Gd$ of the generalized group-groupoids and their homomorphisms.

3. Topological generalized crossed modules

We will present here the topological version of the concept of generalized crossed module. Also we will define the homomorphism of topological generalized crossed modules. Thus we will reach to the category of topological generalized crossed modules.

Definition 3.1. An action of a topological generalized group G on a topological space X is a continuous map $^{\cdot} : G \times X \to X$, $(g, x) \mapsto g^{\cdot}x$ satisfying the conditions
i) $(g_1g_2)^{\cdot}x = g_1 {}^{\cdot}(g_2 {}^{\cdot}x)$, for all $g_1, g_2 \in G$ and $x \in X$.
ii) For all $x \in X$, there is $e(g) \in G$ such that $e(g) {}^{\cdot}x = x$.

After this definition, let us state the topological generalized action of a topological generalized group G on a topological generalized group H.

Definition 3.2. Let G and H be topological generalized groups. Then, if the generalized action $^{\cdot} : G \times H \to H$, $(g, h) \mapsto g {}^{\cdot}h$ is continuous, we say it *topological generalized action.*

Proposition 3.1. *The semidirect product of two topological generalized groups is again a topological generalized group.*

Proof. Let H and G be topological generalized groups. Let us have a topological generalized action of G on H below:

$$\cdot : G \times H \to H,\ (g, h) \mapsto g \cdot h$$

Then, the semi-direct product $G \ltimes H$ with the multiplication

$$(g, h)(g_1, h_1) = (gg_1, h(g \cdot h_1))$$

is a generalized group (see [8] for details). Obviously, the set $G \ltimes H$ is a Hausdorff space with the subspace topology induced from the product topology of $G \times H$. Thus, the semi-direct product $G \times H$ is a topological generalized group.

□

Now we can state the definition of a topological generalized crossed module.

Definition 3.3. Let C, G be two topological generalized groups and let $\eta : C \to G$ be a homomorphism of topological generalized groups. Let us have a topological generalized action of G on C. Then, we say that the triple (C, G, η) is a topological generalized crossed module if the following conditions hold:
TGCM1) $\eta(c \cdot g) = g\eta(c)g^{-1}$, $\forall c \in C$ and $\forall g \in G$
TGCM2) $\eta(c) \cdot c' = cc'c^{-1}$, $\forall c, c' \in C$.

Example 3.1. Let G be a topological generalized group and let N be its topological generalized normal subgroup. Then we construct a topological generalized crossed module $(G, N, \eta = i)$ with the inclusion

$$\eta = i : N \to G, n \mapsto n$$

and the topological generalized action

$$G \times N \to N, (g, n) = g^{\cdot}n = gng^{-1}.$$

Indeed, it is clear that $(G, N, \eta = i)$ is a generalized crossed module by Example 4.2 of [8]. So, it is enough to show the continuity of $\eta = i$ and the generalized action. It is clear that the inclusion map i is continuous. Further, since the operation of generalized action is the operation of the topological generalized group G, it is also continuous. Consequently, the triple $(G, N, \eta = i)$ is a topological generalized crossed module.

Let us now define the notion of topological generalized crossed module homomorphism.

Definition 3.4. Let (H, G, η) and (H', G', η') be topological generalized crossed modules, let $(\tau, \mu) : (H, G, \eta) \to (H', G', \eta')$ be a generalized group homomorphism. If τ and μ are continuous, then (τ, μ) is called a topological generalized group homomorphism.

Therefore, we obtain the category $TGCM$ of the topological generalized crossed modules.

4. Topological generalized group-groupoids

Here we give the topological version of the concept of generalized group-groupoid defined in [8].

Definition 4.1. Let G be a generalized group-groupoid. If G is both a topological groupoid and a topological generalized group such that the structure maps in Definition 2.15 are continuous, then we call G a topological generalized group-groupoid.

Example 4.1. If G is a topological generalized group, then $G \times G$ is a topological generalized group-groupoid.

By Example 2.11, $G \times G$ is a generalized group-groupoid. Furthermore, since G is a Hausdorff space, $G \times G$ is a topological groupoid by [14]. Also, the set $G \times G$ with the product topology and the operation

$$(x, y)(z, t) = (xz, yt)$$

defined by the multiplication of G is a topological generalized group. Finally, since the generalized group structure maps of $G \times G$ are defined by the the structure maps of the topological generalized group G, they are continuous. Thus, $G \times G$ is a topological generalized group-groupoid.

Definition 4.2. Let G and H be topological generalized group-groupoids. A homomorphism of the topological generalized group-groupoids is a continuous homomorphism $f : G \to H$ of the underlying groupoids preserving the generalized group operation.

Thus, we obtain the category $TGG - Gd$ of the topological generalized group-groupoids and their homomorphisms.

The example above defines a functor from the category TGG to the category $TGG - Gd$.

Proposition 4.1. *There is a functor* $\Gamma : TGG \to TGG - Gd$.

Proof. Let G be an object in TGG. By Example 4.1, $G \times G$ is a topological generalized group-groupoid. If $f : G_1 \to G_2$ is a topological generalized group homomorphism, then

$$\begin{aligned} \Gamma(f) : G_1 \times G_1 &\longrightarrow G_2 \times G_2 \\ (a, b) &\longmapsto (f(a), f(b)) \end{aligned}$$

is also a topological generalized group-groupoid homomorphism. Namely, since f is a topological generalized group homomorphism, $\Gamma(f) = (f, f)$ preserves the generalized group structure. Also, since f is continuous, $\Gamma(f) = (f, f)$ is also continuous. On the other hand, from the Example 2.11, we have

$$\Gamma(f)((z, y) \circ (y, x)) = \Gamma(f)(z, y) \circ \Gamma(f)(y, x).$$

Hence, $\Gamma(f)$ is a topological generalized group-groupoid homomorphism. Consequently, Γ is a functor.

□

Remark 4.1. In the next section, we are going to show that the category of topological generalized group-groupoids whose object sets are commutative topological generalized groups is equivalent to the category $TGCM$. For this reason, at this point, we need a warning: The category (denoted by $TGG - Gd^*$) of topological generalized group-groupoids whose object sets are commutative topological generalized groups is full subcategory of the category of topological generalized group-groupoids.

5. Equivalence of the categories

Let us first show how a topological generalized crossed module from a topological generalized group-groupoid G which object space is a commutative topological generalized group is obtained.

Theorem 5.1. *Every topological generalized group-groupoid G which object space is a commutative topological generalized group induces a topological generalized crossed module.*

Proof. A generalized crossed module $\varphi(G) = (A, B, \eta)$ inside a generalized group-groupoid G of which object set is a commutative generalized group is algebraically obtained by carrying out the following steps:
i) $A = CoSt_G e(x)$, $\forall x \in G_0$, is a generalized group.
ii) The set $B = G_0$ is a generalized group.
iii) $\eta : A \to B$ is a generalized group homomorphism.
iv) $\cdot : B \times A \to A$ is a generalized action.

Algebraic details of these steps are available in [8]. Hence, we here only show that each of A and B is a topological generalized group and that both η and the generalized action are continuous.

Since G is a topological generalized group-groupoid, G is a Hausdorff space. Since the property of being a Hausdorff space is hereditary, the set $A \subset G$ is also a Hausdorff space with the subspace topology induced by G. Also, the target generalized group B is clearly a topological generalized group. Because it is determined by G_0.

The boundary map η is the restriction of the target map t of the topological generalized group-groupoid G to A. Hence η is continuous, and so $\eta : A \to B$ is a continuous generalized group homomorphism.

Finally, let us show that the generalized action defined by

$$\cdot : B \times A \to A$$
$$(x, m) \mapsto x \cdot m = 1_x m \overline{1}_x.$$

is continuous. This action can be written as a composition of the maps in the following way:

$$B \times A \xrightarrow{\varepsilon \times I} G \times A \xrightarrow{r} G \xrightarrow{R_{\overline{1}_x}} A$$
$$(x, m) \mapsto (1_x, m) \mapsto 1_x m \mapsto 1_x m \overline{1}_x,$$

where ε is the object map of the groupoid, I is the identity map, r is the restriction of the multiplication of the generalized group $G \times G$ and $R_{\overline{1}_x}$ is right translation. Since each of these maps is continuous, the generalized action $\cdot : B \times A \to A$ is also continuous.

Thus, (A, B, η) is a topological generalized crossed module.

□

Henceforth, we suppose that B is a commutative topological generalized group and A is a topological normal generalized group such that $e(a)b^{-1} = b^{-1}e(a)$ for any $a, b \in A$. Therefore, we have a category $TGCM^*$ of which objects are the topological generalized crossed modules providing the conditions above. It is clearly a full subcategory of $TGCM$.

Theorem 5.2. *Every topological generalized crossed module (A, B, η) gives a topological generalized group-groupoid.*

Proof. From Theorem 5.2 given in [8] we know that the generalized crossed module (A, B, η) is a generalized group-groupoid together with object set B, morphism set

$B \ltimes A$ and groupoid structure maps defined as follows and at the same time generalized group homomorphisms:

The source map : $s : B \ltimes A \to B$, $s(b,a) = b$

The target map : $t : B \ltimes A \to B$, $t(b,a) = \delta(a)b$

The object map : For every $a \in A$, $\epsilon_a : B \to B \ltimes A$, $\epsilon_a(b) = (b, e(a))$

The inverse map : $i : B \ltimes A \to B \ltimes A$, $i(b,a) = (b,a)^{-1} = \big(\eta(a)b, a^{-1}\big)$.

The partial composition : $\circ : B \ltimes A \times B \ltimes A \to B \ltimes A$, $(b_1, a_1) \circ (b, a) = (b, a_1 a)$

Clearly, the object set B of the topological generalized group-groupoid is topological generalized group. Also, by Theorem 3.1, the morphism set $B \ltimes A$ is a topological generalized group.

The source map s is continuous, because it is the first projection. The target map t can be written as

$$B \times A \xrightarrow{I \times \eta} B \times B \longrightarrow B$$
$$(b,a) \mapsto (b, \eta(a)) \mapsto \eta(a)b.$$

Since I is the identity map and η is the topological generalized group homomorphism, the continuity of the target map t is obvious.

The object map ϵ_a is obviously in the form of $(I, c_{e(a)})$, here $c_{e(a)}$ is a constant map. The continuity of ϵ_a immediately follows from the continuities of the identity map I and the constant map $c_{e(a)}$.

The continuity of the inverse map i and the composition map are easily obtained from that of the topological generalized group homomorphism η, generalized action and the multiplication of topological generalized group A.

Thus, $B \ltimes A$ is a topological generalized group-groupoid over B. See [8] for algebraic details.

□

Theorem 5.3. *The categories $TGG - Gd^*$ and $TGCM^*$ are equivalent.*

Proof. Let $M = (A, B, \eta)$ and $M' = (A', B', \eta')$ be topological generalized crossed modules, and $(\tau, \mu) : (A, B, \eta) \to (A', B', \eta')$ be a topological generalized crossed module homomorphism. Thus, by Theorem 5.1, we have a functor $\theta : TGCM^* \to TGG - Gd^*$ which is defined by $\theta(\tau, \mu) = (\mu \times \tau, \mu)$ on morphisms and by $\theta(M) = (B, B \ltimes A)$ on objects.

Let G and H be two topological generalized group-groupoids, and $(f, f_0) : G \to H$ be a topological generalized group-groupoid homomorphism. Thus, by Theorem 5.2, we have a functor $\Gamma : TGG - Gd^* \to TGCM^*$ which is defined by $\Gamma(f, f_0) = (f|_{Kers}, f_0)$ on morphisms and by $\Gamma(G) = (Kers, G_0, t|_{Kers})$ on objects.

It is clear that $\Gamma\theta \simeq 1_{TGCM^*}$ and $\theta\Gamma \simeq 1_{TGG-Gd^*}$. □

6. Conclusions

In this paper, using the concept of generalized group which was given by Molaei, we have constructed the topological versions of the concepts of the crossed module over generalized groups and of the generalized group-groupoid. Also, we have showed that the category of topological generalized crossed modules is equivalent to that of topological generalized group-groupoids whose object sets are commutative topological generalized groups. In algebraic topology, there are a number of categories such as double groupoid, 2-groupoid etc. in which the category of crossed modules over groups is equivalent. We have shown the topological version of one of these equivalents using generalized groups. Other researchers can study these equivalencies based on groups in terms of generalized groups. Therefore, this paper is a preliminary study for these equivalences.

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(Mustafa Habil Gürsoy) Department of Mathematics, Faculty of Art and Science, Inonu University, Malatya, 44280, Turkey
E-mail address: mhgursoy@gmail.com

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Signed double Roman domination numbers in digraphs

Jafar Amjadi and Fatemeh Pourhosseini

Abstract. Let $D = (V, A)$ be a finite simple digraph. A signed double Roman dominating function (SDRD-function) on the digraph D is a function $f : V(D) \to \{-1, 1, 2, 3\}$ satisfying the following conditions: (i) $\sum_{x \in N^-[v]} f(x) \geq 1$ for each $v \in V(D)$, where $N^-[v]$ consist of v and all in-neighbors of v, and (ii) if $f(v) = -1$, then the vertex v must have at least two in-neighbors assigned 2 under f or one in-neighbor assigned 3, while if $f(v) = 1$, then the vertex v must have at least one in-neighbor assigned 2 or 3. The weight of a SDRD-function f is the value $\sum_{x \in V(D)} f(x)$. The signed double Roman domination number (SDRD-number) $\gamma_{sdR}(D)$ of a digraph D is the minimum weight of a SDRD-function on D. In this paper we study the SDRD-number of digraphs, and we present lower and upper bounds for $\gamma_{sdR}(D)$ in terms of the order, maximum degree and chromatic number of a digraph. In addition, we determine the SDRD-number of some classes of digraphs.

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1. Introduction

Let G be a finite and simple graph with vertex set $V(G)$, and let $N_G(v) = N(v)$ be the open neighborhood of the vertex v. A *signed double Roman dominating function* (SDRD-function) on a graph G is defined in [2] as a function $f : V(G) \longrightarrow \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ is adjacent to least two vertices assigned a 2 or to at least one vertex w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $f(N[v]) = \sum_{x \in N[v]} f(x) \geq 1$ holds for each vertex $v \in V(G)$. The *signed double Roman domination number* $\gamma_{sdR}(G)$ of G is the minimum weight of a SDRD-function on G. This parameter has been studied in [1, 3, 7, 9]. A $\gamma_{sdR}(G)$-function is a SDRD-function on G of weight $\gamma_{sdR}(G)$. Following the ideas in [2], we study the SDRD-functions on digraphs D.

Suppose D is a finite simple digraph with vertex set $V(D)$ and arc set $A(D)$ (briefly V and A). The *order* and the *size* of D are integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ respectively. If uv is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v and we also say that x *dominate* y. For each vertex v, the set of in-neighbors and out-neighbors of v are denoted by $N^-(v) = N^-_D(v)$ and $N^+(v) = N^+_D(v)$, respectively. Assume that $N^-_D[v] = N^-[v] = N^-(v) \cup \{v\}$ and $N^+_D[v] = N^+[v] = N^+(v) \cup \{v\}$. We write $d^+(v) = d^+_D(v)$ for the out-degree of a vertex v and $d^-(v) = d^-_D(v)$ for its in-degree. We denote the *minimum* and *maximum in-degree* and the *minimum* and *maximum*

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out-degree of D by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively. A digraph D is called r-out-regular if $\delta^+(D) = \Delta^+(D) = r$. In addition, suppose $\delta = \delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ and $\Delta = \Delta(D) = \max\{\Delta^+(D), \Delta^-(D)\}$ is the *minimum* and *maximum degree* of D, respectively. A digraph D is called *regular* or r-*regular* if $\delta(D) = \Delta(D) = r$. The *distance* $d_D(u, v)$ from a vertex u to a vertex v is the length of a short directed $u - v$ path in D. For every set $X \subseteq V(D)$, $D[X]$ is the subdigraph induced by X. For a real-valued function $f : V \longrightarrow \mathbb{R}$ the weight of f is $\omega(f) = \sum_{v\in V} f(v)$, and for $S \subseteq V$, we write $f(S) = \sum_{v\in S} f(v)$, so $\omega(f) = f(V)$. Consult Haynes et al. [5] for the notation and terminology which are not defined here.

A *signed double Roman dominating function* (SDRD-function) on a digraph D is a function $f : V \longrightarrow \{-1, 1, 2, 3\}$ such that (i) $f(N^-[w]) = \sum_{x\in N^-[w]} f(x) \geq 1$ for each vertex $w \in V$ and (ii) every vertex u for which $f(u) = -1$ has at least one in-neighbor z with $f(z) = 3$ or to at least two in-neighbor v for which $f(v) = 2$, (iii) every vertex v with $f(v) = 1$ has at least one in-neighbor z with $f(z) \geq 2$. The weight of a SDRD-function f on a digraph D is $\omega(f) = \sum_{v\in V(D)} f(v)$. The *signed double Roman domination number (SDRD-number)* $\gamma_{sdR}(D)$ is the minimum weight of a SDRD-function on D. A $\gamma_{sdR}(D)$-function is a SDRD-function on D of weight $\gamma_{sdR}(D)$.

In this paper we initiate the study of the signed double Roman domination number of digraphs, and we establish lower and upper bounds for $\gamma_{sdR}(D)$ in terms of the order, maximum degree and chromatic number of a directed graph. In addition, we determine the SDRD-number of some classes of digraphs.

The *associated digraph* of a graph G, denoted by $D(G) = G^*$, is defined as a digraph obtained from G if each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}[v] = N_G[v]$ for each vertex $v \in V(G) = V(D(G))$, we have the next result.

Remark 1.1. If $D(G)$ is the associated digraph of a graph G, then $\gamma_{sdR}(D(G)) = \gamma_{sdR}(G)$.

In [2], the authors determine the SDRD-number of some classes of graphs including complete graphs, complete bipartite graphs and cycle.

Theorem A. *If $n \neq 4$, then $\gamma_{sdR}(K_n) = 1$ and $\gamma_{sdR}(K_4) = 2$.*

Theorem B. *For $2 \leq m \leq n$,*

$$\gamma_{sdR}(K_{m,n}) = \begin{cases} 3 & \text{if} \quad m = 2 \text{ and } n \geq 3 \\ 4 & \text{if } m \geq 4 \text{ or } m = n = 2 \\ 5 & \text{if } m = 3. \end{cases}$$

Theorem C. *For $n \geq 3$,*

$$\gamma_{sdR}(C_n) = \begin{cases} \frac{n}{3} & \textit{if} \quad n \equiv 0 \ (mod\ 3) \\ \lceil \frac{n}{3} \rceil + 2 & \textit{if} \quad n \equiv 1 \ (mod\ 3) \\ \lceil \frac{n}{3} \rceil + 1 & \textit{if} \quad n \equiv 2 \ (mod\ 3)\ . \end{cases}$$

Using Remark 1.1 and Propositions A, B and C we obtain next result.

Corollary 1.1. (1) *If $n \neq 4$, then $\gamma_{sdR}(K^*_n) = 1$ and $\gamma_{sdR}(K^*_4) = 2$.*

(2) *For* $2 \le m \le n$,

$$\gamma_{sdR}(K^*_{m,n}) = \begin{cases} 3 & \textit{if} \quad m = 2 \textit{ and } n \ge 3 \\ 4 & \textit{if } m \ge 4 \textit{ or } m = n = 2 \\ 5 & \textit{if } m = 3. \end{cases}$$

(3) *For* $n \ge 3$,

$$\gamma_{sdR}(C^*_n) = \begin{cases} \frac{n}{3} & \text{if} \quad n \equiv 0 \pmod 3 \\ \lceil \frac{n}{3} \rceil + 2 & \text{if} \quad n \equiv 1 \pmod 3 \\ \lceil \frac{n}{3} \rceil + 1 & \text{if} \quad n \equiv 2 \pmod 3\ . \end{cases}$$

A *double Roman dominating function* (DRD-function) on D is defined in [6] as a function $f : V \longrightarrow \{0, 1, 2, 3\}$ such that (i) every vertex u for which $f(u) = 0$ has at least one in-neighbor z with $f(z) = 3$ or to at least two in-neighbor v for which $f(v) = 2$, (ii) every vertex v with $f(v) = 1$ has at least one in-neighbor z with $f(z) \ge 2$. The weight of a DRD-function f on a digraph D is $\omega(f) = \sum_{v \in V(D)} f(v)$. The *double Roman domination number (DRD-number)* $\gamma_{sdR}(D)$ is the minimum weight of an DRD-function on D. A $\gamma_{dR}(D)$-function is a DRD-function on D of weight $\gamma_{dR}(D)$. The proof of the next two results can be found in [6].

Theorem D. *For any digraph D, there is a $\gamma_{dR}(D)$-function such that no vertex needs to be assigned the value 1.*

Theorem E. *For any digraph D,*

$$2\gamma(D) \le \gamma_{dR}(D) \le 3\gamma(D).$$

The proof of the following result can be found in Szekeres-Wilf [8].

Theorem F. *For any graph G,*

$$\chi(G) \le 1 + max\{\delta(H) \mid H \textit{ is a subgraph of } G\}.$$

2. Basic Properties

In this section we investigate basic properties of the SDRD-functions and the SDRD-numbers of digraphs. The definitions immediately lead to our first proposition.

Proposition 2.1. *For any SDRD-function $f = (V_{-1}, V_1, V_2, V_3)$ on a digraph D of order n,*
(a) $|V_{-1}| + |V_1| + |V_2| + |V_3| = n$.
(b) $\omega(f) = |V_1| + 2|V_2| + 3|V_3| - |V_{-1}|$.
(c) *$V_2 \cup V_3$ is a dominating set of D. In particular, $|V_2 \cup V_3| \ge \gamma(D)$ where $\gamma(D)$ is the domination number of D.*

Proposition 2.2. *If $f = (V_{-1}, V_1, V_2, V_3)$ is a SDRD-function on a digraph D of order n with maximum out-degree Δ^+ and minimum out-degree δ^+, then*
(i) $(3\Delta^+ + 2)|V_3| + (2\Delta^+ + 1)|V_2| + \Delta^+|V_1| \ge (\delta^+ + 2)|V_{-1}|$.
(ii) $(3\Delta^+ + \delta^+ + 4)|V_3| + (2\Delta^+ + \delta^+ + 3)|V_2| + (\Delta^+ + \delta^+ + 2)|V_1| \ge n(\delta^+ + 2)$.
(iii) $(\Delta^+ + \delta^+ + 2)\omega(f) \ge n(\delta^+ - \Delta^+ + 2) + (\delta^+ - \Delta^+)(2|V_3| + |V_2|)$.
(iv) $\omega(f) \ge n(\delta^+ - 3\Delta^+)/(3\Delta^+ + \delta^+ + 4) + |V_2| + 2|V_3|$